Filomat 38:7 (2024), 2253–2260
https://doi.org/10.2298/FIL2407253M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Condition pseudospectrum of direct sum of operators on sequence Banach spaces

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Abstract. In this paper, we investigate the relationship between the condition pseudospectrum of linear direct sum of operators in the direct sum of Banach space and its coordinate operators. Besides, we give some remarkable examples as applications of our results.

1. Introduction

Let $(X, \|.\|)$ be an infinite-dimensional Banach space. We denote by $\mathcal{L}(X)$ (resp. $\mathcal{C}(X)$) the set of all bounded (resp. closed, densely defined) linear operators from X into X. The set of all compact operators of $\mathcal{L}(X)$ is denoted by $\mathcal{K}(X)$. We denote by $T = I$ the identity operator. let $T \in \mathcal{L}(X)$, the set

$$\rho(T) := \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is injective and } (\lambda - T)^{-1} \in \mathcal{L}(X)\Big\}.$$

The spectrum of T is the set $\sigma(T) := \mathbb{C}\backslash\rho(T)$. The resolvent set $\rho(T)$ is open, whereas the spectrum $\sigma(T)$ of a closed linear operator T is closed. The condition pseudospectrum of $T \in \mathcal{L}(X)$ is usually defined as

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\},$$

and coincides with the set

$$\bigcup \Big\{\sigma(T + D) : D \in \mathcal{L}(X) \text{ and } \|D\| < \varepsilon\|\lambda - T\|\Big\}$$

where, $\varepsilon > 0$ and $\|\lambda - T\|\|(\lambda - T)^{-1}\|$ is assumed to be infinite, if $\lambda - T$ is not invertible. Note that because of this convention, $\sigma(T) \subseteq \sigma_\varepsilon(T)$ for every $\varepsilon > 0$.

For more information and some properties of condition pseudospectrum and it's generalizations of matrices and operators, (see [2, 3]). The theory of condition pseudospectra and pseudospectrum see [5, 7, 8, 10, 11] provides an analytical and graphical alternative for investigating non-normal operators,

2020 *Mathematics Subject Classification*. 47A10, 47A25
Keywords. Condition pseudospectrum, Direct sum of operators, Direct sum of Banach space.
Received: 11 January 2022; Accepted: 09 October 2023
Communicated by Dragan S. Djordjević
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gives a quantitative estimate of departure from non- normality, and also gives some information about the stability, (see [1, 4, 6, 9]) and it's references. It is known that the spectral theory of linear operators in direct sum of Banach spaces should be examined in order to solve many physical problems in life sciences. However, many physical problems of today arising in the modeling of processes of multi particle quantum mechanics , quantum field theory and in the physics of rigid bodies support to study a theory of linear direct sum of operators in the direct sum of Banach space (see [12–14] and references in it). These and other similar reasons led to the emergence of the topic examined in the current paper.

We study the relationship of the condition pseudospectrum of linear direct sum of operators in the direct sum of Banach space. We extend this by considering first a finite family, and then an at most countable family of operators satisfying some conditions. We establish that under suitable assumptions, the condition pseudospectrum of linear direct sum of operators can be decomposed into the union of the condition pseudospectra of some sequence operators in Banach space. Among other things, we illustrate the applicability of this concepts by a considerable number of examples

The main contributions of this paper are as follows. In Section 2, contains preliminary and auxiliary properties that will be necessary in order to prove the main results of the other sections. In Section 3, we establish the connection between the condition pseudspectrum sets of the direct sum of operators and its coordinate operators. Finally, we will apply the results described above to investigate the condition pseudospectrum of linear direct sum of operators in the direct sum of Banach space.

2. Preliminaries.

In this section, we will give auxiliary definitions and results that we will need later. We begin with the following definition.

Definition 2.1. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$. The condition pseudospectrum of T is denoted by $\Sigma_\varepsilon(T)$ and is defined as,*

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\},$$

with the convention that $\|\lambda - T\|\|(\lambda - T)^{-1}\| = \infty$, if $\lambda - T$ is not invertible. The condition pseudoresolvent of T is denoted by $\rho_\varepsilon(T)$ and is defined as,

$$\rho_\varepsilon(T) := \rho(T) \cap \left\{\lambda \in \mathbb{C} : \ \|\lambda - T\|\|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon}\right\}. \qquad \diamond$$

For $0 < \varepsilon < 1$ it can be shown that $\rho_\varepsilon(T)$ is a larger set and is never empty. Here also

$$\rho_\varepsilon(T) \subseteq \rho(T) \text{ for } 0 < \varepsilon < 1.$$

Recall that the usual condition pseudospectral radius $r_\varepsilon(T)$ of $T \in \mathcal{L}(X)$ is defined by

$$r_\varepsilon(T) := \sup\left\{|\lambda| : \lambda \in \sigma_\varepsilon(T)\right\},$$

and the spectral radius of $T \in \mathcal{L}(X)$ is defined as

$$r(T) = \sup\left\{|\lambda| : \lambda \in \sigma(T)\right\}.$$

Throughout the current paper, the norms $\|.\|_p$ in X and $\|.\|_{X_n}$ in X_n, $n \geq 1$ will be denoted by $\|.\|$ and $\|.\|_n$, $n \geq 1$, respectively. If $T_1, T_0 \in \mathcal{L}(X)$, $T_1 \oplus T_2$ is an operator in $\mathcal{L}(X \oplus X)$ specified by the operator matrix

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

The infinite direct sum of Banach spaces X_n, in the sense of l_p, $1 \leq p < \infty$ and the infinite direct sum of linear densely defined closed operators T_n in X_n, $n \geq 1$ are defined as

$$X = \left(\bigoplus_{n=1}^{\infty} X_n \right)_p = \left\{ x = (x_n) : x_n \in X_n, n \geq 1, \|x\|_p = \left(\sum_{n=1}^{\infty} \|x\|_{X_n}^p \right)^{\frac{1}{p}} \right\}$$

and

$$T = \bigoplus_{n=1}^{\infty} T_n, \ T : \mathcal{D}(T) \subset X \to X,$$

$$\mathcal{D}(T) = \{x = (x_n) \in X : x_n \in \mathcal{D}(T_n), n \geq 1, Ax = (T_n x_n) \in X\}$$

respectively.

Theorem 2.2. [14, Theorem 2.3] *Let $T_n \in \mathcal{L}(X_n), n \geq 1$ and $T = \bigoplus_{n=1}^{\infty} T_n : X \to X$. In order to $T \in \mathcal{L}(X)$ the necessary and sufficient condition is*

$$\sup_{n \geq 1} \|T_n\| < \infty.$$

Moreover, in the case of $T \in \mathcal{L}(X)$, the norm of Tis of the form

$$\|T\| = \sup_{n \geq 1} \|T_n\|.$$

3. Condition pseudospectrum of direct sum of operators.

In this section, the connection between the condition pseudspectrum sets of the direct sum of operators and its coordinate operators will be investigated.

Theorem 3.1. *Let X_n be a Banach space, $T_n \in \mathcal{L}(X_n)$ for $n \geq 1$. And, let $X = \bigoplus_{n=1}^{\infty} X_n$ be the direct sum of X_n, $n \geq 1$, $T = \bigoplus_{n=1}^{\infty} T_n \in \mathcal{L}(X)$ with $\sup_{n \geq 1} \|T_n\| < \infty$. Then, for each $\varepsilon > 0$ the condition pseudospectrum set of the operator T is of the form*

$$\sigma_\varepsilon(T) = \bigcup_{n=1}^{\infty} \sigma_\varepsilon(T_n).$$

Consequently, if $T_n \in \mathcal{L}(X_n)$, $n \geq 1$ and $T \in \mathcal{L}(X)$, the condition pseudospectral radius of the operator T is of the form

$$r_\varepsilon(T) = \sup_{n \geq 1} r_\varepsilon(T_n).$$

Proof. From Theorem 2.2, we have that

$$\|\lambda - T\| = \sup_{n \geq 1} \|\lambda - T_n\| \text{ and } \|(\lambda - T)^{-1}\| = \sup_{n \geq 1} \|(\lambda - T_n)^{-1}\|.$$

Now, let $\lambda \in \sigma_\varepsilon(T)$. Then, we will discuss these two cases:
1^{st} *case* : If $\lambda \in \sigma(T)$, then according to [14, Theorem 2.3], we infer that

$$\sigma(T) = \bigcup_{n=1}^{\infty} \sigma(T_n).$$

$\underline{2^{nd}\ case}$: If $\lambda \in \sigma_\varepsilon(T)\backslash\sigma(T)$. Hence,

$$\begin{aligned} \|\lambda - T\|\|(\lambda - T)^{-1}\| &> \frac{1}{\varepsilon} \\ \sup_{n\geq 1}\|\lambda - T_n\|\sup_{n\geq 1}\|(\lambda - T_n)^{-1}\| &> \frac{1}{\varepsilon}. \end{aligned}$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\|\lambda - T_{n_0}\|\|(\lambda - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}.$$

Hence, $\lambda \in \sigma_\varepsilon(T_{n_0})$. Thus,

$$\sigma_\varepsilon(T) \subset \bigcup_{n=1}^{\infty}\sigma_\varepsilon(T_n).$$

Conversely, if $\lambda \in \sigma_\varepsilon(T_{n_0})$ for every $n_0 \in \mathbb{N}$. Then, for $\varepsilon > 0$ we infer that

$$\|\lambda - T_{n_0}\|\|(\lambda - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}.$$

This shows that

$$\sup_{n\geq 1}\|\lambda - T_n\|\sup_{n\geq 1}\|(\lambda - T_n)^{-1}\| > \frac{1}{\varepsilon}.$$

Using Theorem 2.2 again, we conclude that

$$\bigcup_{n=1}^{\infty}\sigma_\varepsilon(T_n) \subset \sigma_\varepsilon(T).$$

Consequently, for $\varepsilon > 0$ we have

$$\sigma_\varepsilon(T) = \bigcup_{n=1}^{\infty}\sigma_\varepsilon(T_n).$$

Now, we may prove that

$$r_\varepsilon(T) = \sup_{n\geq 1} r_\varepsilon(T_n).$$

Since $\sigma_\varepsilon(T_n) \subset \sigma_\varepsilon(T)$ for $n \geq 1$, then

$$r_\varepsilon(T_n) \leq r_\varepsilon(T).$$

It follows that

$$\sup_{n\geq 1} r_\varepsilon(T_n) \leq r_\varepsilon(T).$$

Assume that the assertion fails, if $\sup_{n\geq 1} r_\varepsilon(T_n) < r_\varepsilon(T)$, we must obtain at least one element $\lambda_1 \in \sigma_\varepsilon(T)$ such that

$$\sup_{n\geq 1} r_\varepsilon(T_n) < |\lambda_1| \leq r_\varepsilon(T).$$

Then there is $n_1 \geq 1$ such that

$$\lambda_1 \in \sigma_\varepsilon(T_{n_1}).$$

Therefore,

$$r_\varepsilon(T_{n_1}) < |\lambda_1|.$$

Which is a contradiction. Then, for all $\varepsilon > 0$ we have

$$r_\varepsilon(T) = \sup_{n\geq 1} r_\varepsilon(T_n),$$

which completes the proof of theorem. $\square$

In Theorems 3.3, if the location of T block change as desired, the results do not change. Thus, we can give the following theorem.

Corollary 3.2. *Let $\varphi : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto function. Also, let X_n be a Banach space, $T_n \in \mathcal{L}(X_n)$ for $n \geq 1$. And, let $X = \bigoplus_{n=1}^{\infty} X_n$ be the direct sum of X_n, $n \geq 1$, $T = \bigoplus_{n=1}^{\infty} T_{\varphi(n)} \in \mathcal{L}(X)$ with $\sup_{n\geq 1} \|T_{\varphi(n)}\| < \infty$. Then, for each $\varepsilon > 0$ the condition pseudospectrum set of the operator T is of the form*

$$\sigma_\varepsilon(T) = \bigcup_{n=1}^{\infty} \sigma_\varepsilon(T_n).$$

In the case of $T_n \in \mathcal{L}(X_n)$, $n \geq 1$ and $T \in \mathcal{L}(X)$, the condition pseudospectral radius of the operator T is of the form

$$r_\varepsilon(T) = \sup_{n\geq 1} r_\varepsilon(T_n). \tag{1}$$

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous function. Also, let X_n be a Banach space, $T_n \in \mathcal{L}(X_n)$ for $n \geq 1$. And, let $X = \bigoplus_{n=1}^{\infty} X_n$ be the direct sum of X_n, $n \geq 1$,*

$$f(T) = \bigoplus_{n=1}^{\infty} f(T_n) \in \mathcal{L}(X)$$

with $\sup_{n\geq 1} \|f(T_n)\| < \infty$. Then, for each $\varepsilon > 0$ the condition pseudospectrum set of the operator T is of the form

$$\sigma_\varepsilon(f(T)) = \bigcup_{n=1}^{\infty} \sigma_\varepsilon(f(T_n)).$$

Consequently, if $T_n \in \mathcal{L}(X_n)$, $n \geq 1$ and $T \in \mathcal{L}(X)$, the condition pseudospectral radius of the operator T is of the form

$$r_\varepsilon(f(T)) = \sup_{n\geq 1} r_\varepsilon(f(T_n)).$$

Proof. The argument is analogous to the one in Theorem 3.3, with use being made of Theorem 2.2. □

In the following we gives a precise information about the condition pseudospectrum of bounded linear operator under linear transformation.

Proposition 3.4. *Let X_n be a Banach space, $T_n \in \mathcal{L}(X_n)$ for $n \geq 1$. And, let $X = \bigoplus_{n=1}^{\infty} X_n$ be the direct sum of X_n, $n \geq 1$, $T = \bigoplus_{n=1}^{\infty} T_n \in \mathcal{L}(X)$ with $\sup_{n\geq 1} \|T_n\| < \infty$. Then, for all $0 < \varepsilon < 1$, we have*

$$\begin{aligned} \sigma(T) &= \bigcap_{0<\varepsilon<1} \bigcup_{n=1}^{\infty} \sigma_\varepsilon(T_n) \\ &= \bigcap_{0<\varepsilon<1} \bigcup_{n=1}^{\infty} \bigcup \Big\{ \sigma(T_n + D_n) : D_n \in \mathcal{L}(X_n) \ \text{ and } \ \|D_n\| < \varepsilon \|\lambda - T_n\| \Big\}. \end{aligned}$$

4. Applications.

In this section, we give some remarkable examples as applications of our results.

Application 4.1. *Let $X_n = \mathbb{C},\ n \geq 1$. One-dimensional Euclidian space, $X = \left(\bigoplus_{n=1}^{\infty} \mathbb{C}\right)_p$, $1 \leq p < \infty$.*

$$T_n = \alpha_n I : \mathbb{C} \to \mathbb{C}, \alpha_n \neq 0, n \geq 1, \alpha_n \in l_p(\mathbb{C}), 1 \leq p < \infty.$$

Then, $T = \bigoplus_{n=1}^{\infty} T_n \in \mathcal{L}(X)$ is the infinite diagonal block operator matrices. We can estimate the quantity

$$\|(\lambda - T_n)^{-1}\| = \frac{1}{|\lambda - \alpha_n|} \text{ and } \|\lambda - T_n\| = |\lambda - \alpha_n|.$$

Furthermore, for all $n \geq 1$, we have

$$\sigma(T_n) = \{\alpha_n\} \text{ and } r(T_n) = |\alpha_n|.$$

Now applying Theorem 3.3, we have for all $n \geq 1$,

$$\begin{aligned}
\sigma_\varepsilon(T) &= \bigcup_{n=1}^{\infty} \sigma_\varepsilon(T_n) \\
&= \bigcup_{n=1}^{\infty} \left(\sigma(T_n) \cup \left\{\lambda \in \mathbb{C} : \ \|\lambda - T_n\| \|(\lambda - T_n)^{-1}\| > \frac{1}{\varepsilon}\right\}\right) \\
&= \bigcup_{n=1}^{\infty} \sigma(T_n) \cup \bigcup_{n=1}^{\infty} \left\{\lambda \in \mathbb{C} : \ \|\lambda - T_n\| \|(\lambda - T_n)^{-1}\| > \frac{1}{\varepsilon}\right\} \\
&= \bigcup_{n=1}^{\infty} \sigma(T_n) = \overline{\bigcup_{n=1}^{\infty} \{\alpha_n\}}.
\end{aligned}$$

And using [2, Lemma 2.1], *we get*

$$\begin{aligned}
r(T_n) &\leq r_\varepsilon(T_n) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\|T_n\| \\
\sup_{n\geq 1} r(T_n) &\leq \sup_{n\geq 1} r_\varepsilon(T_n) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\sup_{n\geq 1} \|T_n\| \\
\sup_{n\geq 1} |\alpha_n| &\leq r_\varepsilon(T) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\sup_{n\geq 1} |\alpha_n|.
\end{aligned}$$

Application 4.2. *Let $X_n = \mathbb{C}^2,\ n \geq 1$. Two-dimensional Euclidian space, $X = \left(\bigoplus_{n=1}^{\infty} \mathbb{C}^2\right)_p$, $1 \leq p < \infty$. Now, we consider*

$$T_n = \begin{pmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2,$$

with $\alpha_n \neq \beta_n \neq 0,\ \sup_{n\geq 1} \alpha_n < \infty$ and $\sup_{n\geq 1} \beta_n < \infty$. Then,

$$T = \bigoplus_{n=1}^{\infty} T_n \in \mathcal{L}(X)$$

is the infinite diagonal block operator matrices. A direct computation shows that

$$\|(\lambda - T_n)^{-1}\| = \max\left\{\frac{1}{|\lambda-\beta_n|}, \frac{1}{|\lambda-\alpha_n|}\right\}$$

and

$$\|\lambda - T_n\| = \max\left\{|\lambda-\beta_n|, |\lambda-\alpha_n|\right\}.$$

Hence, for all $n \geq 1$, we have

$$\sigma(T_n) = \{\alpha_n, \beta_n\}, \;\; r(T_n) = \max\{|\alpha_n|, |\beta_n|\},$$

and

$$\sigma_\varepsilon(T_n) = \{\alpha_n, \beta_n\} \bigcup \left\{\lambda \in \mathbb{C} : \frac{|\lambda-\alpha_n|}{|\lambda-\beta_n|} > \frac{1}{\varepsilon}\right\} \cup \left\{\lambda \in \mathbb{C} : \frac{|\lambda-\beta_n|}{|\lambda-\alpha_n|} > \frac{1}{\varepsilon}\right\}$$

Also, using Theorem 3.3, we have for all $n \geq 1$,

$$\sigma_\varepsilon(T) = \bigcup_{n=1}^{\infty} \sigma_\varepsilon(T_n) = \overline{\bigcup_{n=1}^{\infty}\{\alpha_n, \beta_n\} \cup \left\{\lambda \in \mathbb{C} : \frac{|\lambda-\alpha_n|}{|\lambda-\beta_n|} > \frac{1}{\varepsilon}\right\} \cup \left\{\lambda \in \mathbb{C} : \frac{|\lambda-\beta_n|}{|\lambda-\alpha_n|} > \frac{1}{\varepsilon}\right\}}.$$

And

$$\sup_{n\geq 1} \max\{|\alpha_n|, |\beta_n|\} \quad \leq \quad r_\varepsilon(T) \leq \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big) \sup_{n\geq 1} \max\{|\alpha_n|, |\beta_n|\}.$$

Application 4.3. *Let $X_n = \mathbb{C}^n$, T_n be a linear operator on $\mathcal{L}(\mathbb{C}^n)$ and $X = \left(\bigoplus_{n=1}^{\infty} \mathbb{C}^n\right)_p$, $1 \leq p < \infty$. Given for all $k \geq 2$ with the infinity norm by*

$$\begin{cases} T_n : \mathbb{C}^n \to \mathbb{C}^n, \\ \quad e_i \longmapsto T_n e_i, \\ \text{where, } T_n e_i = e_{(k+1)-i}. \end{cases}$$

It is easily checked that

$$\begin{cases} T_n = T_n^{-1}, \\ \|T_n\| = 1, \\ \sigma(T_n) = \{-1, 1\}. \end{cases}$$

We will check that

$$\|(\lambda - T_n)e_i\| = \left|\lambda e_i - e_{(k+1)-i}\right|, \text{ then } \|\lambda - T_n\| = |\lambda| + 1$$

and

$$\|(\lambda - T_n)^{-1}e_i\| = \left|\frac{\lambda e_i - e_{(k+1)-i}}{\lambda^2 - 1}\right| \text{ then, } \|(\lambda - T_n)^{-1}\| = \frac{|\lambda| + 1}{|\lambda^2 - 1|}.$$

Moreover, for $0 < \varepsilon < 1$ we obtain

$$\sigma_\varepsilon(T_n) \quad = \quad \{-1, 1\} \cup \left\{\lambda \in \mathbb{C} : \frac{(|\lambda| + 1)^2}{|\lambda^2 - 1|} > \frac{1}{\varepsilon}\right\}.$$

It is easy to verify that, for all

$$\begin{aligned} \sigma_\varepsilon(T) \quad &= \quad \bigcup_{n=1}^{\infty} \sigma_\varepsilon(T_n) = \{-1, 1\} \cup \left\{\lambda \in \mathbb{C} : \frac{(|\lambda| + 1)^2}{|\lambda^2 - 1|} > \frac{1}{\varepsilon}\right\} \\ 1 \quad &\leq \quad r_\varepsilon(T) \leq \frac{1+\varepsilon}{1-\varepsilon}. \end{aligned}$$

Acknowledgments:

The authors are very grateful for the comments and corrections of the referee, whose detailed attention allowed us to make very worthwhile improvements.

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