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On the invertible completions for relation matrices

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Abstract. Let H and K be separable Hilbert spaces. In this paper, for $A \in \mathcal{BR}(H)$, $B \in \mathcal{BR}(K)$ and $C \in \mathcal{BR}(K,H)$, a necessary and sufficient condition is given for relation matrices $M_X = \left(\begin{smallmatrix} A & C \\ X & B \end{smallmatrix}\right)$ to be right (left) invertible and invertible relation for some $X \in \mathcal{B}(H,K)$ ($X \in \mathcal{BR}(H,K)$). Moreover, some relevant properties and illustrating examples are also given.

1. introduction

A linear relation $T : H \to K$ is any mapping having domain $\operatorname{dom} T$ a nonempty subspace of H, and taking values in the collection of nonempty subspaces of K, and $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all $x_1, x_2 \in \operatorname{dom} T$ and nonzero scalars $\alpha, \beta \in \mathbb{C}$. We denote by $\mathcal{LR}(H,K)$ the class of linear relations everywhere defined and we write $\mathcal{LR}(H) := \mathcal{LR}(H,H)$ (see [15]).

The graph $G(T)$ of T is

$$G(T) = \{(u,v) \in H \oplus K : u \in \operatorname{dom} T, v \in T(u)\}.$$

The inverse of T is the relation T^{-1} given by $G(T^{-1}) = \{(v,u) \in K \oplus H : (u,v) \in G(T)\}$. The closure of T, denoted by $\overline{T}$, is the linear relation defined by $G(\overline{T}) := \overline{G(T)}$. T is called closed if its graph is a closed subspace of $H \oplus K$. The set of all closed linear relations is denoted by $\mathcal{CR}(H,K)$. The class of linear bounded operators, closed operators and compact operators from H into K is denoted by $\mathcal{B}(H,K)$, $\mathcal{C}(H,K)$ and $\mathcal{K}(H,K)$, respectively. We denote the range and the kernel of T by $\operatorname{ran} T := T(\operatorname{dom} T)$ and $\ker T := \{x \in H : (x,0) \in G(T)\}$, respectively. If $\operatorname{ran} T = K$, then T is called surjective and if $\ker T = \{0\}$, then T is called injective. Clearly, $\operatorname{dom} T^{-1} = \operatorname{ran} T$ and $\operatorname{dom} T = \operatorname{ran} T^{-1}$. T is injective if and only if $T^{-1}T = I_{\operatorname{dom} T}$. We write $n(T) = \dim \ker T$, $d(T) = \dim \operatorname{ran} T^{\perp}$. For $T \in \mathcal{CR}(H,K)$ with closed range $\operatorname{ran} T$, T is said to be left Fredholm, if $n(T) < \infty$; while if $d(T) < \infty$, we say T is right Fredholm. If T is both left and right Fredholm, then it is Fredholm. In addition, we assume T is Fredholm, if $\mathrm{i}(T) = 0$, i.e., $n(T) - d(T) = 0$, relation T is called Weyl. The quotient map from K to $K/\overline{T(0)}$ is denoted by Q_T. It is easy to see that $Q_T T$ is single valued so that we can define $\| Tx \| := \| Q_T T x \|$ for all $x \in \operatorname{dom} T$ and $\| T \| := \| Q_T T \|$. A linear relation T is said to be continuous if for any neighborhood $V \in \operatorname{ran} T$, the inverse image $T^{-1}(V)$ is a neighborhood in H. It can be

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shown that T is continuous if and only if $\| T \| < +\infty$. If T is an everywhere defined linear relation such that $\| T \| < +\infty$ then T is said to be bounded. The class of such relation from H into K is denoted by $\mathcal{BR}(H, K)$, and we denote by $\mathcal{BCR}(H, K)$ the class of bounded closed relation everywhere defined from H into K.

Let $T \in \mathcal{LR}(H, K)$, then the adjoint relation $T^* \in \mathcal{LR}(H, K)$ is defined by

$$G(T^*) = \{(v, v') \in K \oplus H : \langle u', v \rangle = \langle u, v' \rangle \ \textit{for all}\ (u, u') \in G(T)\}.$$

Clearly, if T is densely defined, then T^* is closed single valued relation. Assume $T \in \mathcal{CR}(H, K)$, then ran$T$ is closed if and only if ranT^* is closed (see [9], Theorem III.4.4).

For $T \in \mathcal{LR}(H, K)$, we have several equalities as follows:

$$\ker T^* = \operatorname{ran} T^{\perp};\ \ T^*(0) = \operatorname{dom} T^{\perp};\ \ \ker \overline{T} = \operatorname{ran} (T^*)^{\perp};\ \ \overline{T}(0) = \operatorname{dom} (T^*)^{\perp}.$$

Let $T \in \mathcal{B}(H, K)$, linear operator $T^+ : H \to K$ is said to be the Moore-Penrose generalized inverse of T if T^+ satisfies $\operatorname{dom} T^+ = \operatorname{ran} T \oplus \operatorname{ran} T^{\perp}$ and the four Moore-Penrose equations:

$$TT^+T = T,\ \ T^+T = I - P_{\ker T},\ \ T^+TT^+ = T^+,\ \ TT^+ = P_{\overline{\operatorname{ran} T}} |_{\operatorname{dom} T^+}\ .$$

The Moore-Penrose generalized inverse T^+ is uniquely determined and is a closed linear operator. In particular, for any $y \in \operatorname{ran} T$ we have $y = TT^+y$.

Definition 1.1. *A relation $T \in \mathcal{BCR}(H, K)$ is called a left (right) invertible relation if there exists a bounded operator $S \in \mathcal{B}(K, H)$ such that $ST = I_H$ ($TS = I_K + T(0)$). If T is both left and right invertible relation, then T is invertible relation.*

The right spectrum, left spectrum, spectrum, left essential spectrum, right essential spectrum and Weyl spectrum are defined, respectively, as follows:

$$\begin{aligned}
&\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right invertible relation}\};\\
&\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left invertible relation}\};\\
&\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible relation}\};\\
&\sigma_{le}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Fredholm relation}\};\\
&\sigma_{re}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right Fredholm relation}\};\\
&\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl relation}\}.
\end{aligned}$$

Let $M \subseteq H$ be a subspace, $A \in \mathcal{BR}(H)$, $B \in \mathcal{BR}(K, H)$. The notation A_M denotes the relation given by $G(A_M) = \{(x, y) \in H \oplus H :\ y \in Ax + M\}$. Write $\mathcal{N}(A \mid B) := \{G \in \mathcal{B}(K, H) : \operatorname{ran} AG + B(0) \subseteq \operatorname{ran} B + A(0)\}$ and $A_{[\perp]} := P_{A(0)^{\perp}} A$.

A linear relation is a generalization of a linear operator in multivalued case. If linear relation T maps the points of its domain to singletons, then T is said to be a single valued or simply an operator. The concept of linear relation is mentioned first by J.von Neumann to study the adjoins of non-densely defined linear differential equations[23]. Recently, the linear relations have been studied by numerous articles[1–5, 7, 13, 15, 20, 21, 25, 27].

Operator matrices, as we all know, have always been a hot topic for many scholars and have been studied by a lot of papers[6, 10, 12, 14, 16–19, 22, 26], of which articles [16, 22] discuss the invertibility of operator matrices $M_X \in \mathcal{B}(H \oplus K)$. In this paper, we extent the results in [16, 22] and study the invertibility of relation matrices

$$M_X = \begin{pmatrix} A & C \\ X & B \end{pmatrix} \in \mathcal{BR}(H \oplus K)$$

for an unknown element $X \in \mathcal{B}(H, K)$ ($X \in \mathcal{BR}(H, K)$), where $A \in \mathcal{BR}(H)$, $B \in \mathcal{BR}(K)$ and $C \in \mathcal{BR}(K, H)$. The main difference between relation $T \in \mathcal{BR}(H, K)$ and operator $T \in \mathcal{B}(H, K)$ is the existence of multi-valued part $T(0)$. This paper makes full use of the relationship between $T \in \mathcal{BR}(H, K)$ and $Q_T T \in \mathcal{B}(H, K/T(0))$ to deal with the multi-valued part of linear relations well. We obtain mainly the necessary and sufficient condition for relation matrices M_X to be right (left) invertible and invertible relation for some $X \in \mathcal{B}(H, K)$ ($X \in \mathcal{BR}(H, K)$) by means of space decompositions.

2. Auxiliary results

In the section, we collect some fundamental results, which are useful in later proofs. We start with several results of bounded operators.

Lemma 2.1 (see [11]). *Let H_1 and K_1 be infinite dimensional Hilbert spaces and $T \in \mathcal{B}(H_1, K_1)$, then T is compact if and only if* $\operatorname{ran} T$ *contains no closed infinite dimensional subspaces.*

Lemma 2.2 (see [24]). *Let X and Y be Banach spaces and $T \in \mathcal{B}(X, Y)$ with* $\operatorname{ran} T$ *closed. Then* $\operatorname{ran}(T|_M)$ *is closed for any closed subspace $M \subset X$ if and only if* $\ker T + M$ *is closed.*

Lemma 2.3 (see [16]). *Let $S \in \mathcal{B}(H)$ and $T \in \mathcal{C}(H, K)$. If* $\operatorname{ran} S \subseteq \operatorname{dom} T$, *then $TS \in \mathcal{B}(H, K)$.*

Lemma 2.4 (see [8]). *Let $T \in \mathcal{B}(H, K)$ be a right (left) Fredholm operator and $F \in \mathcal{B}(H, K)$ be a compact operator. Then $T + F$ is a right (left) Fredholm operator and* $\mathrm{i}(T + F) = \mathrm{i}(T)$.

Lemma 2.5 (see [22]). *Let row operator $(S \;\; T) : H \oplus K \to K$ be right invertible.*
(i) *If S is Weyl, then there exists $L \in \mathcal{B}(H, K)$ such that $S + TL$ is invertible;*
(ii) *If T is not compact, then there exists $L \in \mathcal{B}(H, K)$ such that $S + TL$ is invertible if and only if $\mathcal{N}(S \mid T)$ contains a non compact operator.*

Here are some properties of linear relations.

Lemma 2.6 (see [9]). *Let $M \subseteq H$ is a subspace and let J_M denote the natural injection of M into H, i.e.,* $\operatorname{dom} J_M = M$ *and $J_M x = x$ for all $x \in M$. Then $(Q^H_M)^* = J^H_{M^\perp}$ and $(J^H_M)^* = Q^H_{M^\perp}$.*

Lemma 2.7 (see [9]). *Let H_1, H_2 and H_3 be Hilbert spaces, $T \in \mathcal{LR}(H_1, H_2)$ and $S \in \mathcal{LR}(H_2, H_3)$. Then $G(T^*S^*) \subseteq G((ST)^*)$. Furthermore, $(ST)^* = T^*S^*$ if at least one of the following statements is fulfilled:*
(i) $\operatorname{ran} T^* = H_1$ *and* $\operatorname{dom} S \subseteq \operatorname{ran} T$;
(ii) $\operatorname{dom} S^* = H_3$ *and* $\operatorname{ran} T \subseteq \operatorname{dom} S$.

Lemma 2.8 (see [1]). *Let $T \in \mathcal{BCR}(H)$. Then*
(i) *$T \in \Phi_+(H)$ if and only if $Q_T T \in \Phi_+(H, H/T(0))$, and* $\mathrm{i}(T) = \mathrm{i}(Q_T T)$;
(ii) *$T \in \Phi_-(H)$ if and only if $Q_T T \in \Phi_-(H, H/T(0))$, and* $\mathrm{i}(T) = \mathrm{i}(Q_T T)$.

Next, we obtain some auxiliary theorems, which are all necessary in the proofs of the later main results and of interest by themselves.

Theorem 2.9. *Let $T \in \mathcal{BCR}(H)$, then*
(i) *T is a left invertible relation if and only if T is injective and* $\operatorname{ran} T$ *is closed;*
(ii) *T is a right invertible relation if and only if T is surjective.*

Proof. (i) Assume that T is injective and $\operatorname{ran} T$ is closed. Take $S := T^{-1}$. Evidently, S is a bounded operator and $ST = I_H$, so T is left invertible relation. Conversely, let T is left invertible relation, then there exists a bounded operator $S \in \mathcal{B}(K, H)$ such that $ST = I_H$, it is clear that T is injective. Moreover, it follows from $ST(0) = 0$ that $T(0) \subseteq \ker S$. Let $y_n \in \operatorname{ran} T$ and $y_n \to y_0$ as $n \to \infty$, then, for any $n \in \mathbb{N}$, there is $x_n \in H$ such that $y_n \in Tx_n$. This together with $ST = I_H$, we have $STx_n = x_n$, i.e., $S(y_n + T(0)) = Sy_n = x_n$. Note that $y_n \to y_0$ as $n \to \infty$, then the boundedness of S means that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and hence there exists $x_0 \in H$ such that $x_n \to x_0$ as $n \to \infty$. It follows from the boundedness of T that $P_{T(0)^\perp} T$ is bounded. From the equality $Tx_n = P_{T(0)^\perp} Tx_n + T(0)$, we can see that

$$Tx_n \to P_{T(0)^\perp} Tx_0 + T(0),$$

which shows $Tx_n \to Tx_0$ as $n \to \infty$. It follows that $y_0 \in Tx_0$ and thus $\operatorname{ran} T$ is closed.

(ii) Suppose that T is surjective. Take $S := P_{\ker T^\perp} T^{-1}$, then it is clear that S is a bounded operator and $TS = I_K + T(0)$, so T is a right invertible relation. Conversely, let T be a right invertible relation, then there exists a bounded operator $S \in \mathcal{B}(K, H)$ such that $TS = I_K + T(0)$). It is clear that T is surjective. □

Theorem 2.10. *Let $A \in \mathcal{LR}(H)$, $B \in \mathcal{LR}(K)$, $C \in \mathcal{LR}(K,H)$ and $X \in \mathcal{LR}(H,K)$, then*

$$Q_{M_X}M_X = \begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \\ Q_{(X\ B)}X & Q_{(X\ B)}B \end{pmatrix}.$$

Proof. Assume that $(\binom{x}{y}, \binom{u}{v}) \in G(M_X)$, then there exist $u_1 \in Ax$, $u_2 \in Cy, v_1 \in Xx$ and $v_2 \in By$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$. Clearly,

$$Q_{M_X}M_X \begin{pmatrix} x \\ y \end{pmatrix} = Q_{M_X} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Note that $\binom{u'}{v'} \in Q_{M_X}\binom{u}{v}$ if and only if $\binom{u'}{v'} - \binom{u}{v} \in \overline{M_X(0)}$, i.e., $u' - u \in \overline{A(0) + C(0)}$ and $v' - v \in \overline{X(0) + B(0)}$, which are equivalent to $u' \in Q_{(A\ C)}u = Q_{(A\ C)}(u_1 + u_2) = Q_{(A\ C)}u_1 + Q_{(A\ C)}u_2$ and $v' \in Q_{(X\ B)}v = Q_{(X\ B)}(v_1 + v_2) = Q_{(X\ B)}v_1 + Q_{(X\ B)}v_2$, respectively. Hence

$$Q_{M_X} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Q_{(A\ C)}u_1 + Q_{(A\ C)}u_2 \\ Q_{(X\ B)}v_1 + Q_{(X\ B)}v_2 \end{pmatrix}.$$

Since $u_1 \in Ax$, $u_2 \in Cy$, $v_1 \in Xx$ and $v_2 \in By$, we have $Q_{(A\ C)}u_1 = Q_{(A\ C)}Ax$, $Q_{(A\ C)}u_2 = Q_{(A\ C)}Cy$, $Q_{(X\ B)}v_1 = Q_{(X\ B)}Xx$ and $Q_{(X\ B)}v_2 = Q_{(X\ B)}By$. Therefore

$$Q_{M_X}M_X \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Q_{(A\ C)}Ax + Q_{(A\ C)}Cy \\ Q_{(X\ B)}Xx + Q_{(X\ B)}By \end{pmatrix} = \begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \\ Q_{(X\ B)}X & Q_{(X\ B)}B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

□

Theorem 2.11. *Let $A \in \mathcal{BR}(H)$, $B \in \mathcal{BR}(K)$, $C \in \mathcal{BR}(K,H)$ and $X \in \mathcal{BR}(K,H)$, then the relation M_X is closed if and only if $A(0) + C(0)$ and $X(0) + B(0)$ are closed.*

Proof. Assume that $A(0) + C(0)$ and $X(0) + B(0)$ are closed. Equivalently, $M_X(0)$ is closed. It suffices to prove that $Q_{M_X}M_X$ is closed. Note that $Q_{M_X}M_X$ is a single valued relation, and $Q_{M_X}M_X = \left(\begin{smallmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \\ Q_{(X\ B)}X & Q_{(X\ B)}B \end{smallmatrix}\right)$ by Theorem 2.10. Since A is an everywhere defined bounded relation, we have

$$\|Q_{(A\ C)}Ax\| \leq \|Q_AAx\| \leq \|A\|\|x\|, \quad x \in H$$

and hence $Q_{(A\ C)}A \in \mathcal{B}(H)$. Similarly, $Q_{(A\ C)}C \in \mathcal{B}(K,H)$, $Q_{(X\ B)}X \in \mathcal{B}(H,K)$ and $Q_{(X\ B)}B \in \mathcal{B}(K)$ are also clear. Then $Q_{M_X}M_X$ is a bounded everywhere defined operator, which is obviously closed.

Conversely, the closedness of M_C implies that $M_C(0)$ is closed, and hence $A(0) + C(0)$ and $X(0) + B(0)$ are closed. □

Theorem 2.12. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$, $C \in \mathcal{BCR}(K,H)$ and $X \in \mathcal{BCR}(K,H)$ with $A(0) + C(0)$ and $X(0) + B(0)$ closed, then the adjoint of M_X is the single valued relation, and*

$$M_X^* = \begin{pmatrix} A^* & X^* \\ C^* & B^* \end{pmatrix} : (A(0) + C(0))^\perp \oplus (X(0) + B(0))^\perp \to H \oplus K.$$

Proof. Since $A(0) + C(0)$ and $X(0) + B(0)$ are closed, M_X is closed according to Theorem 2.11, and hence $\operatorname{dom} M_C^* = M_C(0)^\perp = (A(0)+C(0))^\perp \oplus (X(0)+B(0))^\perp$ and $\operatorname{dom}\left(\begin{smallmatrix} A^* & X^* \\ C^* & B^* \end{smallmatrix}\right) = (\operatorname{dom} A^* \cap \operatorname{dom} C^*) \oplus (\operatorname{dom} X^* \cap \operatorname{dom} B^*) = (A(0)^\perp \cap C(0)^\perp) \oplus (X(0)^\perp \cap B(0)^\perp)$. This together with $(A(0) + C(0))^\perp = A(0)^\perp \cap C(0)^\perp$ and $(X(0) + B(0))^\perp = X(0)^\perp \cap B(0)^\perp$, we have that

$$\operatorname{dom} M_X^* = \operatorname{dom}\begin{pmatrix} A^* & X^* \\ C^* & B^* \end{pmatrix}.$$

Let $\binom{x}{y} \in H \oplus K$ and $\binom{x^*}{y^*} \in \operatorname{dom} M_X^* = \operatorname{dom} \begin{pmatrix} A^* & X^* \\ C^* & B^* \end{pmatrix}$. By the definition of the adjoint relation on Hilbert spaces we have that

$$\begin{aligned}\langle M_X^* \tbinom{x^*}{y^*}, \tbinom{x}{y} \rangle &= \langle \tbinom{x^*}{y^*}, \tbinom{Ax+Cy}{Xx+By} \rangle \\ &= \langle x^*, Ax + Cy\rangle + \langle y^*, Xx + By\rangle \\ &= \langle A^*x^*, x\rangle + \langle C^*x^*, y\rangle + \langle X^*y^*, x\rangle + \langle B^*y^*, y\rangle \\ &= \langle A^*x^* + X^*y^*, x\rangle + \langle C^*x^* + B^*y^*, y\rangle \\ &= \langle \begin{pmatrix} A^* & X^* \\ C^* & B^* \end{pmatrix} \tbinom{x^*}{y^*}, \tbinom{x}{y} \rangle.\end{aligned}$$

Hence

$$M_X^* = \begin{pmatrix} A^* & X^* \\ C^* & B^* \end{pmatrix}.$$

Moreover, obviously, $A^*(0) = \operatorname{dom} A^\perp = \{0\}$. Similarly, we can obtain that $B^*(0) = C^*(0) = X^*(0) = \{0\}$, it means that $M_X^*(0) = \{0\}$, i.e., M_X^* is single valued relation. $\square$

3. Main results

In this section, we mainly investigate the invertible completions for relation matrices, i.e., Theorems 3.1, 3.4, 3.7, 3.10, 3.13, 3.16. As their corollaries, some related properties are also mentioned. And some examples are given to illustrate the results. We first establish the following perturbation result.

Theorem 3.1. *Let $A \in \mathcal{BR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BR}(K, H)$ with $A(0) + C(0)$ closed, then there is $X \in \mathcal{B}(H, K)$ such that M_X is a right invertible relation if and only if $(A \;\; C)$ is right invertible and at least one of the following statements is fulfilled:*

(i) *$\mathcal{N}(A \mid C)$ contains non compact operators;*

(ii) *$M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a right Fredholm relation and $d(M_0) \le n(A_{C(0)}) + \dim(\operatorname{ran} P_{(A(0)+C(0))^\perp} A \cap \operatorname{ran} C\,|_{\ker B})$.*

Proof. Suppose that the assertion (i) holds true. Clearly, Hilbert spaces H and K are infinite dimensional. By hypothesis, there is a non compact operator $G \in \mathcal{B}(K, H)$ such that $\operatorname{ran} AG + C(0) \subseteq \operatorname{ran} C + A(0)$. It follows from Lemma 2.1 that there exists closed infinite dimensional subspace $M \subseteq H$ for which

$$\operatorname{ran} A\,|_M + C(0) \subseteq \operatorname{ran} C + A(0),$$

and hence $\operatorname{ran} P_{(A(0)+C(0))^\perp} AP_M \subseteq \operatorname{ran} P_{(A(0)+C(0))^\perp} C \subseteq \operatorname{dom}(P_{(A(0)+C(0))^\perp} C)^+$. Note that $P_{(A(0)+C(0))^\perp} AP_M \in \mathcal{B}(H)$, by virtue of Lemma 2.3, we can obtain that

$$(P_{(A(0)+C(0))^\perp} C)^+ P_{(A(0)+C(0))^\perp} AP_M \in \mathcal{B}(H, K).$$

Since $\dim M = \infty$, then there exists a surjective operator $T \in \mathcal{B}(H, K)$ so that $\ker T = M^\perp$. Write operator $X_0 := T + B_{[\perp]}(P_{(A(0)+C(0))^\perp} C)^+ P_{A(0)+C(0))^\perp} AP_M$, then M_{X_0} is a right invertible relation. In fact, since $(A \;\; C)$ is right invertible, $\operatorname{ran} A + \operatorname{ran} C = H$ from Theorem 2.9. Let $\binom{u}{v} \in H \oplus K$, since $\operatorname{ran} A + \operatorname{ran} C = H$ and $\operatorname{ran} A\,|_M + C(0) \subseteq \operatorname{ran} C + A(0)$, there are $x_1 \in M^\perp$ and $y_1 \in K$ such that $u \in Ax_1 + Cy_1$. Moreover, the right invertibility of T implies that there exists $x_2 \in M$ such that $v \in Tx_2 + By_1$. Take $x_0 = x_1 + x_2$ and

$y_0 = y_1 - (P_{(A(0)+C(0))^\perp}C)^+P_{(A(0)+C(0))^\perp}Ax_2$, then

$$\begin{aligned}
&\begin{pmatrix} A & C \\ X_0 & B \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\
=&\begin{pmatrix} Ax_1 + Cy_1 + Ax_2 - \\ (P_{(A(0)+C(0))^\perp}C + P_{A(0)+C(0)}C)(P_{(A(0)+C(0))^\perp}C)^+P_{(A(0)+C(0))^\perp}Ax_2 \\ Tx_2 + By_1 + B_{[\perp]}(P_{(A(0)+C(0))^\perp}C)^+P_{(A(0)+C(0))^\perp}AP_Mx_2 - \\ (B_{[\perp]} + B - B)(P_{(A(0)+C(0))^\perp}C)^+P_{(A(0)+C(0))^\perp}Ax_2 \end{pmatrix} \\
=&\begin{pmatrix} Ax_1 + Cy_1 + Ax_2 - P_{(A(0)+C(0))^\perp}Ax_2 \\ Tx_2 + By_1 \end{pmatrix} \\
=&\begin{pmatrix} Ax_1 + Cy_1 + Ax_2 - P_{(A(0)+C(0))^\perp}Ax_2 - P_{(A(0)+C(0))}Ax_2 \\ Tx_2 + By_1 \end{pmatrix} \\
=&\begin{pmatrix} Ax_1 + Cy_1 \\ Tx_2 + By_1 \end{pmatrix}.
\end{aligned}$$

Evidently,

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \begin{pmatrix} A & C \\ X_0 & B \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Now assume that assertion (ii) is valid. Since relation B is closed, $B(0)$ is closed. As a relation from $H \oplus K$ to $(A(0) + C(0))^\perp \oplus (A(0) + C(0)) \oplus B(0)^\perp \oplus B(0)$, M_X has the matrix form

$$M_X = \begin{pmatrix} A_1 & C_1 \\ A_2 & C_2 \\ X_1 & B_{[\perp]} \\ X_2 & B - B \end{pmatrix}.$$

To prove that M_X is right invertible relation for some $X \in \mathcal{B}(H, K)$, it is enough to show that $\hat{M}_X$ is right invertible relation for some $X_1 \in \mathcal{B}(H, K)$, where

$$\hat{M}_X = \begin{pmatrix} A_1 & C_1 \\ X_1 & B_{[\perp]} \end{pmatrix} : H \oplus K \to (A(0) + C(0))^\perp \oplus B(0)^\perp,$$

since $\operatorname{ran} M_X = \operatorname{ran} \hat{M}_X \oplus (A(0) + C(0)) \oplus B(0)$. Note that $d(M_0) \le n(A_{C(0)}) + \dim(\operatorname{ran} P_{(A(0)+C(0))^\perp}A \cap \operatorname{ran} C\mid_{\ker B})$, and clearly, for single valued relation $\hat{M}_0 : H \oplus K \to (A(0) + C(0))^\perp \oplus B(0)^\perp$,

$$d(\hat{M}_0) \le n(A_{C(0)}) + \dim(\operatorname{ran} P_{(A(0)+C(0))^\perp}A \cap \operatorname{ran} C\mid_{\ker B}),$$

then there exists subspace $N \subseteq H$ such that $\dim N = d(\hat{M}_0)$ and $\operatorname{ran} A_1\mid_N \subseteq \operatorname{ran} C_1\mid_{\ker B}$. Since M_0 is right Fredholm relation, $B_{[\perp]} : K \to B(0)^\perp$ is a right Fredholm relation and hence $\operatorname{ran} B_{[\perp]}$ is closed. Note that $\ker B_{[\perp]} = \ker B$. As a relation from $H \oplus \ker B \oplus \ker B^\perp$ to $(A(0) + C(0))^\perp \oplus \operatorname{ran} B_{[\perp]} \oplus (B(0)^\perp \ominus \operatorname{ran} B_{[\perp]})$, $\hat{M}_0$ has the following matrix form

$$\hat{M}_0 = \begin{pmatrix} A_1 & C_1' & C_1'' \\ 0 & 0 & B_{[\perp]}' \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

It is clear that $B_{[\perp]}'$ is invertible. Put $F = \operatorname{ran} A_1 + \operatorname{ran} C_1'$, i.e., $F = \operatorname{ran} A_1 + \operatorname{ran} C_1\mid_{\ker B}$. The invertibility of $B_{[\perp]}'$ implies that F is closed and $\dim((A(0) + C(0))^\perp \ominus F) = d(\hat{M}_0) - \dim(B(0)^\perp \ominus \operatorname{ran} B_{[\perp]}) < \infty$ according to the expression (1). As a relation from $H \oplus \ker B \oplus \ker B^\perp$ to $F \oplus ((A(0) + C(0))^\perp \ominus F)$, $(A_1\ \ C_1)$ admits the following matrix form

$$(A_1\ \ C_1) = \begin{pmatrix} A_{11} & C_{11} & C_{12} \\ 0 & 0 & C_{13} \end{pmatrix}.$$

Note that $(\ker C_{13})^\perp \subseteq \ker B^\perp$, then $(\ker C_{13})^\perp + \ker B$ is closed, which together with the closedness of $\operatorname{ran} B_{[\perp]}$ implies that $\operatorname{ran} B_{[\perp]} |_{(\ker C_{13})^\perp}$ is closed according to Lemma 2.2. Take $M := (B(0)^\perp \ominus \operatorname{ran} B_{[\perp]}) \oplus \operatorname{ran} B_{[\perp]} |_{(\ker C_{13})^\perp}$, it is clear that M is closed. The right invertibility of $(A\ \ C)$ means that so is $(A_1\ \ C_1)$ and hence $\operatorname{ran} C_{13} = (A(0)+C(0))^\perp \ominus F$, which together with $(\ker C_{13})^\perp \subseteq \ker B^\perp$ ensures that $\dim((A(0)+C(0))^\perp \ominus F) = \dim(\ker C_{13})^\perp = \dim \operatorname{ran} B_{[\perp]} |_{(\ker C_{13})^\perp}$. Then, from equality $\dim((A(0)+C(0))^\perp \ominus F) = d(\hat{M}_0) - \dim(B(0)^\perp \ominus \operatorname{ran} B_{[\perp]}) < \infty$, we see

$$\dim M = d(\hat{M}_0) = \dim N.$$

Define a surjective operator $J : H \to M$ and $\ker J = N^\perp$. Take $X_1 = \begin{pmatrix} J \\ 0 \end{pmatrix} : H \to M \oplus (B(0)^\perp \ominus M)$.

Based on the space decomposition

$$\begin{aligned} H \oplus K &= H \oplus \ker B \oplus (\ker C_{13})^\perp \oplus (\ker B^\perp \ominus (\ker C_{13})^\perp), \\ H \oplus K &= F \oplus ((A(0)+C(0))^\perp \ominus F) \oplus M \oplus (B(0)^\perp \ominus M), \end{aligned}$$

$\hat{M}_X$ can be written as

$$\hat{M}_X = \begin{pmatrix} A_{11} & C_{11} & C_{121} & C_{122} \\ 0 & 0 & C_{131} & 0 \\ J & 0 & B^1_{[\perp]} & 0 \\ 0 & 0 & 0 & B^2_{[\perp]} \end{pmatrix}.$$

From the equality $\operatorname{ran} C_{13} = (A(0)+C(0))^\perp \ominus F$, we see that C_{131} is invertible. It follows from the closedness $\operatorname{ran} B_{[\perp]}$ that $B^2_{[\perp]}$ is invertible. Then there exists invertible operator $U \in \mathcal{B}(F \oplus ((A(0)+C(0))^\perp \ominus F) \oplus M \oplus (B(0)^\perp \ominus M))$ such that

$$U\hat{M}_X = \begin{pmatrix} A_{11} & C_{11} & 0 & 0 \\ 0 & 0 & C_{131} & 0 \\ J & 0 & 0 & 0 \\ 0 & 0 & 0 & B^2_{[\perp]} \end{pmatrix}.$$

So the right invertibility of $\hat{M}_X$ is equivalent to that of $\begin{pmatrix} A_{11} & C_{11} \\ J & 0 \end{pmatrix} : H \oplus \ker B \to F \oplus M$. It will be shown that $\begin{pmatrix} A_{11} & C_{11} \\ J & 0 \end{pmatrix}$ is right invertible. For any $u \in F$ and $v \in M$, since J is right invertible, there is $x_1 \in N$ such that $Jx_1 = v$. Note that $\operatorname{ran} A_1 |_N \subseteq \operatorname{ran} C_1 |_{\ker B}$, then there exist $x_2 \in N^\perp$ and $y_1,\ y_2 \in \ker B$ such that $A_{11}x_2 + C_{11}y_1 = u$ and $A_{11}x_1 + C_{11}y_2 = 0$. Then

$$\begin{pmatrix} A_{11} & C_{11} \\ J & 0 \end{pmatrix} \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Conversely, assume that there is $X \in \mathcal{B}(H,K)$ such that M_X is a right invertible relation. It is clear that $\operatorname{ran} M_X \subseteq \operatorname{ran}(A\ \ C) \oplus K$, which together with $\operatorname{ran} M_X = H \oplus K$, we have that $\operatorname{ran}(A\ \ C) = H$, i.e., $\operatorname{ran} A + \operatorname{ran} C = H$, and hence $(A\ \ C)$ is right invertible from Theorem 2.9. Let $K_1 = (\ker P_{(A(0)+C(0))^\perp} C \cap \ker B)^\perp$. Since B is closed, $B(0)$ and $\ker B$ are closed. Then as a relation from $H \oplus K_1^\perp \oplus K_1$ to $(A(0)+C(0))^\perp \oplus (A(0)+C(0)) \oplus B(0)^\perp \oplus B(0)$, M_X can be written as

$$M_X = \begin{pmatrix} A_1 & 0 & C_1 \\ A_2 & C_3 & C_2 \\ X_1 & 0 & B^1_{[\perp]} \\ X_2 & B-B & B-B \end{pmatrix}.$$

Evidently, $M'_X := \begin{pmatrix} A_1 & C_1 \\ X_1 & B^1_{[\perp]} \end{pmatrix} : H \oplus K_1 \to (A(0)+C(0))^\perp \oplus B(0)^\perp$ is a right invertible operator. It follows that $\ker C_1 \cap \ker B^1_{[\perp]} = \{0\}$, similar to the mean of space decompositions in (1) for M'_0, we can obtain that

$$\begin{aligned} n(M'_0) &= n(A_1) + \dim(\operatorname{ran} A_1 \cap \operatorname{ran} C_1 |_{\ker B^1_{[\perp]}}) \\ &= n(A_1) + \dim(\operatorname{ran} A_1 \cap \operatorname{ran} C_1 |_{\ker B}) \\ &= n(A_{C(0)}) + \dim(\operatorname{ran} P_{(A(0)+C(0))^\perp} A \cap \operatorname{ran} C |_{\ker B}). \end{aligned}$$

There are two possible cases depending on the dimension of $B(0)^{\perp}$.

Case 1: Assume that $\dim B(0)^{\perp} < \infty$. Then X_1 is a compact operator and hence $M_0' = \begin{pmatrix} A_1 & C_1 \\ 0 & B^1_{[\perp]} \end{pmatrix} : H \oplus K_1 \to (A(0)+C(0))^{\perp} \oplus B(0)^{\perp}$ is right Fredholm operator according to Lemma 2.4. Note that $\operatorname{ran} M_X = \operatorname{ran} M_X' \oplus (A(0)+C(0)) \oplus B(0)$ and then $M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a right Fredholm relation, utilizing Lemma 2.4, we have

$$d(M_0) = d(M_0') \leq n(M_0').$$

Therefore,

$$d(M_0) \leq n(A_{C(0)}) + \dim\left(\operatorname{ran} P_{(A(0)+C(0))^{\perp}} A \cap \operatorname{ran} C\,|_{\ker B}\right).$$

Case 2: Assume that $\dim B(0)^{\perp} = \infty$. Note that $M_X' = \begin{pmatrix} A_1 & C_1 \\ X_1 & B^1_{[\perp]} \end{pmatrix} : H \oplus K_1 \to (A(0)+C(0))^{\perp} \oplus B(0)^{\perp}$ is right invertible operator, then there exists a bounded linear operator $\begin{pmatrix} Q & S \\ R & T \end{pmatrix} : (A(0)+C(0))^{\perp} \oplus B(0)^{\perp} \to H \oplus K_1$ such that

$$\begin{pmatrix} A_1 & C_1 \\ X_1 & B^1_{[\perp]} \end{pmatrix}\begin{pmatrix} Q & S \\ R & T \end{pmatrix} = \begin{pmatrix} I_{(A(0)+C(0))^{\perp}} & 0 \\ 0 & I_{B(0)^{\perp}} \end{pmatrix}.$$

Then $X_1 S + B^1_{[\perp]} T = I_{B(0)^{\perp}}$ and $A_1 S + C_1 T = 0$, which means that $\begin{pmatrix} S \\ T \end{pmatrix} : B(0)^{\perp} \to H \oplus K_1$ is left invertible operator and $\operatorname{ran}\begin{pmatrix} S \\ T \end{pmatrix} \subseteq \ker(A_1 \;\; C_1)$. It is easy to see that $n(A_1 \;\; C_1) = \infty$. Put $\begin{pmatrix} G \\ F \end{pmatrix}$ is an invertible operator from $B(0)^{\perp}$ onto $\ker(A_1 \;\; C_1)$. It is easy to see that $A_1 G = -C_1 F$.

We first assume that G is not compact. The equality $A_1 G = -C_1 F$ implies that $\operatorname{ran} A_1 G \subseteq \operatorname{ran} C_1$ and then

$$\operatorname{ran} AG + C(0) \subseteq \operatorname{ran} C + A(0).$$

Hence $\mathcal{N}(A \mid C)$ contains a non compact operator.

Now suppose that G is compact. Define $\begin{pmatrix} Y \\ Z \end{pmatrix} := ((A_1 \;\; C_1)\,|_{\ker(A_1 \;\; C_1)^{\perp}})^{-1} : (A(0)+C(0))^{\perp} \to \ker(A_1 \;\; C_1)^{\perp}$. Then $\operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} = \ker(A_1 \;\; C_1)^{\perp}$ and $A_1 Y + C_1 Z = I_{(A(0)+C(0))^{\perp}}$. Take

$$L = \begin{pmatrix} Y & G \\ Z & F \end{pmatrix} : (A(0)+C(0))^{\perp} \oplus B(0)^{\perp} \to H \oplus K_1.$$

Then L is an invertible operator. Indeed, since $\begin{pmatrix} G \\ F \end{pmatrix}$ is invertible operator, there is an operator $(D \;\; E) : H \oplus K_1 \to B(0)^{\perp}$ such that $DG + EF = I_{B(0)^{\perp}}$. Since $A_1 Y + C_1 Z = I_{(A(0)+C(0))^{\perp}}$, we have

$$\begin{pmatrix} A_1 & C_1 \\ D & E \end{pmatrix} L = \begin{pmatrix} A_1 & C_1 \\ D & E \end{pmatrix}\begin{pmatrix} Y & G \\ Z & F \end{pmatrix} = \begin{pmatrix} I_{(A(0)+C(0))^{\perp}} & 0 \\ DY + EZ & I_{B(0)^{\perp}} \end{pmatrix}$$

is an invertible operator, hence L is a left invertible operator.

In addition, note that $\operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} = \ker(A_1 \;\; C_1)^{\perp}$ and $\operatorname{ran}\begin{pmatrix} G \\ F \end{pmatrix} = \ker(A_1 \;\; C_1)$, we have

$$\operatorname{ran} L = \operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} + \operatorname{ran}\begin{pmatrix} G \\ F \end{pmatrix} = H \oplus K_1,$$

so that L is right invertible operator. This means that L is invertible operator. Note that $A_1 G = -C_1 F$, we have

$$M_X' L = \begin{pmatrix} A_1 & C_1 \\ X_1 & B^1_{[\perp]} \end{pmatrix}\begin{pmatrix} Y & G \\ Z & F \end{pmatrix} = \begin{pmatrix} I_{(A(0)+C(0))^{\perp}} & 0 \\ X_1 Y + B^1_{[\perp]} Z & X_1 G + B^1_{[\perp]} F \end{pmatrix}.$$

It follows from the right invertibility of M_X' that $X_1 G + B^1_{[\perp]} F$ is right invertible. The compactness of G implies that $B^1_{[\perp]} F$ is right Fredholm operator and $d(B^1_{[\perp]} F) \leq n(B^1_{[\perp]} F)$ by Lemma 2.4. This together with

$$\begin{pmatrix} I_{(A(0)+C(0))^{\perp}} & 0 \\ -B^1_{[\perp]} Z & I_{B(0)^{\perp}} \end{pmatrix} M_0' L = \begin{pmatrix} I_{(A(0)+C(0))^{\perp}} & 0 \\ 0 & B^1_{[\perp]} F \end{pmatrix},$$

we have $M_0' = \begin{pmatrix} A_1 & C_1 \\ 0 & B^1_{[\perp]} \end{pmatrix} : H \oplus K_1 \to (A(0)+C(0))^\perp \oplus B(0)^\perp$ is right Fredholm operator and $d(M_0) = d(M_0') = d(B^1_{[\perp]}F) \le n(B^1_{[\perp]}F) = n(M_0')$, which means that M_0 is a right Fredholm operator and

$$d(M_0) \le n(A_{C(0)}) + \dim(\operatorname{ran} P_{(A(0)+C(0))^\perp} A \cap \operatorname{ran} C\,|_{\ker B}).$$

□

Corollary 3.2. *Let $A \in \mathcal{BR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BR}(K,H)$ with $A(0)+C(0)$ closed, then*

$$\begin{aligned}\bigcap_{X\in\mathcal{B}(H,K)} \sigma_r(M_X) = \{\lambda \in \mathbb{C} : \operatorname{ran}(A-\lambda I) + \operatorname{ran} C \ne H\} \\ \cup\{\lambda \in \mathbb{C} :\ \lambda \in \sigma_{re}(M_0),\ \mathcal{N}(A-\lambda I \mid C) \subseteq \mathcal{K}(K,H)\} \\ \cup\{\lambda \in \mathbb{C} :\ \mathcal{N}(A-\lambda I \mid C) \subseteq \mathcal{K}(K,H), \\ d(M_0) > n((A-\lambda I)_{C(0)}) + \dim(\operatorname{ran} P_{(A(0)+C(0))^\perp}(A-\lambda I) \cap \operatorname{ran} C\,|_{\ker(B-\lambda I)})\}.\end{aligned}$$

Corollary 3.3. *Let $A \in \mathcal{BR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) \subseteq C(0)$, then there is $X \in \mathcal{B}(H,K)$ such that M_X is a right invertible relation if and only if $(A\ \ C)$ is right invertible and at least one of the following statements is fulfilled:*

(i) *$\mathcal{N}(A \mid C)$ contains non compact operators;*

(ii) *$M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a right Fredholm relation and $d(M_0) \le n(A_{C(0)}) + \dim(\operatorname{ran} P_{C(0)^\perp} A \cap \operatorname{ran} C\,|_{\ker B})$.*

Theorem 3.4. *Let $A \in \mathcal{BR}(H)$, $B \in \mathcal{BR}(K)$ and $C \in \mathcal{BR}(K,H)$, then there is $X \in \mathcal{BR}(H,K)$ such that M_X is a right invertible relation if and only if $(A\ \ C)$ is right invertible.*

Proof. Assume that $(A\ \ C)$ is right invertible. We write $Xx = K$ for all $x \in H$, then it is easy to see that M_X is a right invertible relation. Conversely, assume that there is $X \in \mathcal{BR}(H,K)$ such that M_X is a right invertible relation. From the proof of Theorem 3.1, the conclusion is valid. □

Corollary 3.5. *Let $A \in \mathcal{BR}(H)$, $B \in \mathcal{BR}(K)$ and $C \in \mathcal{BR}(K,H)$, then*

$$\bigcap_{X\in\mathcal{BR}(H,K)} \sigma_r(M_X) = \{\lambda \in \mathbb{C} : \operatorname{ran}(A-\lambda I) + \operatorname{ran} C \ne H\}.$$

Corollary 3.6. *([16, Theorem 2.1]) Let $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$ and $C \in \mathcal{B}(K,H)$, then there is $X \in \mathcal{B}(H,K)$ such that M_X is a right invertible operator if and only if $(A\ \ C)$ is right invertible and at least one of the following statements is fulfilled:*

(i) *$\mathcal{N}(A \mid C)$ contains non compact operators;*

(ii) *$M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a right Fredholm operator and $d(M_0) \le n(A) + \dim(\operatorname{ran} A \cap \operatorname{ran} C\,|_{\ker B})$.*

Theorem 3.7. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0)+C(0)$ closed, then there is $X \in \mathcal{B}(H,K)$ such that M_X is a left invertible relation if and only if $(B^*\ \ C^*\,|_{(A(0)+C(0)^\perp)})$ is right invertible and at least one of the following statements is fulfilled:*

(i) *$\mathcal{N}(B^* \mid C^*\,|_{(A(0)+C(0)^\perp)})$ contains non compact operators;*

(ii) *$M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a left Fredholm relation and $n(M_0) \le d(B) + \dim(\ker B^\perp \cap \operatorname{ran} C^*\,|_{\operatorname{ran} A^\perp \cap (A(0)+C(0))^\perp})$.*

Proof. Note that relation B is closed and thus $B(0)$ is closed, which together with $A(0)+C(0)$ is closed, we have that M_X is closed by Theorem 2.11. It is not hard to notice that $M_X = \begin{pmatrix} A & C \\ X & B \end{pmatrix}$ is a left invertible relation equivalent to $M_X^* = \begin{pmatrix} B^* & C^* \\ X^* & A^* \end{pmatrix} : B(0)^\perp \oplus (C(0)+A(0))^\perp \to K \oplus H$ is a right invertible operator according to Theorem 2.12. From Theorem 3.1, the conclusion is valid. □

Corollary 3.8. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0)+C(0)$ closed, then*

$$\begin{aligned}\bigcap_{X\in\mathcal{B}(H,K)} \sigma_l(M_X) = \{\lambda \in \mathbb{C} : \operatorname{ran}(B^*-\overline{\lambda} I) + \operatorname{ran} C^*\,|_{(A(0)+C(0))^\perp} \ne K\} \\ \cup\{\lambda \in \mathbb{C} : \lambda \in \sigma_{le}(M_0),\ \mathcal{N}(B^*-\overline{\lambda} I \mid C^*\,|_{(A(0)+C(0))^\perp}) \subseteq \mathcal{K}(H,K)\} \\ \cup\{\lambda \in \mathbb{C} :\ \mathcal{N}(B^*-\overline{\lambda} I \mid C^*\,|_{(A(0)+C(0))^\perp}) \subseteq \mathcal{K}(H,K), \\ n(M_0-\lambda I) > d(B-\lambda I) + \dim(\ker(B-\lambda I)^\perp \cap \operatorname{ran} C^*\,|_{\operatorname{ran}(A-\lambda I)^\perp \cap (A(0)+C(0))^\perp})\}.\end{aligned}$$

Corollary 3.9. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) \subseteq C(0)$, then there is $X \in \mathcal{B}(H,K)$ such that M_X is a left invertible relation if and only if $(B^* \quad C^*)$ is right invertible and at least one of the following statements is fulfilled:*

(i) *$\mathcal{N}(B^* \mid C^*)$ contains non compact operators;*

(ii) *$M_0 = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is a left Fredholm relation and $n(M_0) \leq d(B) + \dim(\ker B^{\perp} \cap \operatorname{ran} C^* \mid_{\operatorname{ran} A^{\perp} \cap C(0)^{\perp}})$.*

Theorem 3.10. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) + C(0)$ closed, then there is $X \in \mathcal{BR}(H,K)$ with $X(0) + B(0)$ closed such that M_X is a left invertible relation if and only if there exists a constant relation $T \in \mathcal{BR}(K)$ such that $T(0) + B(0)$ is closed and $(B^* \mid_{(B(0)+T(0))^{\perp}} \quad C^* \mid_{(A(0)+C(0))^{\perp}})$ is right invertible, and at least one of the following statements is fulfilled:*

(i) *$\mathcal{N}(B^* \mid_{(B(0)+T(0))^{\perp}} \mid C^* \mid_{(A(0)+C(0))^{\perp}})$ contains non compact operators;*

(ii) *$M_0^T = \left(\begin{smallmatrix} A & C \\ 0 & B+T \end{smallmatrix}\right)$ is a left Fredholm relation and*

$$n(M_0^T) \leq n(B^* \mid_{(B(0)+T(0))^{\perp}}) + \dim(\operatorname{ran} B^* \mid_{(B(0)+T(0))^{\perp}} \cap \operatorname{ran} C^* \mid_{\operatorname{ran} A^{\perp} \cap (A(0)+C(0))^{\perp}}).$$

Proof. We first prove the sufficiency. Denote $M_X^T := \left(\begin{smallmatrix} A & C \\ X & B+T \end{smallmatrix}\right) \in \mathcal{BR}(H \oplus K)$. Note that $(M_0^T)^* = \left(\begin{smallmatrix} B^* & C^* \\ 0 & A^* \end{smallmatrix}\right)$: $(B(0) + T(0))^{\perp} \oplus (A(0) + C(0))^{\perp} \to K \oplus H$, then we can see that there exists $X_1 \in \mathcal{B}(H,K)$ such that

$$M_{X_1}^T = \begin{pmatrix} A & C \\ X_1 & B+T \end{pmatrix} \in \mathcal{BR}(H \oplus K)$$

is left invertible by replacing B by $B + T$ in Theorem 3.7. Take $X := X_1 + X - X$, where $(X - X)x = T(0)$ for all $x \in H$. It is clear that M_X is left invertible and hence the sufficiency is valid.

We next assume that there is $X \in \mathcal{BR}(H,K)$ with $X(0) + B(0)$ closed such that M_X is left invertible relation. Put $Tx := X(0)$ for all $x \in K$, it is clear that $T(0) + B(0)$ is closed. It follows from the left invertibility of M_X that $M_X^T = \left(\begin{smallmatrix} A & C \\ X & B+T \end{smallmatrix}\right)$ is a left invertible relation, which means that there is $X \in \mathcal{BR}(H,K)$ such that

$$(M_X^T)^* = \begin{pmatrix} B^* & C^* \\ X^* & A^* \end{pmatrix} : (B(0) + T(0))^{\perp} \oplus (C(0) + A(0))^{\perp} \to K \oplus H$$

is a right invertible operator. From Theorem 3.1, we can obtain the conclusion. □

Corollary 3.11. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) + C(0)$ closed, then*

$$\begin{aligned}
&\textstyle\bigcap_{X \in \mathcal{BR}(H,K), \overline{X(0)+B(0)} = X(0)+B(0)} \sigma_l(M_X) \\
&= \{\lambda \in \mathbb{C} : \text{for any constant relation } T \in \mathcal{BR}(K) \text{ with } T(0) + B(0) \text{ closed}, \\
&\qquad \operatorname{ran}(B^* - \overline{\lambda} I) \mid_{(A(0)+T(0))^{\perp}} + \operatorname{ran} C^* \mid_{(A(0)+C(0))^{\perp}} \neq K, \text{ or} \\
&\qquad \lambda \in \sigma_{le}(M_0^T) \text{ and } \mathcal{N}((B^* - \overline{\lambda} I) \mid_{(A(0)+T(0))^{\perp}} \mid C^* \mid_{(A(0)+C(0)^{\perp})}) \subseteq \mathcal{K}(H,K) \\
&\quad \cup \{\lambda \in \mathbb{C} : \text{for any constant relation } T \in \mathcal{BR}(K) \text{ with } T(0) + B(0) \text{ closed}, \\
&\qquad \operatorname{ran}(B^* - \overline{\lambda} I) \mid_{(A(0)+T(0))^{\perp}} + \operatorname{ran} C^* \mid_{(A(0)+C(0))^{\perp}} \neq K, \text{ or } n(M_0^T - \lambda I) > n((B^* - \overline{\lambda} I) \mid_{(B(0)+T(0))^{\perp}}) + \\
&\qquad \dim(\operatorname{ran}(B^* - \overline{\lambda} I) \mid_{(B(0)+T(0))^{\perp}} \cap \operatorname{ran} C^* \mid_{\operatorname{ran} A^{\perp} \cap (A(0)+C(0))^{\perp}}) \text{ and} \\
&\qquad \mathcal{N}((B^* - \overline{\lambda} I) \mid_{(A(0)+T(0))^{\perp}} \mid C^* \mid_{(A(0)+C(0)^{\perp})}) \subseteq \mathcal{K}(H,K)\}.
\end{aligned}$$

Corollary 3.12. *[16, Theorem 2.7]) Let $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$ and $C \in \mathcal{B}(K,H)$, then there is $X \in \mathcal{B}(H,K)$ such that M_X is a left invertible operator if and only if $(B^* \quad C^*)$ is right invertible and at least one of the following statements is fulfilled:*

(i) *$\mathcal{N}(B^* \mid C^*)$ contains non compact operators;*

(ii) *$M_0 = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is a left Fredholm operator and $n(M_0) \leq d(B) + \dim(\ker B^{\perp} \cap \operatorname{ran} C^* \mid_{\operatorname{ran} A^{\perp}})$.*

Next, we turn our attention to the invertibility of relation matrices.

Theorem 3.13. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0)+C(0)$ closed, then there is $X \in \mathcal{B}(H,K)$ such that M_X is an invertible relation if and only if $(A\ \ C)$ and $(B^*\ \ C^*\mid_{(A(0)+C(0))^\perp})$ are right invertible, and at least one of the following statements is fulfilled:*

(i) *Both $\mathcal{N}(A \mid C)$ and $\mathcal{N}(B^* \mid C^*\mid_{(A(0)+C(0))^\perp})$ contain non compact operators;*

(ii) *$M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a Weyl relation.*

Proof. Let assertion (i) hold true. Then we see that $\dim\ker(A\ \ C) = \dim K = \infty$, and thus we can write a left invertible operator $\begin{pmatrix} E \\ F \end{pmatrix} : K \to H \oplus K$ for which $\operatorname{ran}\begin{pmatrix} E \\ F \end{pmatrix} = \ker(A\ \ C)$. It is clear that $Q_{(A\ C)}(A\ \ C) = (Q_{(A\ C)}A\ \ Q_{(A\ C)}C)$, which together with $A(0)+C(0)$ is closed, we know

$$\ker(A\ \ C) = \ker Q_{(A\ C)}(A\ \ C) = \ker(Q_{(A\ C)}A\ \ Q_{(A\ C)}C).$$

Then we have that $(Q_{(A\ C)}A)E + (Q_{(A\ C)}C)F = 0$. Note that $(A\ \ C)$ is right invertible and thus $Q_{(A\ C)}(A\ \ C)$ is right invertible, that is $(Q_{(A\ C)}A\ \ Q_{(A\ C)}C)$ is right invertible. Hence there exists an operator $\begin{pmatrix} Y \\ Z \end{pmatrix}$: $H/(A(0)+C(0)) \to H \oplus K$ such that $Q_{(A\ C)}AY + Q_{(A\ C)}CZ = I_{H/(A(0)+C(0))}$. Write

$$W = \begin{pmatrix} Y & E \\ Z & F \end{pmatrix} : H/(A(0)+C(0)) \oplus K \to H \oplus K.$$

Since $\begin{pmatrix} E \\ F \end{pmatrix} : K \to H \oplus K$ is a left invertible operator, and hence there is an operator $(Q\ \ R) : H \oplus K \to K$ such that $QE + RF = I_K$. Evidently,

$$\begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \\ Q & R \end{pmatrix}\begin{pmatrix} Y & E \\ Z & F \end{pmatrix} = \begin{pmatrix} I_{H/(A(0)+C(0))} & 0 \\ QY + RZ & I_K \end{pmatrix},$$

and thus W is left invertible. In addition, observing that $Q_{(A\ C)}AY + Q_{(A\ C)}CZ = I_{H/(A(0)+C(0))}$, then

$$\operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} + \ker\begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \end{pmatrix} = H \oplus K.$$

In fact, if $\operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} + \ker(Q_{(A\ C)}A\ \ Q_{(A\ C)}C) \neq H \oplus K$, then there exists $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in H \oplus K$ such that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \notin \operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} + \ker(Q_{(A\ C)}A\ \ Q_{(A\ C)}C)$. Evidently,

$$\operatorname{ran}\begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \end{pmatrix}\mid_{\operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix}} = H/(A(0)+C(0)),$$

and hence there is $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix}$ such that

$$\begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

this means that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \ker(Q_{(A\ C)}A\ \ Q_{(A\ C)}A)$, which contradicts the assumption $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \notin \operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} + \ker(Q_{(A\ C)}A\ \ Q_{(A\ C)}C)$. Note that $\operatorname{ran}\begin{pmatrix} E \\ F \end{pmatrix} = \ker(Q_{(A\ C)}A\ \ Q_{(A\ C)}C)$, we see that $\operatorname{ran} W = \operatorname{ran}\begin{pmatrix} Y \\ Z \end{pmatrix} + \operatorname{ran}\begin{pmatrix} E \\ F \end{pmatrix} = H \oplus K$, and therefore W is right invertible. Hence W is invertible. According to Theorem 2.10, we know

$$Q_{M_X} M_X W = \begin{pmatrix} I_{H/(A(0)+C(0))} & 0 \\ (Q_B X)Y + (Q_B B)Z & (Q_B X)E + (Q_B B)F \end{pmatrix}. \tag{2}$$

Since $\mathcal{N}(A \mid C)$ contains a non compact operator, there exists closed infinite dimensional subspace $M \subseteq H$ such that $\operatorname{ran} A\mid_M + C(0) \subseteq \operatorname{ran} C + A(0)$, which means that $M \subseteq \operatorname{ran} E$, and thus E is a non compact operator by Lemma 2.1. Note that $(B^*\ \ C^*\mid_{(A(0)+C(0))^\perp})$ are right invertible, then $\operatorname{ran} B^* + \operatorname{ran} C^*\mid_{(A(0)+C(0))^\perp} = K$, we have that $\operatorname{ran} B^* + \operatorname{ran} C^* J_{(A(0)+C(0))^\perp} = K$ is valid. It follows from Lemmas 2.7 and 2.6 that $(Q_{(A\ C)}C)^* = C^* J_{(A(0)+C(0))^\perp}$, which together with $\operatorname{ran} B^* = \operatorname{ran}(Q_B B)^*$, we have $\operatorname{ran}(Q_B B)^* + \operatorname{ran}(Q_{(A\ C)}C)^* = K$. Therefore, $\begin{pmatrix} Q_{(A\ C)}C \\ Q_B B \end{pmatrix}$ is left invertible. This together with $\begin{pmatrix} E \\ F \end{pmatrix}$ is left invertible, we have that there exist $(G_1\ \ L_1) : H/(A(0)+C(0)) \oplus K/B(0) \to K$ and $(G_2\ \ L_2) : H \oplus K \to K$ such that $G_1 Q_{(A\ C)}C + L_1 Q_B B = I_K$ and $G_2 E + L_2 F = I_K$. Clearly, $G_1(Q_{(A\ C)}C)F + L_1(Q_B B)F = F$, from $(Q_{(A\ C)}A)E = -(Q_{(A\ C)}C)F$, we can know that

$$G_2 E - L_2 G_1(Q_{(A\ C)}A)E + L_2 L_1(Q_B B)F = G_2 E + L_2 F = I_K,$$

that is $(G_2 - L_2 G_1(Q_{(A\ C)}A))E + L_2 L_1(Q_B B)F = I_K$, which means that $\begin{pmatrix} E \\ (Q_B B)F \end{pmatrix}$ is left invertible. Therefore, $(((Q_B B)F)^*\ \ E^*)$ is right invertible. For a non compact operator $G \in \mathcal{N}(B^* \mid C^* |_{(A(0)+C(0))^\perp})$, we know that $B^* G = C^* |_{(A(0)+C(0))^\perp} L$ for some $L \in \mathcal{B}(H)$. Note that $\operatorname{dom} B^* = B(0)^\perp$, and hence $B^* J_{B(0)^\perp} G = C^* J_{(A(0)+C(0))^\perp} L$, that is $(Q_B B)^* G = (Q_{(A\ C)}C)^* L$. Therefore,

$$((Q_B B)F)^* G = F^*(Q_B B)^* G = F^*(Q_{(A\ C)}C)^* L = -E^*(Q_{(A\ C)}A)^* L,$$

which implies that $\mathcal{N}(((Q_B B)F)^* \mid E^*)$ contains the non compact operator G. Hence, there exists an operator $X_1 : H \to K/B(0)$ such that $(Q_B B)F + X_1 E$ is invertible according to Lemma 2.5 (ii). Let $x \in H$ and $[y] = X_1 x$, then we denote $X : H \to K$ by

$$Xx = P_{B(0)^\perp} y, \quad x \in H,$$

it is clear that $X_1 = Q_B X$. From (2), we have that there exists $X \in \mathcal{B}(H,K)$ such that $Q_{M_X} M_X$ is invertible, and hence M_X is invertible.

We now assume assertion (ii) is valid. Let $\begin{pmatrix} E \\ F \end{pmatrix} : K_1 \to H \oplus K$ be a left invertible operator and

$$\operatorname{ran}\begin{pmatrix} E \\ F \end{pmatrix} = \ker\begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \end{pmatrix},$$

where K_1 is a new Hilbert space with $\dim K_1 = \dim \ker(A\ \ C)$. From the proof of assertion (i), we know that there exists an operator $\begin{pmatrix} Y \\ Z \end{pmatrix} : H/(A(0)+C(0)) \to H \oplus K$ such that

$$W = \begin{pmatrix} Y & E \\ Z & F \end{pmatrix} : H/(A(0)+C(0)) \oplus K_1 \to H \oplus K$$

is an invertible operator. Applying Lemma 2.8, the Weylness of M_0 implies that $Q_{M_0} M_0$ is a Weyl operator and hence $Q_B BF$ is Weyl operator from equality (2), which means $\dim K_1 = \dim K/B(0)$. We may suppose $K_1 := K/B(0)$. From the proof above, row operator $(((Q_B B)F)^*\ \ E^*)$ is right invertible. Note that $(Q_B B)F$ is Weyl operator, utilizing Lemma 2.5 (i), there exists $X_1 \in \mathcal{B}(H, K/B(0))$ for which $(Q_B B)F + X_1 E$ is invertible. Similar to the proof of assertion (i), we have that there exists $X \in \mathcal{B}(H,K)$ such that M_X is invertible.

Conversely, assume that there exists $X \in \mathcal{B}(H,K)$ such that M_X is invertible. From the proof of Theorem 3.1, we know $(A\ \ C)$ is right invertible. Note that M_X is closed, the invertibility of M_X^* is equivalent to that of M_X and hence $M_X^* = \begin{pmatrix} B^* & C^* \\ X^* & A^* \end{pmatrix}$ is invertible. It follows that $\operatorname{ran} B^* + \operatorname{ran} C^* |_{(A(0)+C(0))^\perp} = K$ and so $(B^*\ \ C^* |_{(A(0)+C(0))^\perp})$ is right invertible.

Next we claim that $\mathcal{N}(A \mid C)$ consists of compact operators only. we use here the operator $\begin{pmatrix} E \\ F \end{pmatrix} : K_1 \to H \oplus K$ defined in the proof above. From (2), the invertibility of M_X implies $\dim K_1 = \dim K/B(0)$. If $\dim K/B(0) = \infty$, then we see $\dim K_1 = \dim K = \infty$. Note that

$$\operatorname{ran}\begin{pmatrix} E \\ F \end{pmatrix} = \ker\begin{pmatrix} Q_{(A\ C)}A & Q_{(A\ C)}C \end{pmatrix} = \ker\begin{pmatrix} A & C \end{pmatrix},$$

it is easy to see that there exists an unitary operator $V : K \to K_1$ such that $EV \in \mathcal{N}(A \mid C)$, and then E is compact since $\mathcal{N}(A \mid C)$ consists of compact operators only. If, however, $\dim K/B(0) < \infty$ and then $\dim K_1 < \infty$, which means that $E : K_1 \to H$ remains compact. From the identity (2), the invertibility of M_X shows that $(Q_B X)E + (Q_B B)F$ is invertible. Since E is compact, utilizing Lemma 2.4, $(Q_B B)F$ is Fredholm and

$i((Q_B B)F) = 0$. Take $X = 0$, then, from (2), it is clear that $Q_{M_0}M_0$ is a Weyl relation and hence M_0 is a Weyl relation according to Lemma 2.8. For the case when $\mathcal{N}(B^* \mid C^* \mid_{(A(0+C(0)))^\perp})$ consists of compact operators only, we only need to replace the M_X by $M_X^* : B(0)^\perp \oplus (A(0) + C(0)^\perp) \to K \oplus H$ in the proof above. Similarly, we can obtain that M_0 is a Weyl relation. □

Corollary 3.14. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) + C(0)$ be closed, then*

$$\begin{aligned}\bigcap_{X\in\mathcal{B}(H,K)} \sigma(M_X) = \{&\lambda \in \mathbb{C} : \operatorname{ran}(A - \lambda I) + \operatorname{ran} C \neq H\}\\ &\cup\{\lambda \in \mathbb{C} : \operatorname{ran}(B^* - \overline{\lambda} I) + \operatorname{ran} C^* \mid_{(A(0)+C(0))^\perp}) \neq K\}\\ &\cup\{\lambda \in \mathbb{C} : \ \lambda \in \sigma_w(M_0),\ \mathcal{N}(A - \lambda I \mid C) \subseteq \mathcal{K}(K,H)\}\\ &\cup\{\lambda \in \mathbb{C} : \ \lambda \in \sigma_w(M_0),\ \mathcal{N}(B^* - \overline{\lambda} I \mid C^* \mid_{(A(0)+C(0))^\perp}) \subseteq \mathcal{K}(H,K)\}.\end{aligned}$$

Corollary 3.15. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) \subseteq C(0)$, then there is $X \in \mathcal{B}(H,K)$ such that M_X is an invertible relation if and only if $(A \ \ C)$ and $(B^* \ \ C^*)$ are right invertible, and at least one of the following statements is fulfilled:*

(i) *Both $\mathcal{N}(A \mid C)$ and $\mathcal{N}(B^* \mid C^*)$ contain non compact operators;*

(ii) *$M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a Weyl relation.*

Theorem 3.16. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) + C(0)$ closed, then there is $X \in \mathcal{BR}(H,K)$ with $X(0) + B(0)$ closed such that M_X is an invertible relation if and only if there exists a constant relation $T \in \mathcal{BR}(K)$ such that $T(0) + B(0)$ is closed and $(B^* \mid_{(T(0)+B(0))^\perp} \ \ C^* \mid_{(A(0)+C(0))^\perp})$ and $(A \ \ C)$ are right invertible, and at least one of the following statements is fulfilled:*

(i) *Both $\mathcal{N}(A \mid C)$ and $\mathcal{N}(B^* \mid_{(T(0)+B(0))^\perp} \mid C^* \mid_{(A(0)+C(0))^\perp})$ contain non compact operators;*

(ii) *$M_0^T = \begin{pmatrix} A & C \\ 0 & B+T \end{pmatrix}$ is a Weyl relation.*

Proof. The proof of the sufficiency is similar to that of Theorem 3.10. For the necessity, assume that there is $X \in \mathcal{BR}(H,K)$ with $X(0) + B(0)$ closed such that M_X is an invertible relation. Take $Tx = X(0)$ for all $x \in K$, it follows from the closedness of $X(0) + B(0)$ that $T(0) + B(0)$ is closed. The invertibility of M_X implies that $M_X^T := \begin{pmatrix} A & C \\ X & B+T \end{pmatrix}$ is invertible. Similar to the proof of Theorem 3.13, we can obtain that $(B^* \mid_{(T(0)+B(0))^\perp} \ \ C^* \mid_{(A(0)+C(0))^\perp})$ and $(A \ \ C)$ are right invertible. Again, similar to the proof of Theorem 3.13, if assume that $\mathcal{N}(A \mid C)$ contains compact operators only, then we can obtain that

$$M_{X-X} = \begin{pmatrix} A & C \\ X - X & B \end{pmatrix}$$

is a Weyl relation. Note that T is a constant relation and $T(0) = X(0)$, then it follows from the Weylness of M_{X-X} that $M_0^T = \begin{pmatrix} A & C \\ 0 & B+T \end{pmatrix}$ is a Weyl relation. Similarly, we can obtain that M_0^T is a Weyl relation if $\mathcal{N}(B^* \mid_{(T(0)+B(0))^\perp} \mid C^* \mid_{(A(0+C(0)))^\perp})$ consists of compact operators only. □

Corollary 3.17. *Let $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{BCR}(K,H)$ with $A(0) + C(0)$ closed, then*

$$\begin{aligned}&\bigcap\nolimits_{X\in\mathcal{BR}(H,K),\overline{X(0)+B(0)}=X(0)+B(0)} \sigma(M_X)\\ &= \{\lambda \in \mathbb{C} : \text{ for any constant relation } T \in \mathcal{BR}(K) \text{ with } T(0) + B(0) \text{ closed,}\\ &\quad \operatorname{ran}(B^* - \overline{\lambda} I) \mid_{(A(0)+T(0))^\perp} + \operatorname{ran} C^* \mid_{(A(0)+C(0))^\perp} \neq K \text{ or } \operatorname{ran}(\mathrm{A} - \lambda \mathrm{I}) + \operatorname{ran}\mathrm{C} \neq \mathrm{H}\},\ \text{or}\\ &\quad \lambda \in \sigma_w(M_0^T) \text{ and } \mathcal{N}(A - \lambda I \mid C) \subseteq \mathcal{K}(K,H)\\ &\cup\{\lambda \in \mathbb{C} : \text{for any constant relation } T \in \mathcal{BR}(K) \text{ with } T(0) + B(0) \text{ closed,}\\ &\quad \operatorname{ran}(B^* - \overline{\lambda} I) \mid_{(A(0)+T(0))^\perp} + \operatorname{ran} C^* \mid_{(A(0)+C(0))^\perp} \neq K \text{ or } \operatorname{ran}(\mathrm{A} - \lambda \mathrm{I}) + \operatorname{ran}\mathrm{C} \neq \mathrm{H}\},\ \text{or}\\ &\quad \lambda \in \sigma_w(M_0^T) \text{ and } \mathcal{N}((B^* - \overline{\lambda} I) \mid_{(T(0)+B(0))^\perp} \mid C^* \mid_{(A(0)+C(0))^\perp}) \subseteq \mathcal{K}(H,K)\}.\end{aligned}$$

Corollary 3.18. *[22, Theorem 1]) Let $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$ and $C \in \mathcal{B}(K,H)$, then there is $X \in \mathcal{B}(H,K)$ such that M_X is invertible relation if and only if $(A \ \ C)$ and $(B^* \ \ C^*)$ are right invertible, and at least one of the following statements is fulfilled:*

(i) *Both $\mathcal{N}(A \mid C)$ and $\mathcal{N}(B^* \mid C^*)$ contain non compact operators;*

(ii) *$M_0 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a Weyl operator.*

4. Applications and examples

We begin with some propositions obtained by applying the above conclusions.

Proposition 4.1. *Let $A \in \mathcal{BR}(H)$ and $C \in \mathcal{BR}(K,H)$ with $A(0) + C(0)$ closed.*

(i) *If $P_{(A(0)+C(0))^{\perp}}C$ is compact, then there is $X \in \mathcal{B}(H,K)$ such that $A + CX$ is a right invertible relation if and only if $(A\ \ C)$ is right invertible;*

(ii) *If $P_{(A(0)+C(0))^{\perp}}C$ is non compact, then there is $X \in \mathcal{B}(H,K)$ such that $A + CX$ is a right invertible relation if and only if $(A\ \ C)$ is right invertible and $\mathcal{N}(A \mid C)$ contains non compact operators.*

Proof. First we prove assertion (i). The necessity is clear, we next assume $(A\ \ C)$ is right invertible. Note that

$$(A\ \ C) = \begin{pmatrix} A_1 & C_1 \\ A_2 & C_2 \end{pmatrix} : \begin{pmatrix} H \\ K \end{pmatrix} \to \begin{pmatrix} (A(0) + C(0))^{\perp} \\ A(0) + C(0) \end{pmatrix}. \tag{3}$$

It is clear that the right invertibility of $(A\ \ C)$ is equivalent to that of $(A_1\ \ C_1)$, so $(A_1\ \ C_1)$ is right Fredholm. Since C_1 is compact, A_1 is right Fredholm and hence $\left(\begin{smallmatrix} A_1 & C_1 \\ 0 & I \end{smallmatrix}\right)$ is right Fredholm. The right Fredholmness of A_1 means that $d(\left(\begin{smallmatrix} A_1 & C_1 \\ 0 & I \end{smallmatrix}\right)) = d(\left(\begin{smallmatrix} A_1 & 0 \\ 0 & I \end{smallmatrix}\right)) = d(A_1) \leq n(A_1) + \dim \operatorname{ran} A_1$, by Theorem 3.1, there is $X \in \mathcal{B}(H,K)$ such that $\begin{pmatrix} A_1 & C_1 \\ -X & I \end{pmatrix}$ is right invertible. Note that

$$\begin{pmatrix} I & -C_1 \\ 0 & I \end{pmatrix}\begin{pmatrix} A_1 & C_1 \\ -X & I \end{pmatrix}\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} A_1 + C_1X & 0 \\ 0 & I \end{pmatrix}, \tag{4}$$

which means that $A_1 + C_1X$ is right invertible and hence $A + CX$ is right invertible.

For assertion (ii), we first assume that $(A\ \ C)$ is right invertible and $\mathcal{N}(A \mid C)$ contains non compact operators, which means that $\mathcal{N}(A_1 \mid C_1)$ contains non compact operators, where A_1 and C_1 are defined in equality (3). In virtue of Theorem 3.1, there exists X such that $\left(\begin{smallmatrix} A_1 & C_1 \\ -X & I \end{smallmatrix}\right)$ is right invertible and hence $A_1 + C_1X$ is right invertible by (4). It follows that $A + CX$ is right invertible. We next prove the necessity. Assume that there is $X \in \mathcal{B}(H,K)$ such that $A + CX$ is a right invertible relation. The right invertibility of $(A\ \ C)$ is clear. It is easy to see that $P_{(A(0)+C(0))^{\perp}}(A + CX)$ is a right invertible relation and hence $A_1 + C_1X$ is right invertible by (3). Equality (4) implies that

$$\begin{pmatrix} A_1 & C_1 \\ -X & I \end{pmatrix}$$

is right invertible. It follows that A_1 is right invertible, then

$$(A_1\ \ C_1) = \begin{pmatrix} A_{11} & 0 & C_1 \end{pmatrix} : \begin{pmatrix} \ker A_1^{\perp} \\ \ker A_1 \\ K \end{pmatrix} \to (A(0) + C(0))^{\perp}.$$

Since C_1 is non compact, $\operatorname{ran} C_1$ contains infinite dimensional closed subspaces M. Obviously, A_{11} is invertible, so $\operatorname{ran}(A_{11}^{-1}|_M)$ is closed. Define an operator $J \in \mathcal{B}(K,H)$ such that $\operatorname{ran} J = \operatorname{ran}(A_{11}^{-1}|_M)$, then it is clear that $\operatorname{ran} A_1 J \subseteq \operatorname{ran} C_1$, which means that $\operatorname{ran} A_1 J + \operatorname{ran} P_{(A(0)+C(0))}AJ + A(0) + C(0) \subseteq \operatorname{ran} C_1 + \operatorname{ran} P_{(A(0)+C(0))}C + A(0) + C(0)$, i.e., $\operatorname{ran} AJ + C(0) \subseteq \operatorname{ran} C + A(0)$. Hence $J \in \mathcal{N}(A \mid C)$, it follows that $\mathcal{N}(A \mid C)$ contains non compact operators. □

Similar to the proof of Proposition 4.1, the propositions below can be obtained from Corollary 3.12 and Theorem 3.13, respectively.

Proposition 4.2. *Let $A \in \mathcal{B}(H)$ and $C \in \mathcal{B}(K,H)$.*

(i) *If C is compact, then there is $X \in \mathcal{B}(H,K)$ such that $A + CX$ is left invertible if and only if A is left Fredholm and $n(A) \leq \dim(\operatorname{ran} C^* |_{\operatorname{ran} A^{\perp}})$;*

(ii) *If C is non compact, then there is $X \in \mathcal{B}(H,K)$ such that $A + CX$ is left invertible.*

Proposition 4.3. *Let $A \in \mathcal{BCR}(H)$ and $C \in \mathcal{BCR}(K, H)$ with $A(0) + C(0)$ closed.*

(i) *If $P_{(A(0)+C(0))^\perp}C$ is compact, then there is $X \in \mathcal{B}(H, K)$ such that $A + CX$ is an invertible relation if and only if $(A \ \ C)$ is right invertible and $P_{(A(0)+C(0))^\perp}A$ is Weyl ;*

(ii) *If $P_{(A(0)+C(0))^\perp}C$ is non compact, then there is $X \in \mathcal{B}(H, K)$ such that $A + CX$ is an invertible relation if and only if $(A \ \ C)$ is right invertible and $\mathcal{N}(A \mid C)$ contains non compact operators.*

Next, we end this section with three examples to illustrate the previous results. Assume here that the underlying spaces $H = \ell^2 = K$.

Example 4.4. *Let $A \in \mathcal{BR}(\ell^2)$, $B \in \mathcal{BCR}(\ell^2)$ and $C \in \mathcal{BCR}(\ell^2)$ with $A(0) \subseteq C(0)$. If $(A \ \ C)$ is right invertible and $\mathcal{N}(A \mid C)$ contains non compact operators, then we claim that there is $X \in \mathcal{B}(\ell^2)$ such that M_X is right invertible.*

Indeed, as a relation from $\ell^2 \oplus \ell^2$ to $C(0)^\perp \oplus C(0) \oplus \ell^2$, M_X has the following matrix form $M_X = \begin{pmatrix} A_1 & C_{[\perp]} \\ A_2 & C-C \\ X & B \end{pmatrix}$. It is clear that the right invertibility of M_X is equivalent to that of $\begin{pmatrix} A_1 & C_{[\perp]} \\ X & B \end{pmatrix}$. Define bounded linear operator

$$X := T + B_{[\perp]}(C_{[\perp]})^+ A_1 P_M,$$

where T and M are defined in the proof of Theorem 3.1. Note that

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ -B_{[\perp]}C^+_{[\perp]} & I \end{pmatrix}\begin{pmatrix} A_1 & C_{[\perp]} \\ X & B \end{pmatrix}\begin{pmatrix} I & 0 \\ -C^+_{[\perp]}A_1P_M & I \end{pmatrix} \\ = &\begin{pmatrix} A_1P_{M^\perp} & C_{[\perp]} \\ T + B_{[\perp]}C^+_{[\perp]}A_1P_{M^\perp} + B - B & B_{[\perp]}P_{\ker C} + B - B \end{pmatrix} := L, \end{aligned}$$

and relation L can be written as

$$L = \begin{pmatrix} 0 & E & C_{[\perp]} \\ F & G & B_{[\perp]}P_{\ker C} + B - B \end{pmatrix} : M \oplus M^\perp \oplus \ell^2 \to C(0)^\perp \oplus \ell^2,$$

where $E = (A_1P_{M^\perp})|_{M^\perp}$, $F = (T + B_{[\perp]}C^+_{[\perp]}A_1P_{M^\perp} + B - B)|_M = (T + B - B)|_M$, $G = (T + B_{[\perp]}C^+_{[\perp]}A_1P_{M^\perp} + B - B)|_{M^\perp}$. Since T is surjective and $\ker T^\perp = M$, $F : M \to \ell^2$ is surjective. Note that $(A \ \ C)$ is right invertible and $\operatorname{ran} A \mid_M + C(0) \subseteq \operatorname{ran} C$, then $(E \ \ C_{[\perp]}) : M^\perp \oplus \ell^2 \to C(0)^\perp$ is surjective. Hence M_X is right invertible. This shows the correctness of Corollary 3.3.

Example 4.5. *Let $A, B, C \in \mathcal{BR}(\ell^2)$ are given by*

$$\begin{aligned} &Ax = (x_1, 0, x_2, 0, x_3, 0, \cdots), \ \ Bx = (x_1, x_2, 0, 0, 0, 0 \cdots), \\ &Cx = (0, x_2, 0, x_4, 0, x_6 \cdots) \end{aligned}$$

for all $x = (x_1, x_2, x_3, \cdots) \in \ell^2$, respectively. Then we claim that there exists $X \in \mathcal{BR}(\ell^2)$ such that M_X is right invertible.

Indeed, it is clear that $(A \ \ C)$ is right invertible, then Theorem 3.4 ensures that there exists $X \in \mathcal{BR}(\ell^2)$ such that M_X is right invertible. Alternatively, define $Xx = \ell^2$ for all $x \in \ell^2$. It is clear that $\operatorname{ran} M_X = \ell^2 \oplus \ell^2$, which means that M_X is right invertible. This shows the correctness of Theorem 3.4.

Example 4.6. *Let $A, B, C \in \mathcal{BCR}(\ell^2)$ be given by*

$$\begin{aligned} &Ax = (0, x_2, 0, x_3, 0, x_4, 0, x_5, \cdots), \ \ Bx = (0, x_1, 0, x_3, 0, x_5, 0, x_7 \cdots) + B(0), \\ &Cx = (x_2, 0, 0, 0, x_4, 0, 0, 0, x_6, 0, 0, 0, x_8 \cdots) + C(0), \end{aligned}$$

for all $x = (x_1, x_2, x_3, \cdots) \in \ell^2$, respectively, where

$$\begin{aligned} &B(0) = \{(0, 0, x_1, 0, x_2, 0, x_3, \cdots) : (x_1, x_2, x_3, \cdots) \in \ell^2\}, \\ &C(0) = \{(0, 0, x_1, 0, 0, 0, x_2, 0, 0, 0, x_3, \cdots) : (x_1, x_2, x_3, \cdots) \in \ell^2\}. \end{aligned}$$

Then there exists $X \in \mathcal{B}(K, H)$ such that M_X is invertible.

Evidently, $\operatorname{ran} A + \operatorname{ran} C = \ell^2$ and $\operatorname{ran} B^* + \operatorname{ran} C^* = \ell^2$. And M_0 is Fredholm and $n(M_0) = d(M_0) = 1$. From Theorem 3.13, then there exists $X \in \mathcal{B}(H, K)$ such that M_X is invertible. Indeed, For all $x = (x_1, x_2, x_3, \cdots) \in \ell^2$, we define $Xx = (x_1, 0, 0, 0, 0, \cdots)$. Then we have

$$M_X = \begin{pmatrix} 0 & 0 & C_1 & C - C \\ A_1 & 0 & 0 & 0 \\ 0 & 0 & B - B & B_1 \\ 0 & X_1 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \ker A^{\perp} \\ \ker A \\ \ker C^{\perp} \\ \ker C \end{pmatrix} \to \begin{pmatrix} \operatorname{ran} C \\ \operatorname{ran} C^{\perp} \\ \operatorname{ran} B \\ \operatorname{ran} B^{\perp} \end{pmatrix}.$$

It is clear that M_X is invertible.

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