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Localization property of generalized Besov-type spaces, Triebel-Lizorkin-type spaces and their associated multiplier spaces

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Abstract. In this paper, we study the localization property of generalized Besov-type spaces, Triebel-Lizorkin-type spaces and their associated multiplier spaces, defined from function $v : [0,\infty) \to]0,\infty)$ satisfying the inequality $v(ts) \geq c\, t^{-\mu} v(s)$ for $0 < t, s \leq 1$ and some real μ.

1. Introduction

El Baraka in [5, 6], Dachun and Yuan in [20, 21] introduced new classes of Besov-type spaces $B^{\mu,\tau}_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin-type spaces $F^{\mu,\tau}_{p,q}(\mathbb{R}^n)$, which unify and generalize the Besov spaces $B^{\mu}_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $F^{\mu}_{p,q}(\mathbb{R}^n)$ (see, for example, [22, Lemma 2.1.]). The theory of these function spaces has been considered by many researchers. For a complete treatment of $B^{\mu,\tau}_{p,q}(\mathbb{R}^n)$ spaces and $F^{\mu,\tau}_{p,q}(\mathbb{R}^n)$ spaces we refer the reader to the work [22].

In this paper, we define a generalized Besov-type space and Triebel–Lizorkin-type space by using a function $v : [0,\infty) \to]0,\infty)$ that satisfies the following property

$$\sup_{0<t<1} t^{-\mu} \sup_{0<s\leq 1} \frac{v(s)}{v(ts)} < \infty, \qquad (\mu \in \mathbb{R}). \tag{1}$$

There is a huge literature nowadays about generalized spaces of Besov $B^{v}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin $F^{v}_{p,q}(\mathbb{R}^n)$ also using conditions of type (1), see for example the papers of Hartzstein and Viviani [9, 10] and the book of Triebel [19, p.53 and p.108]. We study in generalized Besov-type spaces and generalized Triebel-Lizorkin-type spaces the localization property. In the context of intersections, we want to extend the results given in [2] and [12] for $B^{s}_{p,q}(\mathbb{R}^n)$, to the case of $B^{v,\tau}_{p,q}(\mathbb{R}^n)$ and in [7] and [18] for $F^{s}_{p,q}(\mathbb{R}^n)$, to the case of $F^{v,\tau}_{p,q}(\mathbb{R}^n)$.

Next, we study the localization of pointwise multipliers in the Besov-type space $M(B^{v,\tau}_{p,q})$ and in the Triebel-Lizorkin-type space $M(F^{v,\tau}_{p,q})$. Let us recall that if $v(t) = t^{-\mu}$ and $\tau = 0$, we have the following results:

$M(F^{\mu,0}_{p,2}) = (F^{\mu}_{p,2})_{\ell^\infty}$ $(1 < p < \infty,\ \mu > \frac{n}{p})$ (Strichartz [13]),

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$M(B^{\mu,0}_{p,p}) = (B^{\mu}_{p,p})_{\ell^\infty}$ $(1 \le p \le \infty,\ \mu > \frac{n}{p})$ (Peetre [11]),
$M(F^{\mu,0}_{p,q}) = (F^{\mu}_{p,q})_{\ell^\infty}$ $(1 \le p < \infty,\ 1 \le q \le \infty,\ \mu > \frac{n}{p})$ (Franke [8]).

2. Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space and by $\mathbb{N}$ the collection of all natural numbers. We write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y, where X and Y are quasi-normed spaces. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. By supp f we denote the support of the function f. The symbol $A \lesssim B$ means that there exists a positive constant C, independent of the main parameters, such that $A \le CB$. If $A \lesssim B \lesssim A$, we then write $A \sim B$. For any function f and any $a \in \mathbb{R}^n$, we set $\tau_a f = f(\cdot - a)$. As usual for any y in $\mathbb{R}$, $[y]$ stands for the largest integer smaller than or equal to y.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. If $a \in \mathbb{Z}$, then $a_+ = \max(0, a)$.

By ℓ^q, $q \in (0, \infty]$, we denote the discrete Lebesgue space equipped with the usual quasi-norm. Mostly we will deal with sequences defined either on $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z}^n$.

By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$. The topology in the complete locally convex space $\mathcal{S}(\mathbb{R}^n)$ is generated by

$$p_m(\varphi) := \sup_{x \in \mathbb{R}^n} (1 + |x|)^m \sum_{|\alpha| \le m} \left| D^\alpha \varphi(x) \right|, \quad m \in \mathbb{N}$$

We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}(f)(\xi) = \hat{f} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$. Its inverse is denoted by $\mathcal{F}^{-1} f = \check{f}$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

For any measurable subset $\Omega \subset \mathbb{R}^n$ and $p \in (0, \infty]$, the $L^p(\Omega)$ consists of all measurable functions for which

$$\left\| f | L^p(\Omega) \right\| = \left(\int_\Omega \left| f(x) \right|^p dx \right)^{1/p}$$

is finite, with the obvious modification made when $p = \infty$.

By $\ell^q(\mathbb{Z}^n)$, $0 < q \le \infty$, we denote the space of all (complex) sequences $\{a_k\}_{k \in \mathbb{Z}^n}$ equipped with the quasi-norm

$$\|\{a_k\}_{k \in \mathbb{Z}^n} \mid \ell^q(\mathbb{Z}^n)\| = \left(\sum_{k \in \mathbb{Z}^n} |a_k|^q \right)^{1/q}$$

(with the usual modification if $q = \infty$).

Let $0 < w \le p \le \infty$. The Morrey space $\mathcal{M}^p_w(\mathbb{R}^n)$ is defined to be the set of all $w - $*locally* Lebesgue integrable function f of $\mathbb{R}^n$ such that

$$\left\| f | \mathcal{M}^p_w(\mathbb{R}^n) \right\| = \sup_{x \in \mathbb{R}^n,\ r > 0} r^{n(\frac{1}{p} - \frac{1}{w})} \left(\int_{B(x,r)} \left| f(x) \right|^w dx \right)^{1/w} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge \frac{3}{2}$. We put $\varphi_0(x) = \psi(x)$, $\varphi_1(x) = \psi(x/2) - \psi(x)$ and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x) \quad \text{for} \quad j = 2, 3, ...$$

Then we have $\operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 3 \cdot 2^{j-1}\}$, $\varphi_j(x) = 1$ for $3 \cdot 2^{j-2} \leq |x| \leq 2^j$ and $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. The system of functions $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is called a smooth dyadic resolution of unity. We define the convolution operators Δ_j by the following:

$$\Delta_j f = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), \qquad j \in \mathbb{N}, \qquad \Delta_0 f = \mathcal{F}^{-1}(\psi(2^{-k}\cdot)\mathcal{F}f), \qquad f \in \mathcal{S}'(\mathbb{R}^n).$$

Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j * f$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

For $\mu \in \mathbb{R}$ and $0 < p, q \leq \infty$, the Besov space $B^{\mu}_{p,q}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\left\| f|B^{\mu}_{p,q}(\mathbb{R}^n)\right\| = \left(\sum_{j=0}^{\infty} 2^{j\mu q} \left\|\mathcal{F}^{-1}\varphi_j * f|L^p(\mathbb{R}^n)\right\|^q\right)^{1/q}$$

is finite, with the obvious modification made when $q = \infty$.

For $\mu \in \mathbb{R}$ and $0 < p, q \leq \infty$, the homogeneous Besov space $\dot{B}^{\mu}_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)$ for which

$$\left\| f|\dot{B}^{\mu}_{p,q}(\mathbb{R}^n)\right\| = \left(\sum_{j\in\mathbb{Z}} 2^{j\mu q}\left\|\mathcal{F}^{-1}\varphi_j * f|L^p(\mathbb{R}^n)\right\|^q\right)^{1/q}$$

is finite, with the obvious modification made when $q = \infty$.

For $\mu \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leq \infty$. The Triebel-Lizorkin space $F^{\mu}_{p,q}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\left\| f|F^{\mu}_{p,q}(\mathbb{R}^n)\right\| = \left\|\left(\sum_{j=0}^{\infty} 2^{j\mu q} \left|\mathcal{F}^{-1}\varphi_j * f\right|^q\right)^{1/q} |L^p(\mathbb{R}^n)\right\|$$

is finite, with again the obvious modification made when $q = \infty$. The theory of the spaces $B^{\mu}_{p,q}$ and $F^{\mu}_{p,q}$ has been developed in detail in [16], [17] and [19] but has a longer history already including many contributors; we do not want to discuss this here.

A Banach space of distributions (*B.s.d.*) in $\mathcal{D}'(\mathbb{R}^n)$ is a vector subspace E of $\mathcal{D}'(\mathbb{R}^n)$ with a complete norm $\|\cdot\|_E$ such that the canonical injection $E \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ is continuous. We associate on the space E the following hypothesis.

(i) $\left\|\tau_k f|E\right\| = \left\|f|E\right\|$ for $k \in \mathbb{Z}^n$,

(ii) For all $f \in E$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have that $\varphi \cdot f \in E$.

We say that $g : \mathbb{R}^n \to \mathbb{C}$ is a multiplier of E (we note $g \in M(E)$), if for all $f \in C^{\infty} \cap E$, we have $g \cdot f \in E$ and $\left\|g \cdot f\right\|_E \lesssim \left\|f\right\|_E$. We equip $M(E)$ with the norm

$$\left\|g|M(E)\right\| = \sup\left\{\left\|g \cdot f|E\right\| : \ f \in E\,,\ \left\|f|E\right\| = 1\right\}.$$

Recall that, for all $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, its Hardy-Littlewood maximal function Mg is defined as follows:

$$Mg(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|g(y)\right| \mathrm{d}y\,, \qquad (x \in \mathbb{R}^n)$$

where the supremum is taken over all balls B containing x. If $t \in (0,\infty)$, then we define $M_t g(x) = \left(M\left|g\right|^t\right)^{1/t}(x)$ for all $x \in \mathbb{R}^n$.

3. Generalized Besov-type and Triebel-Lizorkin-type spaces

In this section, we define the generalized Besov-type space and Triebel–Lizorkin-type space by using the function that satisfies the following propriety

$$v(2^{-k}) \leq c\, 2^{-\mu j}\, v(2^{-j-k}), \qquad (k = 0, 1, \ldots,\ j = 1, 2, \ldots,\ \mu \in \mathbb{R}),$$

the constant c is independent of k and j. There exist some examples of functions v satisfying (1):

$$t^{-\mu},\ t^{-\mu}(1 + (\log t)_+),\ \max(t^{-\beta}, t^{-\mu}) \text{ with } \beta < \mu, \text{ and } \min(t^{-\beta}, t^{-\mu}) \text{ with } \beta > \mu.$$

Definition 3.1. *Let $\mu \in \mathbb{R}$, $r \in \mathbb{Z}$, $(p, q) \in (0, \infty]^2$, (resp., $p \in (0, \infty)$) and $\Omega \subset \mathbb{R}^n$. Let v be a positive function satisfied (1). The space $\ell^v_{q,r^+}(L^p(\Omega))$ (resp., $L^p(\Omega, \ell^v_{q,r^+})$) is the set of the sequences $\{f_k\}_{k \geq r^+} \subset \mathcal{S}'$ such that*

$$\left\| \{f_j\}_{j \geq r^+} | \ell^v_{q,r^+}(L^p(\Omega)) \right\| = \left(\sum_{j=r^+}^{\infty} \left(\int_\Omega \left(v(2^{-j}) \left| f_j(x) \right| \right)^p dx \right)^{q/p} \right)^{1/q} < \infty,$$

$$(resp., \quad \left\| \{f_j\}_{j \geq r^+} | L^p(\Omega, \ell^v_{q,r^+}) \right\| = \left(\int_\Omega \left(\sum_{j=r^+}^{\infty} \left(v(2^{-j}) \left| f_j(x) \right| \right)^q \right)^{p/q} dx \right)^{1/p} < \infty).$$

Note that when $v(t) = t^{-\mu}$, $r = 0$ and $\Omega = \mathbb{R}^n$ we have $\ell^v_{q,0}(L^p(\mathbb{R}^n)) = \ell^\mu_q(L^p(\mathbb{R}^n))$ and $L^p(\mathbb{R}^n, \ell^v_{q,0}) = L^p(\mathbb{R}^n, \ell^\mu_q)$.

Now, we define the spaces $B^{v,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{v,\tau}_{p,q}(\mathbb{R}^n)$ which will be our main object of study.

Definition 3.2. *Let $\mu \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$. Let v be a positive function satisfies (1).*

(i) Let $p \in (0, \infty]$. The Besov-type space $B^{v,\tau}_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f | B^{v,\tau}_{p,q}(\mathbb{R}^n) \right\| = \sup_{x \in \mathbb{R}^n, r \in \mathbb{Z}} \frac{1}{|B(x, 2^{-r})|^\tau} \left\| \{\Delta_j f\}_{j \geq r^+} | \ell^v_{q,r^+}(L^p(B(x, 2^{-r}))) \right\| < \infty.$$

In the limiting case, $p = \infty$ ($q = \infty$) the usual modification is required.

(ii) Let $p \in (0, \infty)$. The Triebel-Lizorkin-type space $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f | F^{v,\tau}_{p,q}(\mathbb{R}^n) \right\| = \sup_{x \in \mathbb{R}^n, r \in \mathbb{Z}} \frac{1}{|B(x, 2^{-r})|^\tau} \left\| \{\Delta_j f\}_{j \geq r^+} | L^p(B(x, 2^{-r}), \ell^v_{q,r^+}) \right\| < \infty.$$

In the limiting case, $q = \infty$ the usual modification is required.

Remark 3.3.

- *The spaces $B^{v,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{v,\tau}_{p,q}$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces if $p \geq 1$, $q \geq 1$).*
- *The classical different properties of $B^{\mu,\tau}_{p,q}$ and $F^{\mu,\tau}_{p,q}$ (obtained here by taking $v(t) = t^{-\mu}$), as equivalent norms, embeddings ..., can be found in [21, 22].*
- *The particular case $v(t) = t^{-\mu}$ and $\tau = 0$ yields the Besov space $B^\mu_{p,q}(\mathbb{R}^n)$ and the Triebel-Lizorkin space $F^\mu_{p,q}(\mathbb{R}^n)$.*
- *In particular, if $v(t) = 1$, we have $F^{1, \frac{1}{p} - \frac{1}{w}}_{p,2}(\mathbb{R}^n) = \mathcal{M}^p_w(\mathbb{R}^n)$ with $1 < w \leq p \leq \infty$.*

Now, we will recall some estimates of Yamazaki's type for the convergent series.

Lemma 3.4. *Let $0 < b < 1$. Let $\{\varepsilon_j\}_{j \geq r^+}$ be a sequence of real positive numbers in $\ell^q(\mathbb{Z})$. Then we have*

$$\left\| \left\{ \sum_{j=0}^{k} b^{(k-j)} \varepsilon_j \right\}_{k \geq r^+} | \ell^q(\mathbb{Z}) \right\| + \left\| \left\{ \sum_{j=k}^{\infty} b^{(j-k)} \varepsilon_j \right\}_{k \geq r^+} | \ell^q(\mathbb{Z}) \right\| \lesssim \left\| \{\varepsilon_k\}_{k \geq r^+} | \ell^q(\mathbb{Z}) \right\|.$$

Lemma 3.5. *Let $\mu \in \mathbb{R}$, $q \in [1,\infty)$ and $1 < w \le p < \infty$. Let v be a positive function satisfied (1) and $\{g_j\}_{j=0}^{\infty} \subset L^1_{\text{loc}}(\mathbb{R}^n)$. If $0 < t < \min\{w,q\} \le w \le p \le \infty$, then*

$$\left\| \{M_t g_j\}_{j\ge 0} \,|\mathcal{M}^p_w(B(x,a),\ell^v_{q,0})\right\| \lesssim \left\| \{g_j\}_{j\ge 0} \,|\mathcal{M}^p_w(B(x,a),\ell^v_{q,0})\right\|.$$

Proposition 3.6. *Let a symbol $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and a function $g \in C^\infty(\mathbb{R}^n)$ be given such that, for $A > 0$ and $R \ge 1$,*

$$\operatorname{supp} \mathcal{F} g \subset B(0,AR) \quad \textit{and} \quad \operatorname{supp} b \subset B(0,A).$$

Let $t \in (0,1]$. Then there exists a positive constant C such that

$$\left|\mathcal{F}^{-1}(\varphi\mathcal{F} g)(x)\right| \le C(RA)^{\frac{n}{t}-n}\left\|\varphi|\dot{B}^s_{p,q}(\mathbb{R}^n)\right\| M_t g(x).$$

Here C can be taken as a function of t only.

The proof of Lemma 3.4 is immediate by using Young's inequality in ℓ^q. However, the proof of Lemma 3.5 can be found in [14] and the proof of Proposition 3.6 can be found in [22, Proposition 6.1 page 150].

Proposition 3.7. *Let $\gamma > 1$, $q \in (0,\infty]$ and v be a positive function satisfies (1). Let $\{g_j\}_{j\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{F} g_j$ is supported by the ball $|\xi| \le \gamma 2^j$.*
(i) *Let $p \in (0,\infty]$, $\tau \in [0,\infty)$ and $\mu \in \mathbb{R}$. Then, the following inequality*

$$\left\|\sum_{j=0}^{\infty} g_j | B^{v,\tau}_{p,q}\right\| \lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left\|g_j|L^p(B(x,2^{-r}))\right\|\right)^q\right)^{1/q}$$

holds.
(ii) *Let $p \in (0,\infty)$, $\tau \in (0,1/p)$ and $\mu > (n/\min\{p,q\} - n)_+$. Then, the following inequality*

$$\left\|\sum_{j=0}^{\infty} g_j | F^{v,\tau}_{p,q}\right\| \lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left\|\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left|g_j\right|\right)^q\right)^{1/q} |L^p B(x,2^{-r})\right\|$$

holds.

Proof. We observe that there exists $N_1 = \left[\log_2 \gamma\right]$, $N_2 = \left[\log_2 3\gamma\right]$ in $\mathbb{N}$ such that

$$\Delta_k\left(\sum_{j=0}^{\infty} g_j\right) = \sum_{j=k-N_1}^{k+N_2} \Delta_k g_j.$$

For (i). Let $x_0 \in \mathbb{R}^n$, for all $x \in B(x_0,2^{-r})$ we have

$$\Delta_k g_j(x) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\varphi_k)(y)\, g_j(x-y)\, dy,$$

then

$$\begin{aligned}
\left\|\Delta_k g_j \mid L^p(B(x_0, 2^{-r}))\right\|^p &\lesssim \int_{B(x_0,2^{-r})} \left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| \left|g_j(x-y)\right| dy \right)^p dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\chi_{B(x_0,2^{-r})}(x)\right)^{1/p} \left|g_j(x-y)\right| \right. \\
&\quad \left. \times \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy \right)^p dx \\
&\lesssim \int_{\mathbb{R}^n} \left(\int_{B(x_0,2^{-r})} \left|g_j(x-y)\right|^p dx \right)^{1/p} \\
&\quad \times \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy \\
&\lesssim \int_{\mathbb{R}^n} \left(\int_{B(x_0-y,2^{-r})} \left|g_j(z)\right|^p dz \right)^{1/p} \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy \\
&= \int_{\mathbb{R}^n} \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\| \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy.
\end{aligned}$$

We define

$$I_{k,r} = \int_{\mathbb{R}^n} v(2^{-k}) \sum_{j=k-N_1}^{k+N_2} \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\| \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy,$$

then we have by Fubini's Theorem

$$\begin{aligned}
\left(\sum_{k=r^+}^{\infty} I_{k,r}^q\right)^{1/q} &\lesssim \left(\sum_{k=r^+}^{\infty} \left(\int_{\mathbb{R}^n} v(2^{-k}) \sum_{j=k-N_1}^{k+N_2} \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\| \right.\right. \\
&\quad \left.\left. \times \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy \right)^q \right)^{1/q} \\
&\lesssim \int_{\mathbb{R}^n} \left(\sum_{k=r^+}^{\infty} \left(v(2^{-k}) \sum_{j=k-N_1}^{k+N_2} \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\| \right)^q \right)^{1/q} \\
&\quad \times \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy.
\end{aligned}$$

We put

$$H_{k,r} = v(2^{-k}) \sum_{j=k-N_1}^{k+N_2} \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\|,$$

then by (1) and according to sign of μ, we have

$$H_{k,r} \lesssim \begin{cases} 2^{k\mu} \sum_{j=k-N_1}^{\infty} 2^{-\mu j} v(2^{-j}) \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\| & \text{if} \quad \mu > 0\,, \\ \left(\sum_{j=k-N_1}^{k+N_2} 1\right)^{1/q'} \left(\sum_{j=k-N_1}^{k+N_2} \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\|^q\right)^{1/q} & \text{if} \quad \mu = 0\,, \\ 2^{k\mu} \sum_{j=0}^{k+N_2} 2^{-\mu j} v(2^{-j}) \left\|g_j \mid L^p(B(x_0-y, 2^{-r}))\right\| & \text{if} \quad \mu < 0\,. \end{cases}$$

The lemma 3.4 gives

$$\left(\sum_{k=r^+}^{\infty} I_{k,r}^q\right)^{1/q} \lesssim \int_{\mathbb{R}^n} \left(\sum_{k=r^+}^{\infty} v(2^{-k}) \left\|g_k \mid L^p(B(x_0-y, 2^{-r}))\right\|^q \right)^{1/q} \left|\mathcal{F}^{-1}(\varphi_k)(y)\right| dy,$$

then

$$\begin{aligned}\left|B(x_0-y,2^{-r})\right|^{-\tau}\left(\sum_{k=r^+}^{\infty} I_{k,r}^q\right)^{1/q} &\lesssim \left|B(x_0-y,2^{-r})\right|^{-\tau}\int_{\mathbb{R}^n}\left(\sum_{k=r^+}^{\infty} v(2^{-k})\left\|g_j \mid L^p(B(x_0-y,2^{-r}))\right\|^q\right)^{1/q} \\ &\quad\times\left|\mathcal{F}^{-1}(\varphi_k)(y)\right|dy \\ &\lesssim \int_{\mathbb{R}^n}\left(\sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\left|B(x,2^{-r})\right|^{-\tau}\left(\sum_{k=r^+}^{\infty} v(2^{-k})\left\|g_j \mid L^p(B(x,2^{-r}))\right\|^q\right)\right)^{1/q} \\ &\quad\times\left|\mathcal{F}^{-1}(\varphi_k)(y)\right|dy \\ &\lesssim \left(\sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\left|B(x,2^{-r})\right|^{-\tau}\left(\sum_{k=r^+}^{\infty} v(2^{-k})\left\|g_j \mid L^p(B(x,2^{-r}))\right\|^q\right)\right)^{1/q} \\ &\quad\times\int_{\mathbb{R}^n}\left|\mathcal{F}^{-1}(\varphi_k)(y)\right|dy \\ &\lesssim \left(\sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\left|B(x,2^{-r})\right|^{-\tau}\left(\sum_{k=r^+}^{\infty} v(2^{-k})\left\|g_j \mid L^p(B(x,2^{-r}))\right\|^q\right)\right)^{1/q}.\end{aligned}$$

The proof of (i) is complete.
For (ii). By the proposition 3.6, we find

$$\left|\Delta_k g_j(x)\right| \lesssim 2^{j(\frac{n}{t}-n)}\left\|\varphi(2^k\cdot)|\dot{B}_{1,t}^{t/n}(\mathbb{R}^n)\right\|M_t g_j(x).$$

Since $\left\|\varphi(2^k\cdot)|\dot{B}_{1,t}^{t/n}(\mathbb{R}^n)\right\| = 2^{-k(\frac{n}{t}-n)}\left\|\varphi|\dot{B}_{1,t}^{t/n}(\mathbb{R}^n)\right\|$ (see [16]), then by (1) we have

$$v(2^{-k}) \lesssim 2^{\mu(k-j)}v(2^{-j}),$$

and thus

$$v(2^{-k})\sum_{j=k-N_1}^{k+N_2}\left|\Delta_k g_j(x)\right| \lesssim \sum_{j=k-N_1}^{k+N_2} 2^{(j-k)(\frac{n}{t}-n-\mu)}v(2^{-j})M_t g_j(x).$$

Now, by Lemma 3.4 and as $j \le k$, $\mu > (n/\min\{p,q\}-n)_+$ and $t \in (0,1]$, we obtain

$$\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left|\Delta_k g_j\right|\right)^q\right)^{1/q} \lesssim \left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left[M\left|g_j\right|^t\right]^{1/t}\right)^q\right)^{1/q}.$$

This implies that

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\frac{1}{|B(x,2^{-r})|^\tau}\left\|\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left|\Delta_k g_j\right|\right)^q\right)^{1/q}|L^p(B(x,2^{-r}))\right\|^{1/t} \\ \lesssim\ &\sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\frac{1}{|B(x,2^{-r})|^\tau}\left\|\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left[M\left|g_j\right|^t\right]^{1/t}\right)^q\right)^{1/q}|L^p(B(x,2^{-r}))\right\|^{1/t}.\end{aligned}$$

Using Lemma 3.5, we have

$$\left\|\sum_{j=0}^{\infty} g_j|F_{p,q}^{v,\tau}\right\| \lesssim \sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\frac{1}{|B(x,2^{-r})|^\tau}\left\|\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left|g_j\right|\right)^q\right)^{1/q}|L^pB(x,2^{-r})\right\|.$$

The proposition is proved. □

4. Localization spaces

Motivated by [2], [4], [7], [12] and [18], we give the localization property of Lebesgue spaces, Besov-type spaces, Triebel-Lizorkin-type spaces and their associated multiplier spaces on the $\ell^u(\mathbb{Z}^n)$ spaces.

4.1. Localization of Lebesgue space

In this subsection, we present the localization of Lebesgue spaces $L^p(B(x, 2^{-r}))$ in the norm of $\ell^p(\mathbb{Z}^n)$.

We first need the concept of a smooth dyadic resolution of unity. Let β be a function in $\mathcal{D}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \beta \subset B(0, Q), \quad \text{with} \quad Q > \sqrt{n}$$

and

$$\sum_{k \in \mathbb{Z}^n} \tau_k \beta\,(x) = 1, \quad \forall x \in \mathbb{R}^n \setminus \{0\}\,.$$

Definition 4.1. *Let E be a (B.s.d.). The localized space of E, denoted by $(E)_{\ell^u}$, is the set of $f \in \mathcal{S}'$, such that*

$$\left\| f | (E)_{\ell^u} \right\| = \left(\sum_{k \in \mathbb{Z}^n} \left\| \tau_k \beta \cdot\ f | E \right\|^u \right)^{1/u} < \infty.$$

In this definition, we can replace the function β by another function, say $\theta \in \mathcal{D}(\mathbb{R}^n)$. To do so, it suffices the function θ does not vanish on the support of β (see [2, Proposition 5, page 156]). We will choose the function θ as follows.

Let $Q > 0$ be large enough such that the cube $[-Q, +Q]^n$ includes the support of β. We will assume that θ is a non-negative function, such that $\theta(x) = 1$, for all $x \in [-Q, +Q]^n$.

Proposition 4.2. *Let E be a (B.s.d.). A distribution f belongs to $(E)_{\ell^p}$ if and only if*

$$\left(\sum_{k \in \mathbb{Z}^n} \left\| \tau_k \theta \cdot\ f | E \right\|^p \right)^{1/p} < \infty.$$

The above expression is an equivalent norm in $(E)_{\ell^p}$.

Proof. On one hand, we can write $\beta = g \cdot \theta$ or $g \in \mathcal{D}(\mathbb{R}^n)$ and

$$\left\| \tau_k \beta \cdot f | E \right\| \lesssim \left\| g | M(E) \right\| \left\| \tau_k \theta \cdot f | E \right\|,$$

then

$$\left\| f |\, (E)_{\ell^p} \right\| \lesssim \left(\sum_{k \in \mathbb{Z}^n} \left\| \tau_k \theta \cdot f | E \right\|^p \right)^{1/p}.$$

On the other hand,

$$\begin{aligned} \left\| \tau_k \theta \cdot f | E \right\| &\lesssim \sum_{k' \in \mathbb{Z}^n} \left\| \tau_{k'} \beta \cdot \tau_k \theta \cdot f | E \right\| \\ &\lesssim \sum_{k' \in \mathbb{Z}^n} \left\| \tau_{k'} \lambda \cdot \tau_k \theta \cdot | M(E) \right\| \left\| \tau_{k'} \beta \cdot f | E \right\|, \end{aligned} \tag{2}$$

where $\lambda \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda = 1$ on the support of β. By a change of variables and the fact that $\|\cdot|M(E)\|$ is invariant by translation, we have

$$\left\| \tau_{k'} \lambda \cdot \tau_k \theta | M(E) \right\| = \left\| \tau_{(k-k')} \lambda \cdot \theta | M(E) \right\|. \tag{3}$$

By combining (2), (3) and the discrete Young inequality, we obtain the result. □

Lemma 4.3. *Let $1 \leq p \leq \infty$. Then, there exists a number α with $0 < 1 < \alpha$ such that*

$$1 \leq \sum_{k\in\mathbb{Z}^n} \tau_k\theta(x)^p \leq \alpha, \qquad \text{for all } x \in \mathbb{R}^n.$$

Proof. We put

$$\varrho(x) := \sum_{k\in\mathbb{Z}^n} \tau_k\theta(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

We can easily verify that the function ϱ is C^∞ and $\mathbb{Z}$-periodic. Therefore it is a bounded function, hence the existence of the number α. Of course, nothing prevents us from assuming that $Q > 1$. Consequently, we have

$$\varrho(x) \geq \theta(x)^p = 1, \qquad \forall x \in [0,1]^n.$$

Taking into account the periodicity of ϱ, it comes that

$$\varrho(x) \geq 1 \text{ for all } x \in \mathbb{R}^n.$$

So

$$1 \leq \varrho(x) \leq \alpha \text{ for all } x \in \mathbb{R}^n.$$

This finishes the proof of Lemma. □

The next Proposition gives the localization of Lebesgue spaces $L^p(B(x,2^{-r}))$ in norm of $\ell^p(\mathbb{Z}^n)$.

Proposition 4.4. *Let $p \in [1,\infty)$ and $\tau \in [0,\infty)$. Then*

$$\left\| f | L^p(B(x,2^{-r})) \right\| \sim \left\| f | (L^p(B(x,2^{-r})))_{\ell^p} \right\|.$$

Proof. We have

$$\sum_{k\in\mathbb{Z}^n} \left\| \tau_k\theta \cdot f | L^p(B(x,2^{-r})) \right\|^p = \sum_{k\in\mathbb{Z}^n} \int_{B(x,2^{-r})} |f(x)|^p \tau_k\theta(x)^p \,\mathrm{d}x.$$

According to the lemma 4.3, we obtain

$$\int_{B(x,2^{-r})} |f(x)|^p \,\mathrm{d}x \lesssim \sum_{k\in\mathbb{Z}^n} \int_{B(x,2^{-r})} |f(x)|^p \tau_k\theta(x)^p \,\mathrm{d}x \lesssim \alpha \int_{B(x,2^{-r})} |f(x)|^p \,\mathrm{d}x.$$

This finishes the proof. □

4.2. Localization of Besov-type spaces

In this subsection, we present the localization of Besov-type spaces in the norm of $\ell^r(\mathbb{Z}^n)$.

Theorem 4.5. *Let $\mu \in \mathbb{R}$, $\tau \in [0,\infty)$ and $p, q \in [1,\infty]$. Let v be a positive function satisfying* (1)*. Then*
(i) $B^{v,\tau}_{p,q} \hookrightarrow \left(B^{v,\tau}_{p,q}\right)_{\ell^w}$ *for* $w \geq \max(p,q)$,
(ii) $\left(B^{v,\tau}_{p,q}\right)_{\ell^u} \hookrightarrow B^{v,\tau}_{p,q}$ *for* $u \leq \min(p,q)$.

In particular, $B^{v,\tau}_{p,p}$ is localizable in the $\ell^p(\mathbb{Z}^n)$ norm.

Proof. (i) Let $f \in B^{v,\tau}_{p,q}$. Replacing f by $\sum_{j=0}^{\infty} \Delta_j f$, we find

$$\left\|\tau_k\beta \cdot f|B^{v,\tau}_{p,q}\right\|^w = \left\|\sum_{j\geq 0} \tau_k\beta \cdot \Delta_j f|B^{v,\tau}_{p,q}\right\|.$$

According to Proposition 3.7, we obtain

$$\left\|\tau_k\beta \cdot f|B^{v,\tau}_{p,q}\right\|^w \lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left\|\tau_k\beta \cdot \Delta_j f|L^p(B(x,2^{-r}))\right\|\right)^q\right)^{w/q}.$$

Using Minkowski's inequality (because $w \geq q$), we have

$$\begin{aligned}\left\|f|(B^{v,\tau}_{p,q})_{\ell^w}\right\| &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{k\in\mathbb{Z}^n}\left(\sum_{j=r^+}^{\infty}\left(v(2^{-j})\left\|\tau_k\beta \cdot \Delta_j f|L^p(B(x,2^{-r}))\right\|\right)^q\right)^{w/q}\right)^{1/w} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{j\geq 0} v(2^{-j})^q\left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\theta \cdot \Delta_j f|L^p(B(x,2^{-r}))\right\|^w\right)^{q/w}\right)^{1/q}.\end{aligned}$$

But we have $\ell^p(\mathbb{Z}^n) \subset \ell^w(\mathbb{Z}^n)$ and $L^p(B(x,2^{-r}))$ is localizable in norm $\ell^p(\mathbb{Z}^n)$, so

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\beta \cdot \Delta_j f|L^p(B(x,2^{-r}))\right\|^w\right)^{1/w} &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\beta \cdot \Delta_j f|L^p(B(x,2^{-r}))\right\|^p\right)^{1/p} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left\|\Delta_j f|L^p(B(x,2^{-r}))\right\|. \qquad (4)\end{aligned}$$

Therefore (4) gives

$$\begin{aligned}\left\|f|(B^{v,\tau}_{p,q})_{\ell^w}\right\| &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{j\geq 0} v(2^{-j})^q\left\|\Delta_j f|L^p(B(x,2^{-r}))\right\|_p^q\right)^{1/q} \\ &\lesssim \left\|f|B^{v,\tau}_{p,q}\right\|.\end{aligned}$$

(ii) Let $f \in \left(B^{v,\tau}_{p,q}\right)_{\ell^u}$, since $f = \sum_{k\in\mathbb{Z}^n} \tau_k\beta \cdot f$, we have

$$\begin{aligned}\left\|f|B^{v,\tau}_{p,q}\right\| &= \left\|\sum_{k\in\mathbb{Z}^n} \tau_k\beta \cdot f|B^{v,\tau}_{p,q}\right\| \\ &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{j\geq 0} v(2^{-j})^q\left\|\sum_{k\in\mathbb{Z}^n} \Delta_j\left(\tau_k\beta \cdot f\right)|L^p(B(x,2^{-r}))\right\|^q\right)^{1/q}.\end{aligned}$$

Applying Lemma 4.7 to the second member of the precedent equation, we obtain

$$\left\|f|B^{v,\tau}_{p,q}\right\| \lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^\tau}\left(\sum_{j\geq 0}\left(\sum_{k\in\mathbb{Z}^n}\left(v(2^{-j})\left\|\Delta_j\left(\tau_k\beta \cdot f\right)|L^p(B(x,2^{-r}))\right\|\right)^p\right)^{q/p}\right)^{1/q}.$$

By the inequality of Minkowski (because $q \geq u$), we have

$$\begin{aligned}
\left\| f | B^{v,\tau}_{p,q} \right\| &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^{\tau}} \left(\sum_{j\geq 0} \left(\sum_{k\in\mathbb{Z}^n} \left(v(2^{-j}) \left\| \Delta_j (\tau_k \beta \cdot f) | L^p(B(x,2^{-r})) \right\| \right)^u \right)^{q/u} \right)^{1/q} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^{\tau}} \left(\sum_{k\in\mathbb{Z}^n} \left(\sum_{j\geq 0} \left(v^q(2^{-j}) \left\| \Delta_j (\tau_k \beta \cdot f) | L^p(B(x,2^{-r})) \right\| \right)^q \right)^{q/u} \right)^{1/u} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^{\tau}} \left(\sum_{k\in\mathbb{Z}^n} \left(v(2^{-j}) \left\| \tau_k \beta \cdot f | B^{v,\tau}_{p,q} \right\| \right)^u \right)^{1/u} \\
&= \left\| f | (B^{v,\tau}_{p,q})_{\ell^u} \right\|.
\end{aligned}$$

This shows the inclusion

$$\left(B^{v,\tau}_{p,q} \right)_{\ell^u} \hookrightarrow B^{v,\tau}_{p,q}.$$

□

Remark 4.6. *The particular case $v(t) = t^{-\mu}$ and $\tau = 0$ yields the Besov space $B^{\mu}_{p,q}(\mathbb{R}^n)$, we obtain the results of Bourdaud [2], Ferahtia [7], Sickel [12] and Triebel [18].*

4.3. Localization of Triebel-Lizorkin-type spaces

In this subsection, we present the localization of Triebel-Lizorkin-type spaces in the norm of $\ell^p(\mathbb{Z}^n)$. The following result plays a fundamental role in the proof of Theorem 4.8.

Proposition 4.7. *Let $p \in [1,\infty]$. There exists a constant $c > 0$ such that the inequality*

$$\left\| \sum_{k\in\mathbb{Z}^n} f_k | L^p(B(x,2^{-r})) \right\| \leq c \left(\sum_{k\in\mathbb{Z}^n} \left\| f_k | L^p(B(x,2^{-r})) \right\|^p \right)^{1/p}$$

holds, for all $Q > 1$ and for all family $\{f_k\}_{k\in\mathbb{Z}^n}$ of $\mathcal{S}'$ with supp f_k *contained in the ball $|x-k| \leq Q$.*

Proof. The proof is immediate if we notice that

$$\sum_{k\in\mathbb{Z}^n} f_k = \sum_{k\in\mathbb{Z}^n} \tau_k\theta \cdot f_k = H_\theta \{f_k\}_{k\in\mathbb{Z}^n},$$

where $\theta \in \mathcal{D}(\mathbb{R}^n)$ is chosen by sort that $\theta = 1$ on the ball $|x| \leq Q$. We calculate $H_\theta \{f_k\}_{k\in\mathbb{Z}^n}$ in the norms of $L^1(B(x,2^{-r}))$ and $L^\infty(B(x,2^{-r}))$, we obtain

$$\left\| H_\theta \{f_k\}_{k\in\mathbb{Z}^n} | L^1(B(x,2^{-r})) \right\| \lesssim \left(\int_{B(x,2^{-r})} \left(\sum_{k\in\mathbb{Z}^n} |f_k(x)| \tau_k\theta(x) \right) \mathrm{d}x \right).$$

By Hölder's inequality, we get

$$\begin{aligned}
\left\| H_\theta \{f_k\}_{k\in\mathbb{Z}^n} | L^1(B(x,2^{-r})) \right\| &\lesssim \left\| \theta | \ell^\infty \right\| \sum_{k\in\mathbb{Z}^n} \left\| f_k | L^1(B(x,2^{-r})) \right\| \\
&= \left\| \theta | L^\infty \right\| \left\| \left\| \{f_k\}_{k\in\mathbb{Z}^n} | L^1(B(x,2^{-r})) \right\| | \ell^1(\mathbb{Z}^n) \right\|.
\end{aligned}$$

Thus

$$\begin{aligned}\left\|H_{\theta}\left\{f_k\right\}_{k\in\mathbb{Z}^n}|L^{\infty}(B(x,2^{-r}))\right\| &\lesssim \sup_{x\in\mathbb{R}^n}\left(\sup_{k\in\mathbb{Z}^n}\left\|\{f_k\}_{k\in\mathbb{Z}^n}|L^{\infty}(B(x,2^{-r}))\right\|\right.\\ &\quad\left.\times\Big(\sum_{k\in\mathbb{Z}^n}\tau_k\theta(x)\Big)\right)\\ &\lesssim \left\|\left\|\left\{f_k\right\}_{k\in\mathbb{Z}^n}|L^{\infty}(B(x,2^{-r}))\right\||\ell^{\infty}(\mathbb{Z}^n)\right\|.\end{aligned}$$

Finally, using the complex interpolation of $L^p(\Omega)$,

$$\left[L^1(\Omega),L^{\infty}(\Omega)\right]_{\frac{1}{p}}=L^p(\Omega)\quad\text{(see [1, 1.1])}$$

and

$$\left[\ell^1(L^1(\Omega)),\ell^{\infty}(L^{\infty}(\Omega))\right]_{\frac{1}{p}}=\ell^p(\left[L^1(\Omega),L^{\infty}(\Omega)\right]_{\frac{1}{p}})=\ell^p(L^p(\Omega))\quad\text{(see [15, 1.18.1])}$$

we obtain the result. □

The following result gives the localization property of generalized Triebel-Lizorkin-type spaces on the $\ell^p(\mathbb{Z}^n)$ spaces.

Theorem 4.8. *Let $\mu\in\mathbb{R}$, $p\in[1,\infty)$, $q\in[1,\infty]$ and $\tau\in(0,1/p)$. Let v be a positive function satisfying (1). Then*

$$\left(F^{v,\tau}_{p,q}\right)_{\ell^p}\sim F^{v,\tau}_{p,q}.$$

Proof. Step 1. We first show that

$$\left(F^{v,\tau}_{p,q}\right)_{\ell^p}\hookrightarrow F^{v,\tau}_{p,q}.$$

Let $f\in\left(F^{v,\tau}_{p,q}\right)_{\ell^p}$, then

$$\left\|f|F^{v,\tau}_{p,q}\right\|=\sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\frac{1}{|B(x,2^{-r})|^{\tau}}\left\|\left\{\Delta_j f\right\}_{j\geq r^+}|L^p(B(x,2^{-r}),\ell^v_{q,r^+})\right\|. \tag{5}$$

Replacing f in (5) by $\sum_{k\in\mathbb{Z}^n}\tau_k\beta\cdot f$, we get

$$\begin{aligned}\left\|f|F^{v,\tau}_{p,q}\right\| &= \sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\frac{1}{|B(x,2^{-r})|^{\tau}}\left\|\left(\sum_{j=r^+}^{\infty}\Big(\sum_{k\in\mathbb{Z}^n}v(2^{-j})\left|\Delta_j\left(\tau_k\beta\cdot f\right)\right|\Big)^q\right)^{1/q}|L^p(B(x,2^{-r}))\right\|\\ &= \sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\frac{1}{|B(x,2^{-r})|^{\tau}}\left\|\left\|\left\{\left\|\left\{v(2^{-j})\left|\Delta_j(\tau_k\beta\cdot f)\right|\right\}_{k\in\mathbb{Z}^n}|\ell^1\right\|\right\}_{j\geq r^+}|\ell^q\right\||L^p(B(x,2^{-r}))\right\|.\end{aligned}$$

Thanks to the inequality of Minkowski, Propositions 4.2 and 4.7, we have

$$\begin{aligned}\left\|f|F^{v,\tau}_{p,q}\right\| &\lesssim \sup_{x\in\mathbb{R}^n,r\in\mathbb{Z}}\frac{1}{|B(x,2^{-r})|^{\tau}}\left\|\left\|\left\{\left\|\left\{v(2^{-j})\left|\Delta_j(\tau_k\beta\cdot f)\right|\right\}_{j\geq r^+}|\ell^q\right\|\right\}_{k\in\mathbb{Z}^n}|\ell^1\right\||L^p(B(x,2^{-r}))\right\|\\ &\lesssim \left(\sum_{k\in\mathbb{Z}^n}\left\|\left(\sum_{j=0}^{\infty}\left(v(2^{-j})\left|\Delta_j\left(\tau_k\beta\cdot f\right)\right|\right)^q\right)^{1/q}|L^p(B(x,2^{-r}))\right\|^p\right)^{1/p}\\ &\lesssim \left(\sum_{k\in\mathbb{Z}^n}\left\|\left(\tau_k\beta\cdot f\right)|F^{v,\tau}_{p,q}\right\|^p\right)^{1/p}\\ &\lesssim c\left\|f|(F^{v,\tau}_{p,q})_{\ell^p}\right\|.\end{aligned}$$

Step 2. Let $f \in F^{v,\tau}_{p,q}$. Since $f = \sum_{j=0}^{\infty} \Delta_j f$, we have

$$\Big(\sum_{k\in\mathbb{Z}^n} \big\|\tau_k\beta \cdot f|F^{v,\tau}_{p,q}\big\|^p\Big)^{1/p} \lesssim \Big(\sum_{k\in\mathbb{Z}^n} \Big\|\sum_{j=0}^{\infty} \tau_k\beta \cdot \Delta_j f|F^{v,\tau}_{p,q}\Big\|^p\Big)^{1/p}.$$

By Proposition 3.7, we find

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}^n} \big\|\tau_k\beta \cdot f|F^{s,\tau}_{p,q}\big\|^p\Big)^{1/p} &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^{\tau}}\Big(\sum_{k\in\mathbb{Z}^n} \Big\|\Big(\sum_{j=r^+}^{\infty} \big(v(2^{-j})\,\big|\tau_k\beta \cdot \Delta_j f\big|\big)^q\Big)^{1/q} |L^p(B(x,2^{-r}))\Big\|^p\Big)^{1/p}\\ &\lesssim \sup_{x\in\mathbb{R}^n, r\in\mathbb{Z}} \frac{1}{|B(x,2^{-r})|^{\tau}}\Big(\sum_{k\in\mathbb{Z}^n} \Big\|\tau_k\beta \cdot \Big(\sum_{j=r^+}^{\infty} \big(v(2^{-j})\,\big|\Delta_j f\big|\big)^q\Big)^{1/q} |L^p(B(x,2^{-r}))\Big\|^p\Big)^{1/p}.\end{aligned}$$

According to the localization of the space $L^p(B(x,2^{-r}))$ in norm $\ell^p(\mathbb{Z}^n)$, we obtain

$$\big\|f|(F^{v,\tau}_{p,q})_{\ell^p}\big\| \lesssim \big\|f|F^{v,\tau}_{p,q}\big\|.$$

□

Remark 4.9. *The particular case $v(t) = t^{-\mu}$ and $\tau = 0$ yields the Triebel-Lizorkin space $F^{\mu}_{p,q}(\mathbb{R}^n)$, we find the results of Djeriou [3], Ferahtia [7] and Triebel [18].*

4.4. Localization of the spaces of the multipliers.

In this subsection, we will study the localization of the spaces of point multipliers of Besov-type spaces and Triebel-Lizorkin-type spaces.

Theorem 4.10. *Let $q \in [1,\infty]$ and v be a positive function satisfies* (1). *Then*
(i) *For any $\mu > (n/\min\{p,q\} - n)_+, 1 \le p < \infty$ and $\tau \in [0, 1/p)$, we have*

$$M(F^{v,\tau}_{p,q}) \sim (M(F^{v,\tau}_{p,q}))_{\ell^\infty}$$

and for any $\mu > 0, 1 \le p < \infty$ and $\tau \in [0,\infty)$, we have

$$M(B^{v,\tau}_{p,q}) \sim (M(B^{v,\tau}_{p,q}))_{\ell^\infty}.$$

(ii) *If $\mu > n\max\{\frac{1}{p} - \tau, \frac{1}{q} - 1\}$, we have*

$$M(F^{v,\tau}_{p,q}) \sim (F^{v,\tau}_{p,q})_{\ell^\infty}.$$

Proof. (i) By similarity we prove only the Triebel-Lizorkin-type case in (i). First, we prove $M(F^{v,\tau}_{p,q}) \hookrightarrow (M(F^{v,\tau}_{p,q}))_{\ell^\infty}$.

Let $f \in M(F^{v,\tau}_{p,q})$, then

$$\sup_{k\in\mathbb{Z}^n} \big\|\tau_k\beta \cdot |M(F^{v,\tau}_{p,q})\big\| \le c\,\big\|\beta|M(F^{v,\tau}_{p,q})\big\|\,\big\|f|M(F^{v,\tau}_{p,q})\big\|.$$

Second, we prove $(M(F^{v,\tau}_{p,q}))_{\ell^\infty} \hookrightarrow M(F^{v,\tau}_{p,q})$.

Let $f \in M(F^{v,\tau}_{p,q})$, then for everything $g \in F^{v,\tau}_{p,q}$, we have

$$\big\|f \cdot g|F^{v,\tau}_{p,q}\big\| \lesssim \Big(\sum_{k\in\mathbb{Z}^n} \big\|\tau_k\beta \cdot f \cdot g|F^{v,\tau}_{p,q}\big\|^p\Big)^{1/p}.$$

Because $F^{v,\tau}_{p,q}$ is localizable in norm $\ell^p(\mathbb{Z}^n)$, then

$$\begin{aligned}\left\|f\cdot g|F^{s,\tau}_{p,q}\right\| &\lesssim \left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\beta\cdot\tau_k\lambda\cdot f\cdot g|F^{v,\tau}_{p,q}\right\|^p\right)^{1/p}\\ &\lesssim c\left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\beta\cdot f|M(F^{v,\tau}_{p,q})\right\|^p\left\|\tau_k\lambda\cdot g|F^{v,\tau}_{p,q}\right\|^p\right)^{1/p},\end{aligned}$$

with $\lambda\in\mathcal{D}(\mathbb{R}^n)$ and $\lambda=1$ on the support of β, then

$$\begin{aligned}\left\|f\cdot g|F^{v,\tau}_{p,q}\right\| &\lesssim \sup_{k\in\mathbb{Z}^n}\left\|\tau_k\beta\cdot f|M(F^{v,\tau}_{p,q})\right\|\left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\lambda\cdot g|F^{v,\tau}_{p,q}\right\|^p\right)^{1/p}\\ &\lesssim \left\|f|(M(F^{v,\tau}_{p,q}))_{\ell^\infty}\right\|\left\|g|F^{v,\tau}_{p,q}\right\|,\end{aligned}$$

hence the second inclusion. Then

$$M(F^{v,\tau}_{p,q})\sim(M(F^{v,\tau}_{p,q}))_{\ell^\infty}.$$

Statement (i) is proved.
(ii) We must show that

$$(F^{v,\tau}_{p,q})_{\ell^\infty}\hookrightarrow M(F^{v,\tau}_{p,q})\sim(M(F^{v,\tau}_{p,q}))_{\ell^\infty}\hookrightarrow(F^{v,\tau}_{p,q})_{\ell^\infty}.$$

If $\mu>n\max\left\{\frac{1}{p}-\tau,\frac{1}{q}-1\right\}$, we have $F^{v,\tau}_{p,q}\hookrightarrow M(F^{v,\tau}_{p,q})$. In other words

$$\left\|f\cdot g|F^{v,\tau}_{p,q}\right\|\lesssim\left\|f|F^{v,\tau}_{p,q}\right\|\left\|g|F^{v,\tau}_{p,q}\right\|,\qquad\forall f,g\in F^{v,\tau}_{p,q},$$

we have also

$$(F^{v,\tau}_{p,q})_{\ell^\infty}\hookrightarrow(M(F^{v,\tau}_{p,q}))_{\ell^\infty}=(M(F^{v,\tau}_{p,q})).$$

Conversely, if $f\in M(F^{v,\tau}_{p,q})$ and $\forall g\in F^{v,\tau}_{p,q}$, then

$$\left\|\tau_k\beta\cdot f\cdot g|F^{v,\tau}_{p,q}\right\|\lesssim\left\|\tau_k\beta\cdot f|F^{v,\tau}_{p,q}\right\|\left\|g|F^{v,\tau}_{p,q}\right\|.$$

Hence

$$\left\|\tau_k\beta\cdot f|M(F^{v,\tau}_{p,q})\right\|\lesssim\sup_{k\in\mathbb{Z}^n}\left\|\tau_k\beta\cdot f|F^{s,\tau}_{p,q}\right\|,$$

we obtain

$$(M(F^{v,\tau}_{p,q}))_{\ell^\infty}\hookrightarrow(F^{v,\tau}_{p,q})_{\ell^\infty}.$$

Statement (ii) is proved. □

Remark 4.11. *If $v(t)=t^{-\mu}$, $1\le p<\infty$, $1\le q\le\infty$, $\tau\in(0,1/p)$ and $\mu>n\max\{\frac{1}{p}-\tau,\frac{1}{q}-1\}$, we find the results of Djeriou [3], Franke [8], Peetre [11] and Strichartz [13].*

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