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Composition operators from the space of exponential Cauchy kernels generated by special measures

Ajay K. Sharma[a], Shayesta Farooq[b], M. Mursaleen[c,d,*]

[a]*Department of Mathematics, Central University of Jammu, Raya-Suchani (Bagla), Samba 181143, J&K, India*
[b]*Department of Mathematics, Baba Ghulam Shah Badshah University, Rajouri, J&K-185234, India*
[c]*Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*
[d]*University Centre for Research and Development, Chandigarh University, Mohali 140243, Punjab, India*

Abstract. In this paper, we explicitly characterize bounded and compact composition operators from the space of exponential Cauchy kernels generated by special measures to Bloch and little Bloch type spaces. Moreover, the norm of a composition operator acting between these spaces is also obtained.

1. Introduction

Denote by $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ the unit circle, $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and by m the Lebesgue measure on $\partial\mathbb{D}$ such that $m(\partial\mathbb{D}) = 1$. Let $\mathfrak{M}$ be the space of all complex Borel measures on $\partial\mathbb{D}$ and $\mathfrak{M}^*$ the subset of $\mathfrak{M}$ consisting of probability measures. The family $\mathcal{K}$ of Cauchy transforms is a subspace of $H(\mathbb{D})$ consisting of all those functions which admits a representation of the form

$$f(z) = \int_{\partial\mathbb{D}} \frac{1}{1-\bar{x}z}\, d\mu(x) \quad (z \in \mathbb{D})$$

for some $\mu \in \mathfrak{M}$. Endowed with the norm

$$\|f\|_{\mathcal{K}} = \inf_{\mu\in\mathfrak{M}} \left\{ \|\mu\| \ :\ f(z) = \int_{\partial\mathbb{D}} \frac{1}{1-\bar{x}z}\, d\mu(x) \right\},$$

$\mathcal{K}$ becomes a Banach space, where $\|\mu\|$ is the total variation of the measure μ. By the Lebesgue decomposition theorem $\mathfrak{M} = \mathfrak{M}_a + \mathfrak{M}_s$, where $\mathfrak{M}_a = \{\mu_a \in \mathfrak{M} : \mu_a \ll m\}$ and $\mathfrak{M}_s = \{\mu_s \in \mathfrak{M} : \mu_s \perp m\}$. Therefore, for any $\mu \in \mathfrak{M}$, we have $\mu = \mu_a + \mu_s$, where $\mu_a \in \mathfrak{M}_a$, $\mu_s \in \mathfrak{M}_s$ and $\|\mu\| = \|\mu_a\| + \|\mu_s\|$. Thus we also have the decomposed: $\mathcal{K} = (\mathcal{K})_a + (\mathcal{K})_s$, where $(\mathcal{K})_a$ is isometrically isomorphic to $\mathfrak{M}/\overline{H_0^1}$, the closed subspace of $\mathfrak{M}$ of absolutely continuous measures and $(\mathcal{K})_s$ is isomorphic to $\mathfrak{M}_s$, the closed subspace of $\mathfrak{M}$ consisting of singular measures.

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* Corresponding author: M. Mursaleen
Email addresses: aksju_76@yahoo.com (Ajay K. Sharma), shayestalone668@gmail.com (Shayesta Farooq), mursaleenm@gmail.com (M. Mursaleen)

Recently Yallaoui [22] defined the space of exponential Cauchy transforms as

$$\mathcal{K}_e = \left\{ f \in H(\mathbb{D}) : f(z) = \int_{\partial\mathbb{D}} \exp\left[\frac{1}{1-xz}\right] d\mu(x) \right\}.$$

The space $\mathcal{K}_e$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{K}_e} = \inf_{\mu\in\mathfrak{M}} \left\{ \|\mu\| \ : \ f(z) = \int_{\partial\mathbb{D}} \exp\left[\frac{1}{1-\bar{x}z}\right] d\mu(x) \right\},$$

and can also be decomposed as $\mathcal{K}_e = (\mathcal{K}_e)_a \oplus (\mathcal{K}_e)_s$. Moreover, $\mathcal{K} \subset (\mathcal{K}_e)_a$ and $H^p \subset (\mathcal{K}_e)_a$ for all $p > 0$. For more about these spaces, we refer [1]-[9] and [11]-[22].
The Bloch-type space $\mathcal{B}_\nu(\mathbb{D}) = \mathcal{B}_\nu$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\nu} := |f(0)| + b_\nu(f) = |f(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty,$$

where ν is a positive continuous function on $\mathbb{D}$ generally called a *weight* or a *weight function*. The weight ν is called *typical* if it is radial, i.e. $\nu(z) = \nu(|z|)$, $z \in \mathbb{D}$ and $\nu(|z|)$ decreasingly converges to 0 as $|z| \to 1$. Also the little Bloch-type space

$$\mathcal{B}_{\nu,0}(\mathbb{D}) = \mathcal{B}_{\nu,0} = \{f \in H(\mathbb{D}) : \lim_{|z|\to 1} \nu(z)|f'(z)| = 0\}$$

is a closed subspace of $\mathcal{B}_\nu$.

Recall that for φ be a holomorphic self-map of $\mathbb{D}$, the composition operator C_φ is a linear operator defined as

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

Recently, Abu-Muhanna and Yallaoui [2] characterized bounded and compact composition operators on the space of exponential Cauchy transforms. Motivated by results in [2], we explicitly characterize bounded and compact composition operators from the space of exponential Cauchy kernels generated by special measures to Bloch and little Bloch type spaces. Moreover, we also obtain norm of a composition operator acting between these spaces. For recent study of composition operators on Cauchy transforms, see [1]-[2], [7], [8], [11], [12], [21] and the references therein. Throughout this paper, ν is a typical weight, and any positive constants is denoted by C may not be same at each occurrence. The notation $a \lesssim b$ means that $a \le Cb$ and $a \gtrsim b$, means $a \ge Cb$. Moreover, if $a \lesssim b$ and $b \le Ca$, then we write $a \asymp b$.

2. Main results

In this section, we characterize bounded and compact composition operators from the space of exponential Cauchy kernels to Bloch type spaces.
The following two lemmas will play important role in this paper, see Corollary 1 and Lemma 1 in [1].

Lemma 1. *The family $\{f_x : x \in \partial\mathbb{D}\}$ is a subset of $\mathcal{K}_e$, where*

$$f_x(z) = \exp\left[\frac{1}{1-xz}\right], \quad z \in \mathbb{D}. \tag{2.1}$$

Moreover, $\sup_{x\in\partial\mathbb{D}} \|f\|_{\mathcal{K}_e} = 1$.

Lemma 2. *For each $f \in \mathcal{K}_e$, there exists a measure $\mu \in \mathfrak{M}$ such that*

$$f(z) = \int_{\partial\mathbb{D}} \exp\left[\frac{1}{1-xz}\right] d\mu(x)$$

and $\|f\|_{\mathcal{K}_e} = \|\mu\|$.

The next lemma is proved in [21].

Lemma 3. *Let $\nu : \mathbb{D} \to [0, \infty)$ be a typical weight and $d\lambda(z) = dA(z)/(1-|z|^2)^2$. Then $f \in \mathcal{B}_\nu$ if and only if*

$$\|f\|^2_{\mathcal{B}_\nu} \asymp |f(0)|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \nu^2(z) \Big(1 - \Big|\frac{a-z}{1-\bar{a}z}\Big|^2\Big)^2 d\lambda(z) < \infty, \tag{2.2}$$

Standard arguments from Proposition 3.11 in [10] yields the proof of the next lemma. We omit details.

Lemma 4. *If C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$, then C_φ maps $\mathcal{K}_e$ compactly into $\mathcal{B}_\nu$ if and only if for any norm bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{K}_e$ converging to zero on compact subsets of $\mathbb{D}$, we have that $\lim_{j\to\infty} \|C_\varphi f_j\|_{\mathcal{B}_\nu} = 0$.*

The compactness of a closed subset $F \subset \mathcal{B}_{\nu,0}$ can be characterized as follows, see Lemma 1 in [20].

Lemma 5. *A closed set F in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded with respect to the norm $\|\cdot\|_{\mathcal{B}_\nu}$ and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in F} \nu(z)|f'(z)| = 0.$$

Theorem 1. *C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$ if and only if*

$$M_1 := \sup_{x\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\bar{x}\varphi(z)|^2} \exp\Big[\Re\Big(\frac{1}{1-\bar{x}\varphi(z)}\Big)\Big] < \infty. \tag{2.3}$$

Moreover, if C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$, then

$$\|C_\varphi\|_{\mathcal{K}_e\to\mathcal{B}_\nu} = M_1 + \sup_{x\in\partial\mathbb{D}} \exp\Big[\Re\Big(\frac{1}{1-\bar{x}\varphi(0)}\Big)\Big]. \tag{2.4}$$

Proof. First, suppose that (2.3) holds. Let $f \in \mathcal{K}_e$. Then by Lemma 2, there is a measure $\mu \in \mathfrak{M}$ such that

$$f(z) = \int_{\partial\mathbb{D}} \exp\Big[\frac{1}{1-\bar{x}z}\Big] d\mu(x) \tag{2.5}$$

and $\|\mu\| = \|f\|_{\mathcal{K}_e}$. Thus, we have

$$f'(z) = \int_{\partial\mathbb{D}} \exp\Big[\frac{1}{1-\bar{x}z}\Big] \frac{\bar{x}}{(1-\bar{x}z)^2} d\mu(x)$$

and

$$f'(\varphi(z)) = \int_{\partial\mathbb{D}} \exp\Big[\frac{1}{1-\bar{x}\varphi(z)}\Big] \frac{\bar{x}}{(1-\bar{x}\varphi(z))^2} d\mu(x).$$

Therefore,

$$\begin{aligned} \nu(z)|\varphi'(z)||f'(\varphi(z))| &\le \int_{\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\bar{x}\varphi(z)|^2} \Big|\exp\Big[\frac{1}{1-\bar{x}\varphi(z)}\Big]\Big| d|\mu|(x) \\ &\le \sup_{x\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\bar{x}\varphi(z)|^2} \Big|\exp\Big[\frac{1}{1-\bar{x}\varphi(z)}\Big]\Big| \int_{\partial\mathbb{D}} d|\mu|(x) \end{aligned} \tag{2.6}$$

Using the facts that $\int_{\partial\mathbb{D}} d|\mu|(x) = \|\mu\|$, $\|\mu\| = \|f\|_{\mathcal{K}_e}$ and $|\exp(f(z))| = \exp[\Re(f(z))]$ for any $f \in H(\mathbb{D})$, we have that

$$\sup_{z\in\mathbb{D}} \nu(z)|(C_\varphi f)'(z)| \le \sup_{x\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\bar{x}\varphi(z)|^2} \exp\Big[\Re\Big(\frac{1}{1-\bar{x}\varphi(z)}\Big)\Big] \|f\|_{\mathcal{K}_e}.$$

Taking the supremum over $z \in \mathbb{D}$, we get

$$\sup_{z\in\mathbb{D}} \nu(z)|(C_\varphi f)'(z)| \le M_1 \|f\|_{\mathcal{K}_e}. \tag{2.7}$$

Again

$$\begin{aligned} |(C_\varphi f)(0)| = |f(\varphi(0))| &= \left| \int_{\partial\mathbb{D}} \exp\left[\frac{1}{1-\overline{x}\varphi(0)}\right] d\mu(x) \right| \\ &\le \sup_{x\in\partial\mathbb{D}} \left| \exp\left[\frac{1}{1-\overline{x}\varphi(0)}\right] \right| \left| \int_{\partial\mathbb{D}} d|\mu|(x) \right. \\ &= \sup_{x\in\partial\mathbb{D}} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(0)}\right)\right] \|f\|_{\mathcal{K}_e}. \end{aligned} \tag{2.8}$$

Thus from (2.7) and (2.8), we have

$$\|C_\varphi f\|_{\mathcal{B}_\nu} \le \left\{ M_1 + \sup_{x\in\partial\mathbb{D}} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(0)}\right)\right] \right\} \|f\|_{\mathcal{K}_e}.$$

Hence C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$ and

$$\|C_\varphi\|_{\mathcal{K}_e\to\mathcal{B}_\nu} \le M_1 + \sup_{x\in\partial\mathbb{D}} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(0)}\right)\right]. \tag{2.9}$$

Next suppose that C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$. Let

$$f_x(z) = \exp\left[\frac{1}{1-\overline{x}z}\right], \quad x \in \partial\mathbb{D}. \tag{2.10}$$

Then, by lemma 1, we have $\sup_{x\in\mathbb{D}} \|f_x\|_{\mathcal{K}_e} = 1$ and

$$f_x'(z) = \exp\left[\frac{1}{1-\overline{x}z}\right] \frac{\overline{x}}{(1-\overline{x}z)^2}.$$

From this and the fact that C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$, we have that $\|C_\varphi f_x\|_{\mathcal{B}_\nu} \le \|C_\varphi\|_{\mathcal{K}_e\to\mathcal{B}_\nu}$, for every $x \in \partial\mathbb{D}$ and so

$$M_1 + \sup_{x\in\partial\mathbb{D}} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(0)}\right)\right] \le \|C_\varphi\|_{\mathcal{K}_e\to\mathcal{B}_\nu}. \tag{2.11}$$

Furthermore, from (2.9) and (2.11), (2.4) follows and the proof is accomplished. □

Theorem 2. *C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$ if and only if*

$$M_2 := \sup_{x\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right] \left(1 - \left|\frac{a-z}{1-\overline{a}z}\right|^2\right)^2 d\lambda(z) \tag{2.12}$$

is finite.

Proof. It is sufficient to show that $M_2 \asymp M_1^2$. Claim that $M_2 \gtrsim M_1^2$. For $z \in D(a)$, we have $\nu(a) \asymp \nu(z)$ and $|1-\bar{a}z| \asymp 1-|a|^2$, Thus by the subharmonicity of the function

$$\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right],$$

we can have

$$
\begin{aligned}
M_2 &\geq \sup_{x\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{D(a)}\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4}\exp\Big[\Re\Big(\frac{2}{1-\overline{x}\varphi(z)}\Big)\Big]\Big(1-\Big|\frac{a-z}{1-\overline{a}z}\Big|^2\Big)^2 d\lambda(z)\\
&= \sup_{x\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{D(a)}\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4}\exp\Big[\Re\Big(\frac{2}{1-\overline{x}\varphi(z)}\Big)\Big]\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}dA(z)\\
&\geq \sup_{x\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4}\exp\Big[\Re\Big(\frac{2}{1-\overline{x}\varphi(z)}\Big)\Big] = M_1^2.
\end{aligned}
\tag{2.13}
$$

This settles the claim. Next we show that $M_2 \lesssim M_1^2$. Again

$$\int_{\mathbb{D}}\Big(1-\Big|\frac{a-z}{1-\overline{a}z}\Big|^2\Big)^2 d\lambda(z) \lesssim 1,$$

we have that

$$M_2 \leq M_1^2 \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(1-|\eta_a(z)|^2)^2 d\lambda(z) \lesssim M_1^2. \tag{2.14}$$

Thus the proof is accomplished. □

Theorem 3. *The following statements are equivalent:*

(1) C_φ maps $\mathcal{K}_e$ compactly into $\mathcal{B}_\nu$.

(2) For each of $x\in\mathbb{T}$, the transform Λ defined as

$$\Lambda(x) = \int_{\mathbb{D}}\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4}\exp\Big[\Re\Big(\frac{2}{1-\overline{x}\varphi(z)}\Big)\Big](1-|\eta_a(z)|^2)^2 d\lambda(z)$$

is a continuous function of x.

(3) For each x in $\partial\mathbb{D}$ and E of $\mathbb{D}$, let

$$\nu_x(E) = \int_E\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4}\exp\Big[\Re\Big(\frac{2}{1-\overline{x}\varphi(z)}\Big)\Big]\Big(1-\Big|\frac{a-z}{1-\overline{a}z}\Big|^2\Big)^2 d\lambda(z).$$

Then for any given $\varepsilon > 0$ and a subset E of $\mathbb{D}$, there is a $\delta > 0$ such that $\nu_x(E) < \varepsilon$ for all x in $\partial\mathbb{D}$ whenever

$$\int_{\mathbb{D}}\Big(1-\Big|\frac{a-z}{1-\overline{a}z}\Big|^2\Big)^2 d\lambda(z) < \delta.$$

Proof. (1) ⇒ (2). Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in $\partial\mathbb{D}$ such that x_j converges to x for some x in $\partial\mathbb{D}$ as $j\to\infty$. Let f_{x_j} be defined as in (2.1). Then $\sup_{j\in\mathbb{N}}\|f_{x_j}\|_{\mathcal{K}_e} = 1$ and $f_{x_j}\to f_x$ uniformly on compact subsets of $\mathbb{D}$. Since C_φ maps $\mathcal{K}_e$ compactly into $\mathcal{B}_\nu$, so by Lemma 4, we have that $\|C_\varphi(f_{x_j}-f_x)\|_{\mathcal{B}_\nu}\to 0$ as $j\to\infty$.

$$
\begin{aligned}
|\Lambda(x_j)-\Lambda(x)| &= \Big|\int_{\mathbb{D}}\Big\{\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x_j}\varphi(z)|^4}\exp\Big[\Re\Big(\frac{2}{1-\overline{x_j}\varphi(z)}\Big)\Big]\\
&\quad -\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4}\exp\Big[\Re\Big(\frac{2}{1-\overline{x}\varphi(z)}\Big)\Big]\Big\}\Big(1-\Big|\frac{a-z}{1-\overline{a}z}\Big|^2\Big)^2 d\lambda(z)\Big|\\
&= \Big|\int_{\mathbb{D}}\Big\{|(f_{x_j}\circ\varphi)'(z)|^2-|(f_x\circ\varphi)'(z)|^2\Big\}\nu^2(z)\Big(1-\Big|\frac{a-z}{1-\overline{a}z}\Big|^2\Big)^2 d\lambda(z)\Big|\\
&\lesssim \Big(\int_{\mathbb{D}}|f'_{x_j}(\varphi(z))\varphi'(z)-f'_x(\varphi(z))\varphi'_i(z)|^2\nu^2(z)\Big(1-\Big|\frac{a-z}{1-\overline{a}z}\Big|^2\Big)^2 d\lambda(z)\Big)^{1/2}\\
&= \|C_\varphi(f_{x_j}-f_x)\|_{\mathcal{B}_\nu}\to 0 \text{ as } j\to\infty.
\end{aligned}
$$

Thus $\Lambda(x_j) \to \Lambda(x)$ as $j \to \infty$. Hence the transformation Λ is a continuous function of $x \in \partial\mathbb{D}$.
(2) $\Rightarrow$ (3). If possible, suppose that (2) does not hold. Then there are sequences $\{x_k\}_{k\in\mathbb{N}}$ in $\partial\mathbb{D}$ and $\{E_k\}_{k\in\mathbb{N}}$ in $\mathbb{D}$ such that $x_k \to x$ and

$$\int_{\mathbb{D}} \left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) \to 0$$

as $k \to \infty$, but $\nu_{x_k}(E_k) \gtrsim 1$ for all $k \in \mathbb{N}$. Now

$$\begin{aligned}|\nu_{x_k}(E_k) - \nu_x(E_k)| \le \int_{E_k} \Bigg| &\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x_k}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x_k}\varphi(z)}\right)\right] \\ &- \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right]\Bigg|\left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z). \end{aligned} \tag{2.15}$$

Thus

$$\begin{aligned}\nu_{x_k}(E_k) \le \int_{E_k} \Bigg| &\frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x_k}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x_k}\varphi(z)}\right)\right] \\ &- \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right]\Bigg|\left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) + \nu_x(E_k). \end{aligned} \tag{2.16}$$

Also C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$, so we have that

$$\begin{aligned}\nu_x(E_k) &= \int_{E_k} \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right]\left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) \\ &\le M_1^2 \int_{E_k} \left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) \to 0 \text{ as } k \to \infty. \end{aligned} \tag{2.17}$$

Using (2.16) and (2.17) in (2.15), we have that $\nu_x(E_k) \to 0$, contradiction. Hence (2) $\Rightarrow$ (3) holds.
(3) $\Rightarrow$ (1). Let $\epsilon > 0$ be given. Using the identity:

$$1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2},$$

we can easily see that

$$\int_{\mathbb{D}} \left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) = 1.$$

Therefore, by the Jensen's inequality we have that

$$\begin{aligned}&\int_{\mathbb{D}} \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right]\left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) \\ &\quad\le \|\mu_k\| \int_{\partial\mathbb{D}} \int_{\mathbb{D}} \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right]\left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) d|\mu_k|(x).\end{aligned}$$

Choose a compact set $\Omega \subset \mathbb{D}$ such that

$$\int_{\mathbb{D}\setminus\Omega} \left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^2 d\lambda(z) < \delta.$$

Then

$$\begin{aligned}
&\int_{\mathbb{D}\setminus\Omega} \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right]\left(1-\left|\frac{a-z}{1-\overline{a}z}\right|^2\right)^2 d\lambda(z) \\
&\le \|\mu_k\| \int_{\partial\mathbb{D}} \int_{\mathbb{D}\setminus\Omega} \frac{\nu^2(z)|\varphi'(z)|^2}{|1-\overline{x}\varphi(z)|^4} \exp\left[\Re\left(\frac{2}{1-\overline{x}\varphi(z)}\right)\right]\left(1-\left|\frac{a-z}{1-\overline{a}z}\right|^2\right)^2 d\lambda(z) d|\mu_k|(x) \\
&\le \epsilon\|\mu_k\| \int_{\partial\mathbb{D}} d|\mu_k|(x) = \epsilon\|f_k\|^2_{\mathcal{K}_e} < \epsilon. \qquad (2.18)
\end{aligned}$$

On Ω, there is some k_0 such that $|f_k'(\varphi_i(z))|^2 < \epsilon$ for $k \ge k_0$. Thus for $k \ge k_0$, we have that

$$\begin{aligned}
&\int_K |f_k'(\varphi(z))|^2|\varphi'(z)|^2\nu^2(z)\left(1-\left|\frac{a-z}{1-\overline{a}z}\right|^2\right)^2 d\lambda(z) \\
&\quad \le \epsilon C \int_K |\varphi'(z)|^2\nu^2(z)\left(1-\left|\frac{a-z}{1-\overline{a}z}\right|^2\right)^2 d\lambda(z) < \epsilon C\|\varphi\|^2_{\mathcal{B}_\nu} \qquad (2.19)
\end{aligned}$$

Therefore, by (2.18), (2.19) and the fact that $\varphi \in \mathcal{B}_\nu$, we have that $\|C_\varphi)f_k\|_{\mathcal{B}_\nu} \to 0$ as $k \to \infty$. This completes the proof of (3) $\Rightarrow$ (1).
□

Theorem 4. *C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_{\nu,0}$ if and only if*

$$M_1 := \sup_{x\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{x}\varphi(z)|^2} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(z)}\right)\right] < \infty. \qquad (2.20)$$

and

$$\lim_{|z|\to 1} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{x}\varphi(z)|^2} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(z)}\right)\right] = 0 \qquad (2.21)$$

for every $x \in \partial\mathbb{D}$.

Proof. First suppose that (2.20) and (2.21) hold. Let $f \in \mathcal{K}_e$ be arbitrary. Then using Lemma 2 and proceeding as in Theorem 1, we have

$$\nu(z)|\varphi'(z)||f'(\varphi(z))| \le \int_{\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{x}\varphi(z)|^2}\left|\exp\left[\frac{1}{1-\overline{x}\varphi(z)}\right]\right| d|\mu|(x) \qquad (2.22)$$

By (2.21), the the left hand side in (2.22) tends to zero for every $x \in \partial\mathbb{D}$, as $|z| \to 1$, and it is dominated by M_1, where M_1 is as in Theorem 1. Thus by the Lebesgue-dominated convergence theorem, the integral in (2.22) tends to zero as $|z| \to 1$. Therefore,

$$\lim_{|z|\to 1} \nu(z)|(C_\varphi f)'(z)| = 0.$$

Thus $C_\varphi f \in \mathcal{B}_{\nu,0}$ for every $f \in \mathcal{K}_e$. Hence C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_{\nu,0}$. Conversely, suppose that C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_{\nu,0}$. Then $C_\varphi f_x \in \mathcal{B}_{\nu,0}$ for every function f_x, $x \in \partial\mathbb{D}$, defined in (2.10), that is

$$\lim_{|z|\to 1} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{x}\varphi(z)|^2} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(z)}\right)\right] = 0$$

for every $x \in \partial\mathbb{D}$. Since C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_{\nu,0}$, so C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_\nu$. Therefore, by Theorem 1, (2.20) follows, as desired. □

Theorem 5. *C_φ maps $\mathcal{K}_e$ boundedly into $\mathcal{B}_{\nu,0}$ if and only if*

$$\lim_{|z|\to 1} \sup_{x\in\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{x}\varphi(z)|^2} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(z)}\right)\right] = 0. \tag{2.23}$$

Proof. By Lemma 5, the set $\{C_\varphi f : f \in \mathcal{K}_e, \|f\|_{\mathcal{K}_e} \le 1\}$ has compact closure in $\mathcal{B}_{\nu,0}$ if and only if

$$\lim_{|z|\to 1} \sup\{\nu(z)|(C_\varphi f)'(z)| : f \in \mathcal{K}_e, \|f\|_{\mathcal{K}_e} \le 1\} = 0. \tag{2.24}$$

Let f be a function in the unit ball of $\mathcal{K}_e$. Then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{K}_e}$ and

$$f(z) = \int_{\partial\mathbb{D}} \exp\left[\frac{1}{1-\overline{x}z}\right] d\mu(x).$$

Thus proceeding as in Theorem 1, we have

$$\begin{aligned} \nu(z)|(C_\varphi f)'(z)| &\le \frac{\nu(z)|\varphi'(z)|}{|1-\overline{x}\varphi(z)|^2} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(z)}\right)\right] \\ &\le \frac{\nu(z)|\varphi'(z)|}{|1-\overline{x}\varphi(z)|^2} \exp\left[\Re\left(\frac{1}{1-\overline{x}\varphi(z)}\right)\right]. \end{aligned} \tag{2.25}$$

Using (2.24) in (2.25), we get (2.23). Hence C_φ maps $\mathcal{K}_e$ compactly into $\mathcal{B}_{\nu,0}$. Conversely, suppose that C_φ maps $\mathcal{K}_e$ compactly into $\mathcal{B}_{\nu,0}$. Using the functions in (2.10) in (2.24), we easily have the desired condition in (2.23). □

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