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Computation of Krein space numerical ranges of 2×2 matrices

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Abstract. In this paper, we study some properties of the Krein space J-numerical ranges and compute the Krein space J-numerical ranges of some 2×2 complex matrices with respect to several fundamental symmetries. We also compare the direct computation of Krein space J-numerical ranges with the computation via the joint numerical ranges. Finally, we discuss the Krein space J-numerical ranges of upper triangular matrices.

1. Introduction

A complex vector space $\mathcal{K}$ is called a *Krein space* if it has an indefinite inner product $\langle \cdot, \cdot \rangle_J$ given by $\langle x, y \rangle_J = \langle Jx, y \rangle$, where $\langle \cdot, \cdot \rangle$ is a positive definite inner product that makes $\mathcal{K}$ a Hilbert space and J is a bounded linear operator such that $J = J^{-1} = J^*$. The operator J is called a *fundamental symmetry* and $P_+ = \frac{I+J}{2}$ and $P_- = \frac{I-J}{2}$ are called *fundamental projections*. The direct sum $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ is called the *fundamental decomposition* where $\mathcal{K}_+ := P_+\mathcal{K}$ and $\mathcal{K}_- := P_-\mathcal{K}$. The subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$ are orthogonal to each other with respect to $\langle \cdot, \cdot \rangle_J$. We denote by $\mathcal{L}(\mathcal{K})$ the set of all bounded linear operators on $\mathcal{K}$ and the *J-adjoint* $^\#$ is the Krein space adjoint with respect to the indefinite inner product, whereas * denotes the Hilbert space adjoint on $\mathcal{K}$. We easily observe that the J-adjoint operator $T^\#$ of $T \in \mathcal{L}(\mathcal{K})$ is given by $T^\# = JT^*J$ for a fundamental symmetry J. An operator $T \in \mathcal{L}(\mathcal{K})$ is called *J-selfadjoint* if $T = T^\#$ and *J-unitary* if $TT^\# = T^\#T = I$. In [1, 10], it was shown that the spectrum of a J-selfadjoint operator is symmetric with respect to the real axis. The authors [1, 2] have studied the several spectra of J-selfadjoint operators and the spectra of their Hilbert space adjoint operators.

The numerical range of an operator on a Hilbert space has been studied extensively in many areas. In a finite dimensional case, numerical ranges are used as a rough estimate of eigenvalues of a matrix. The numerical ranges of some bounded linear operators on an infinite dimensional Hilbert space was introduced in order to answer some questions that the spectra fail to address. For more detailed information on the numerical range, we refer [9] to readers. Generalizations of the numerical range have been used to study quantum information theory, for example, higher-rank numerical range was introduced as a tool in quantum

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error correction [7, 8]. In this paper we consider a generalization of a numerical range in a Krein space, which naturally arises in situations where an indefinite inner product has an analytically useful property, in particular, in the theory of interpolation for operator-valued Schur functions [3].

For an operator $T \in \mathcal{L}(\mathcal{K})$, the *Krein space J-numerical range* is defined by

$$W_J(T) = \left\{ \frac{\langle Tx, x\rangle_J}{\langle x, x\rangle_J} : x \in \mathcal{K}, \langle x, x\rangle_J \neq 0 \right\}.$$

We define the positive J-numerical range and the negative J-numerical range as follows;

$$W_J^+(T) = \left\{ \frac{\langle Tx, x\rangle_J}{\langle x, x\rangle_J} : \langle x, x\rangle_J > 0 \right\} \text{ and } W_J^-(T) = \left\{ \frac{\langle Tx, x\rangle_J}{\langle x, x\rangle_J} : \langle x, x\rangle_J < 0 \right\}.$$

We can also write the positive J-numerical range and the negative J-numerical range as follows;

$$W_J^\pm(T) = \{\pm\langle Tx, x\rangle_J : x \in \mathcal{K}, \langle x, x\rangle_J = \pm 1\}.$$

Then it is obvious that

$$W_J(T) = W_J^+(T) \cup W_J^-(T) \quad \text{and} \quad W_{-J}^+(T) = W_J^-(T).$$

Whereas the numerical range in a Hilbert space is bounded, compact and convex, the Krein space J-numerical range may be neither bounded nor convex, but is known to be pseudo-convex for a special class of matrices [13]. The Krein space J-numerical range has been studied by many people [4–6, 11–14]. In particular, Li and Rodman [11, 12] determined the positive J-numerical range using the joint numerical range of three self-adjoint operators. Bebiano et al. [6] have investigated a class of tridiagonal matrices with a hyperbolical numerical range using the Matlab. In [14], Nakazato et al. have investigated the shape of the tracial numerical range $W_C^J(A) = \{\mathrm{tr}(CUAU^{-1}) : UJU^* = J\}$ where $J = \mathrm{diag}(1, -1)$, $\det(U) = 1$, and A, C are two rank-one 2×2-matrices.

In this paper, we study some properties of the Krein space J-numerical range and gives a characterization of a fundamental symmetry on $\mathbb{C}^2$. We compute the Krein space J-numerical ranges of some special 2×2 complex matrices with respect to several fundamental symmetries and compare the direct computation of Krein space J-numerical ranges with the computation via the joint numerical ranges. Finally, we compute and discuss the Krein space J-numerical ranges of upper triangular matrices.

2. Basic properties of Krein space *J*-numerical ranges

The following basic properties of Krein space J-numerical ranges are well known and immediately follow from the definition. For the reader's convenience, we give the proof.

Proposition 2.1. *Let $(\mathcal{K}, \langle\cdot,\cdot\rangle_J)$ be a Krein space and $T \in \mathcal{L}(\mathcal{K})$ where $\mathcal{L}(\mathcal{K})$ is the algebra of all bounded linear operators on $\mathcal{K}$. The following properties of $W_J(T)$ are satisfied;*

(i) $W_J(\alpha I + \beta T) = \alpha + \beta W_J(T)$ *for complex numbers α, β and the identity operator I.*

(ii) $W_J(T^*) = \overline{W_J(T)} = W_J(T^\#)$ *where $\overline{W_J(T)}$ denotes the set of complex conjugates of elements in $W_J(T)$.*

(iii) $W_J(U^\# T U) = W_J(T)$ *for any J-unitary operator U.*

(iv) *If T is self-adjoint, then $W_J(T) \subset \mathbb{R}$.*

(v) *T is J-selfadjoint if and only if $W_J(T) \subset \mathbb{R}$.*

Proof. (i) We have that for arbitrary complex numbers α and β,

$$\langle(\alpha I + \beta T)x, x\rangle_J = \langle \alpha Jx, x\rangle + \langle \beta JTx, x\rangle = \alpha\langle x, x\rangle_J + \beta\langle Tx, x\rangle_J.$$

This implies that $W_J(\alpha I + \beta T) = \alpha + \beta W_J(T)$.

(ii) The first equality follows from that

$$\langle T^* x, x\rangle_J = \langle x, TJx\rangle = \overline{\langle TJx, x\rangle} = \overline{\langle TJx, Jx\rangle_J}.$$

If $\langle x, x\rangle_J = 1$, then we see that $W_J^+(T^*) = \overline{W_J^+(T)}$. Since $W_J^-(T^*) = \overline{W_J^-(T)}$, we also have that $W_J(T^*) = \overline{W_J(T)}$. On the other hand, we obtain that

$$\langle T^\# x, x\rangle_J = \langle x, JTx\rangle = \overline{\langle Tx, x\rangle_J},$$

which implies $W_J(T^\#) = \overline{W_J(T)}$.

(iii) For a J-unitary operator U, we easily see that

$$\langle U^\# TUx, x\rangle_J = \langle U^* JTUx, x\rangle = \langle TUx, Ux\rangle_J.$$

On the other hand, since $U^* JU = J$, we have that $\langle Ux, Ux\rangle_J = \langle x, x\rangle_J = 1$. This completes the proof.

(iv) immediately follows from (ii).

(v) If T is J-selfadjoint, then it follows from (ii) that $W_J(T) \subset \mathbb{R}$. Conversely, assume that $W_J(T) \subset \mathbb{R}$. Since J is self-adjoint, $\langle x, x\rangle_J = \langle Jx, x\rangle$ is a real number. By assumption, we have that

$$\frac{\langle Tx, x\rangle_J}{\langle x, x\rangle_J} \in \mathbb{R} \quad \text{for all } x \in \mathcal{K},$$

so that $\langle Tx, x\rangle_J$ is also a real number for all $x \in \mathcal{K}$. Hence, JT is self-adjoint, which implies that $T = JT^* J = T^\#$. Therefore, T is J-selfadjoint. □

We denote by $M_2(\mathbb{C})$ the set of all 2×2 complex matrices. The following proposition gives a characterization of a fundamental symmetry on the 2-dimensional space.

Proposition 2.2. *Let* $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a fundamental symmetry on* $\mathbb{C}^2$.

(i) If $a + d \neq 0$*, then* $J = \pm I$ *where* I *is the identity matrix in* $M_2(\mathbb{C})$.

(ii) If $a + d = 0$*, then* $J = \begin{pmatrix} \pm\sqrt{1 - |b|^2} & b \\ \overline{b} & \mp\sqrt{1 - |b|^2} \end{pmatrix}$.

Proof. Since $J = J^* = J^{-1}$, we observe that a and d are real numbers, $b = \overline{c}$, and obtain from $J^2 = I$ the following equalities;

$$a^2 + |b|^2 = 1, \ \ |b|^2 + d^2 = 1 \ \text{ and } \ b(a + d) = 0.$$

(i) If $a + d \neq 0$, then b should be the zero. This implies that $a^2 = d^2 = 1$, so that $J = \pm I$.

(ii) If $a + d = 0$, then it is obvious that $J = \begin{pmatrix} \pm\sqrt{1 - |b|^2} & b \\ \overline{b} & \mp\sqrt{1 - |b|^2} \end{pmatrix}$. □

Remark 2.3. In Proposition 2.2, we see that $\det(J) = \begin{cases} 1, & \text{if } a + d \neq 0, \\ -1, & \text{if } a + d = 0. \end{cases}$ □

3. Krein space J-numerical ranges in the 2-dimensional space

Let J be a fundamental symmetry on a Krein space $\mathcal{K}$. For an operator $T \in \mathcal{L}(\mathcal{K})$, let H and G be self-adjoint operators defined by

$$H = \frac{JT + T^*J}{2} \quad \text{and} \quad G = \frac{JT - T^*J}{2i}, \tag{1}$$

so that $JT = H + iG$. In [13], Li, Tsing and Uhlig, introduced the notion of the *joint numerical range* of (H, G, J) given by

$$W(H, G, J) = \{(\langle Hx, x\rangle, \langle Gx, x\rangle, \langle Jx, x\rangle) \in \mathbb{R}^3 : x \in \mathcal{K}, \langle x, x\rangle = 1\}$$

as a generalization of a numerical range in a Hilbert space. Using the joint numerical range of three Hermitian matrices, they gave a detailed description of S-numerical ranges for a Hermitian matrix S. We denote by $K(H, G, J)$ the convex cone generated by $W(H, G, J)$ as follows;

$$\begin{aligned} K(H, G, J) &= \bigcup_{\mu \geq 0} \mu W(H, G, J) \\ &= \{(\langle Hx, x\rangle, \langle Gx, x\rangle, \langle Jx, x\rangle) \in \mathbb{R}^3 : x \in \mathcal{K}\}. \end{aligned}$$

The following lemma gives the relationship between the Krein space J-numerical ranges and the joint numerical ranges.

Lemma 3.1. *[13, Lemma 1.1] Let J be a fundamental symmetry on $\mathbb{C}^n$ and $T \in M_n(\mathbb{C})$. For Hermitian matrices H and G given by* (1)*, we have that*

$$x + iy \in W_J^+(T) \quad \Longleftrightarrow \quad (x, y, 1) \in K(H, G, J).$$

Throughout this paper, $W(T)$ denotes the Hilbert space numerical range of an operator T, unless specified otherwise. From now on, we concentrate on Krein space J-numerical ranges of operators on $\mathbb{C}^2$.

Proposition 3.2. *Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a fundamental symmetry in $M_2(\mathbb{C})$.*

(a) If $\det(J) = 1$*, then $W_J(T) = [0, 1]$.*
(b) If $\det(J) = -1$*, then the following statements hold;*
  (i) If $a = 1$ and $b = 0$, then $W_J(T) = (-\infty, 0] \cup [1, \infty)$.
  (ii) If $a = -1$ and $b = 0$, then $W_J(T) = (-\infty, 0] \cup [1, \infty)$.
  (iii) If $|b| = 1$, then $W_J(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) = \frac{1}{2}\}$.

Proof. (a) Assume that $\det(J) = 1$. Then it follows from Proposition 2.2 that $J = \pm I$. If $J = I$, then $\langle \cdot, \cdot \rangle_J$ becomes a positive-definite inner product, so that $W_J(T) = W_J^+(T) = W(T) = [0, 1]$. On the other hand, if $J = -I$, then we have that

$$\frac{\langle Tx, x\rangle_J}{\langle x, x\rangle_J} = \frac{-\langle Tx, x\rangle}{-\langle x, x\rangle} = \frac{\langle Tx, x\rangle}{\langle x, x\rangle}$$

and since $\langle x, x\rangle_J = -\langle x, x\rangle \neq 0 \Longleftrightarrow \langle x, x\rangle \neq 0$, we get $W_J(T) = [0, 1]$.

(b) If $\det(J) = -1$, then it follows from Proposition 2.2 that $J = \begin{pmatrix} \pm\sqrt{1 - |b|^2} & b \\ \overline{b} & \mp\sqrt{1 - |b|^2} \end{pmatrix}$, where b is a complex number with $|b| \leq 1$. We have that for $\mathbf{x} = (x, y) \in \mathbb{C}^2$,

$$\begin{cases} \langle \mathbf{x}, \mathbf{x}\rangle_J = 2\mathrm{Re}(b\overline{x}y) \pm \sqrt{1 - |b|^2}(|x|^2 - |y|^2) \\ \langle T\mathbf{x}, \mathbf{x}\rangle_J = \overline{b}x\overline{y} \pm \sqrt{1 - |b|^2}|x|^2. \end{cases} \tag{2}$$

(i) If $a = 1$ and $b = 0$, then we have $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\langle \mathbf{x}, \mathbf{x} \rangle_J = |x|^2 - |y|^2$. By (2), we have that

$$W_J^+(T) = \{|x|^2 : |x|^2 - |y|^2 = 1,\ (x, y) \in \mathbb{C}^2\} = [1, \infty),$$
$$W_{-J}^+(T) = \{-|x|^2 : -|x|^2 + |y|^2 = 1,\ (x, y) \in \mathbb{C}^2\} = (-\infty, 0].$$

Since $W_{-J}^+(T) = W_J^-(T)$, we obtain that $W_J(T) = W_J^+(T) \cup W_J^-(T) = (-\infty, 0] \cup [1, \infty)$.

(ii) If $a = -1$ and $b = 0$, then we have $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $\langle \mathbf{x}, \mathbf{x} \rangle_J = -|x|^2 + |y|^2$ and $\langle T\mathbf{x}, \mathbf{x} \rangle_J = -|x|^2$, we have that

$$W_J^+(T) = (-\infty, 0] \quad \text{and} \quad W_J^-(T) = [1, \infty).$$

Thus, we have that $W_J(T) = W_J^+(T) \cup W_J^-(T) = (-\infty, 0] \cup [1, \infty)$.

(iii) If $|b| = 1$, then we have $J = \begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix}$. Since $\langle \mathbf{x}, \mathbf{x} \rangle_J = 2\mathrm{Re}(b\overline{x}y)$ for any $\mathbf{x} = (x, y) \in \mathbb{C}^2$, we obtain that

$$\langle H\mathbf{x}, \mathbf{x} \rangle = \mathrm{Re}(b\overline{x}y) = \frac{1}{2} \quad \text{and} \quad \langle G\mathbf{x}, \mathbf{x} \rangle = -\mathrm{Im}(b\overline{x}y)$$

for $\mathbf{x} \in \mathbb{C}^2$ with $\langle \mathbf{x}, \mathbf{x} \rangle_J = 1$. By Lemma 3.1, this implies that $W_J^+(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) = \frac{1}{2}\}$. Moreover, since $W_J^-(T) = W_{-J}^+(T)$ and

$$z_1 + iz_2 \in W_{-J}^+(T) \iff (z_1, z_2, 1) \in K(-H, -G, -J),$$

we have $W_J^-(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) = \frac{1}{2}\}$, so that

$$W_J(T) = W_J^+(T) \cup W_J^-(T) = \left\{z \in \mathbb{C} : \mathrm{Re}(z) = \frac{1}{2}\right\}.$$

□

In the following example, we consider the case where $b = \frac{1}{\sqrt{2}}$ in (ii) of Proposition 2.2 and compute the Krein space J-numerical range.

Example 3.3. *Let* $J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *be a fundamental symmetry on* $\mathbb{C}^2$ *and* $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

From (2), *we obtain that for* $\mathbf{x} = (x, y) \in \mathbb{C}^2$,

$$\begin{cases} \langle \mathbf{x}, \mathbf{x} \rangle_J = \sqrt{2}Re(\overline{x}y) + \frac{1}{\sqrt{2}}(|x|^2 - |y|^2), \\ \langle T\mathbf{x}, \mathbf{x} \rangle_J = \frac{1}{\sqrt{2}}(|x|^2 + x\overline{y}). \end{cases} \tag{3}$$

If $\langle \mathbf{x}, \mathbf{x} \rangle_J = 1$, *then it follows from equations* (3) *that*

$$\sqrt{2}Re(\overline{x}y) = 1 - \frac{1}{\sqrt{2}}(|x|^2 - |y|^2) \quad \text{and} \quad Re[\langle T\mathbf{x}, \mathbf{x} \rangle_J] = \frac{1}{\sqrt{2}}|x|^2 + \frac{1}{\sqrt{2}}Re(x\overline{y}).$$

Put $x = x_1 + ix_2$ *and* $y = y_1 + iy_2$ *where* x_j *and* y_j *are real numbers for* $j = 1, 2$. *Using the Lagrange multiplier, we will optimize the function* f

$$f(x_1, x_2, y_1, y_2) := Re[\langle T\mathbf{x}, \mathbf{x} \rangle_J] = \frac{1}{\sqrt{2}}(x_1^2 + x_2^2 + x_1y_1 + x_2y_2)$$

subject to the constraint

$$g(x_1, x_2, y_1, y_2) = \frac{1}{\sqrt{2}}(x_1^2 + x_2^2 - y_1^2 - y_2^2) + \sqrt{2}(x_1y_1 + x_2y_2) - 1 = 0.$$

If $\nabla f = \lambda \nabla g$, then we obtain that

$$\sqrt{2}x_j + \frac{1}{\sqrt{2}}y_j = \lambda(\sqrt{2}x_j + \sqrt{2}y_j) \quad \textit{and} \quad \frac{1}{\sqrt{2}}x_j = \lambda(-\sqrt{2}y_j + \sqrt{2}x_j),$$

for each $j = 1, 2$. We have that

$$\begin{cases} (2-2\lambda)x_j + (1-2\lambda)y_j = 0, \\ (1-2\lambda)x_j + 2\lambda y_j = 0. \end{cases} \tag{4}$$

Then we obtain the equation $(8\lambda^2 - 8\lambda + 1)x_j = 0$, so that $\lambda = (2 \pm \sqrt{2})/4$ or $x_j = 0$ $(j = 1, 2)$. If $x_j = 0$, then it follows from (4) that $y_j = 0$ for each $j = 1, 2$, which contradicts to the constraint $g(x_1, x_2, y_1, y_2) = 0$.

If $\lambda = (2 + \sqrt{2})/4$, we obtain from (4) that

$$y_j = (\sqrt{2} - 1)x_j \qquad (j = 1, 2).$$

Since $g(x_1, x_2, y_1, y_2) = 0$, we have that $(4\sqrt{2} - 4)x_1^2 + (4\sqrt{2} - 4)x_2^2 - \sqrt{2} = 0$, so that

$$x_1^2 + x_2^2 = \frac{2 + \sqrt{2}}{4} \quad \textit{and} \quad y_1^2 + y_2^2 = \frac{2 - \sqrt{2}}{4}.$$

Thus, we have that

$$f(x_1, x_2, y_1, y_2) = \frac{1}{\sqrt{2}}\left\{x_1^2 + x_2^2 + (\sqrt{2} - 1)x_1^2 + (\sqrt{2} - 1)x_2^2\right\} = \frac{2 + \sqrt{2}}{4}.$$

If $\lambda = (2 - \sqrt{2})/4$, then we have that

$$y_j = (-\sqrt{2} - 1)x_j \qquad (j = 1, 2).$$

Due to the constraint $g(x_1, x_2, y_1, y_2) = 0$, we have that $(4\sqrt{2} + 4)x_1^2 + (4\sqrt{2} + 4)x_2^2 + \sqrt{2} = 0$, which implies a contradiction. Therefore, we get

$$Re[\langle T\mathbf{x}, \mathbf{x}\rangle_J] \geq \frac{2 + \sqrt{2}}{4} \quad \textit{for } \langle \mathbf{x}, \mathbf{x}\rangle_J = 1,$$

and the equality holds for $|x|^2 = (2 + \sqrt{2})/4$ and $|y|^2 = (2 - \sqrt{2})/4$. In this case, we see that the imaginary part of $\langle T\mathbf{x}, \mathbf{x}\rangle_J$ is zero, which means that

$$W_J^+(T) = \left\{z \in \mathbb{C} : Re(z) \geq \frac{2 + \sqrt{2}}{4}\right\}.$$

Similarly, we have that for $\mathbf{x}$ with $\langle \mathbf{x}, \mathbf{x}\rangle_{-J} = 1$,

$$Re[\langle T\mathbf{x}, \mathbf{x}\rangle_{-J}] \leq \frac{2 - \sqrt{2}}{4}.$$

The equality holds for $|x|^2 = (2 - \sqrt{2})/4$ and $|y|^2 = (2 + \sqrt{2})/4$. In this case, the imaginary part of $\langle T\mathbf{x}, \mathbf{x}\rangle_J$ is also zero. So, we have that

$$W_J^-(T) = W_{-J}^+(T) = \left\{z \in \mathbb{C} : Re(z) \leq \frac{2 - \sqrt{2}}{4}\right\}.$$

Consequently, we get

$$W_J(T) = W_J^+(T) \cup W_J^-(T) = \left\{z \in \mathbb{C} : Re(z) \leq \frac{2 - \sqrt{2}}{4} \quad \textit{or} \quad Re(z) \geq \frac{2 + \sqrt{2}}{4}\right\}.$$

□

We can also compute the Krein space J-numerical range in Example 3.3 using Lemma 3.1.

Remark 3.4. *Let J and T be as in Example 3.3. For any $\mathbf{x} = (x, y) \in \mathbb{C}^2$. We have that*

$$\langle J\mathbf{x}, \mathbf{x}\rangle = \sqrt{2}Re(\bar{x}y) + \frac{1}{\sqrt{2}}(|x|^2 - |y|^2).$$

If $\langle J\mathbf{x}, \mathbf{x}\rangle = 1$, we see that $Re(\bar{x}y) = \frac{1}{\sqrt{2}} - \frac{1}{2}(|x|^2 - |y|^2)$. Then we have that

$$\langle H\mathbf{x}, \mathbf{x}\rangle = \frac{1}{\sqrt{2}}|x|^2 + \frac{1}{\sqrt{2}}Re(\bar{x}y) = \frac{\sqrt{2}}{4}(|x|^2 + |y|^2) + \frac{1}{2}$$

and that

$$\langle G\mathbf{x}, \mathbf{x}\rangle = -\frac{1}{2\sqrt{2}i}\bar{x}y + \frac{1}{2\sqrt{2}i}x\bar{y} = \frac{1}{\sqrt{2}}Im(\bar{x}y).$$

Hence we have

$$W_J^+(T) = \left\{z \in \mathbb{C} : Re(z) \geq \frac{2+\sqrt{2}}{4}\right\}.$$

If $\langle -J\mathbf{x}, \mathbf{x}\rangle = 1$, we also have that $Re(\bar{x}y) = -\frac{1}{\sqrt{2}} - \frac{1}{2}(|x|^2 - |y|^2)$. Thus, we obtain that

$$\langle H\mathbf{x}, \mathbf{x}\rangle = -\frac{\sqrt{2}}{4}(|x|^2 + |y|^2) + \frac{1}{2} \ \ and \ \ \langle G\mathbf{x}, \mathbf{x}\rangle = -\frac{1}{\sqrt{2}}Im(\bar{x}y),$$

so that $W_{-J}^+(T) = \{z \in \mathbb{C} : Re(z) \leq (2-\sqrt{2})/4\}$. Therefore, we have that

$$W_J(T) = \left\{z \in \mathbb{C} : Re(z) \leq \frac{2-\sqrt{2}}{4} \ \ or \ \ Re(z) \geq \frac{2+\sqrt{2}}{4}\right\}.$$

□

Proposition 3.5. *Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a fundamental symmetry in $M_2(\mathbb{C})$.*

- *(a) If $\det(J) = 1$, then $W_J(T) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$.*
- *(b) If $\det(J) = -1$, then the following statements hold;*
  - *(i) If $b = 0$, then $W_J(T) = \mathbb{C}$.*
  - *(ii) If $|b| = 1$, then $W_J(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) \neq 0$ and $\mathrm{Im}(z) \neq 0\}$.*

Proof. (a) If $\det(J) = 1$, then $J = \pm I$ by Proposition 2.2 . For $J = \pm I$, we have that

$$W_J(T) = W_J^{\pm}(T) = \pm W(T).$$

Since $W(T) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$, we see that $W_J(T) = W(T)$ for $J = \pm I$.

(b) If $\det(J) = -1$, then we have $J = \begin{pmatrix} \pm\sqrt{1-|b|^2} & b \\ \bar{b} & \mp\sqrt{1-|b|^2} \end{pmatrix}$ where b is a complex number with $|b| \leq 1$. For $\mathbf{x} = (x, y) \in \mathbb{C}^2$, we have that

$$\begin{cases} \langle \mathbf{x}, \mathbf{x}\rangle_J = 2\mathrm{Re}(b\bar{x}y) \pm \sqrt{1-|b|^2}(|x|^2 - |y|^2), \\ \langle T\mathbf{x}, \mathbf{x}\rangle_J = \bar{b}|y|^2 \pm \sqrt{1-|b|^2}y\bar{x}. \end{cases}$$

Let H and G be Hermitian matrices given by (2).

(i) If $b = 0$, then $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\langle \mathbf{x}, \mathbf{x} \rangle_J = \pm(|x|^2 - |y|^2)$. We have that

$$\langle H\mathbf{x}, \mathbf{x} \rangle = \mathrm{Re}(\overline{x}y) \text{ and } \langle G\mathbf{x}, \mathbf{x} \rangle = \mathrm{Im}(\overline{x}y).$$

It follows from Lemma 3.1 that $W_J^+(T) = \mathbb{C}$, which implies that $W_J(T) = \mathbb{C}$.

(ii) Let $b = b_1 + ib_2$ with $|b| = 1$, so that $J = \begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix}$. Since $\langle \mathbf{x}, \mathbf{x} \rangle_J = 2\mathrm{Re}(b\overline{x}y)$, we have that $\mathrm{Re}(b\overline{x}y) = \frac{1}{2}$ for $\mathbf{x}$ with $\langle \mathbf{x}, \mathbf{x} \rangle_J = 1$. Since

$$\langle H\mathbf{x}, \mathbf{x} \rangle = b_1|y|^2 \text{ and } \langle G\mathbf{x}, \mathbf{x} \rangle = -b_2|y|^2 = -\sqrt{1 - b_1^2}\,|y|^2, \tag{5}$$

we obtain from Lemma 3.1 that $W_J^+(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) \neq 0 \text{ and } \mathrm{Im}(z) < 0\}$. If $\langle \mathbf{x}, \mathbf{x} \rangle_J = -1$, then we have $\mathrm{Re}(b\overline{x}y) = -\frac{1}{2}$. Since

$$\begin{aligned} \alpha + i\beta \in W_{-J}^+(T) &\iff (\alpha, \beta, 1) \in K(-H, -G, -J) \\ &\iff (-\alpha, -\beta, 1) \in K(H, G, -J), \end{aligned}$$

it follows from equations (5) that $W_{-\mathcal{J}}^+(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) \neq 0 \text{ and } \mathrm{Im}(z) > 0\}$. Consequently, we obtain from $W_J^-(T) = W_{-J}^+(T)$ that

$$W_J(T) = W_J^+(T) \cup W_J^-(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) \neq 0 \text{ and } \mathrm{Im}(z) \neq 0\}.$$

□

Like as Example 3.3, we consider the case where $\det(J) = -1$ and $b = \frac{1}{\sqrt{2}}$ in Proposition 3.5 and compute the Krein space J-numerical range.

Example 3.6. *Let* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $J = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *be a fundamental symmetry on* $\mathbb{C}^2$. *We have that for any* $\mathbf{x} = (x, y) \in \mathbb{C}^2$,

$$\langle \mathbf{x}, \mathbf{x} \rangle_J = \sqrt{2}Re(\overline{x}y) + \frac{1}{\sqrt{2}}(|x|^2 - |y|^2), \quad \langle T\mathbf{x}, \mathbf{x} \rangle_J = \frac{1}{\sqrt{2}}(|y|^2 + \overline{x}y).$$

If $\langle \mathbf{x}, \mathbf{x} \rangle_J = 1$, *then we obtain that*

$$\sqrt{2}Re(\overline{x}y) = 1 - \frac{1}{\sqrt{2}}(|x|^2 - |y|^2), \quad Re[\langle T\mathbf{x}, \mathbf{x} \rangle_J] = \frac{1}{\sqrt{2}}|y|^2 + \frac{1}{\sqrt{2}}Re(\overline{x}y).$$

As in Example 3.3, we use the Lagrange multiplier to compute $Re[\langle T\mathbf{x}, \mathbf{x} \rangle_J]$. *First, let* $x = x_1 + ix_2$ *and* $y = y_1 + iy_2$ *where* x_j *and* y_j *are real numbers for* $j = 1, 2$. *We define the functions* f *and* g *by*

$$\begin{aligned} f(x_1, x_2, y_1, y_2) &:= Re[\langle T\mathbf{x}, \mathbf{x} \rangle_J] = \frac{1}{\sqrt{2}}(y_1^2 + y_2^2 + x_1y_1 + x_2y_2), \\ g(x_1, x_2, y_1, y_2) &:= \frac{1}{\sqrt{2}}(x_1^2 + x_2^2 - y_1^2 - y_2^2) + \sqrt{2}(x_1y_1 + x_2y_2) - 1. \end{aligned}$$

We will compute the range of the function f *subject to the constraint* $g(x_1, x_2, y_1, y_2) = 0$. *If* $\nabla f = \lambda \nabla g$, *then we have that for each* $j = 1, 2$.

$$\frac{1}{\sqrt{2}}y_j = \lambda(\sqrt{2}x_j + \sqrt{2}y_j) \text{ and } \frac{1}{\sqrt{2}}(x_j + 2y_j) = \lambda(-\sqrt{2}y_j + \sqrt{2}x_j).$$

Thus, we get the equations

$$2\lambda x_j + (2\lambda - 1)y_j = 0 \;\; and \;\; (2\lambda - 1)x_j - (2\lambda - 2)y_j = 0, \tag{6}$$

for each $j = 1, 2$. *From these equations, we get that* $(8\lambda^2 - 8\lambda + 1)y_j = 0$, *so that either*

$$y_j = 0 \;\; or \;\; \lambda = (2 \pm \sqrt{2})/4 \;\; for\ each\ j = 1, 2.$$

(i) If $y_j = 0$, *then it follows from the equations (6) that* $x_j = 0$ $(j = 1, 2)$, *which contradicts to the constraint* $g(x_1, x_2, y_1, y_2) = 0$.

(ii) If $\lambda = (2 - \sqrt{2})/4$, *then we have that* $y_j = (\sqrt{2} - 1)x_j$ *for* $j = 1, 2$. *From* $g(x_1, x_2, y_1, y_2) = 0$, *we obtain that*

$$(4\sqrt{2} - 4)x_1^2 + (4\sqrt{2} - 4)x_2^2 - \sqrt{2} = 0,$$

so that $x_1^2 + x_2^2 = (2 + \sqrt{2})/4$ *and* $y_1^2 + y_2^2 = (2 - \sqrt{2})/4$. *Therefore, we have that*

$$f(x_1, x_2, y_1, y_2) = \frac{1}{\sqrt{2}}\Big\{y_1^2 + y_2^2 + (\sqrt{2} + 1)y_1^2 + (\sqrt{2} + 1)y_2^2\Big\} = \frac{\sqrt{2}}{4}.$$

(iii) On the other hand, if $\lambda = (2 + \sqrt{2})/4$, *then we see that* $y_j = (-\sqrt{2} - 1)x_j$ *for each* $j = 1, 2$. *By the equation* $g(x_1, x_2, y_1, y_2) = 0$, *we have that* $(4\sqrt{2} + 4)x_1^2 + (4\sqrt{2} + 4)x_2^2 + \sqrt{2} = 0$, *which is a contradiction.*

Therefore, we obtain that

$$Re[\langle T\mathbf{x}, \mathbf{x}\rangle_J] \geq \frac{\sqrt{2}}{4} \quad for\ \langle \mathbf{x}, \mathbf{x}\rangle_J = 1,$$

and that the equality holds for $|x|^2 = (2 + \sqrt{2})/4$ *and* $|y|^2 = (2 - \sqrt{2})/4$. *In this case, the imaginary part of* $\langle T\mathbf{x}, \mathbf{x}\rangle_J$ *is zero, so that*

$$W_J^+(T) = \Big\{z \in \mathbb{C} : Re(z) \geq \frac{\sqrt{2}}{4}\Big\}.$$

If $\langle \mathbf{x}, \mathbf{x}\rangle_{-J} = 1$, *then we also have that* $Re[\langle T\mathbf{x}, \mathbf{x}\rangle_{-J}] \geq -\sqrt{2}/4$, *and the equality holds for* $|x|^2 = (2 - \sqrt{2})/4$ *and* $|y|^2 = (2 + \sqrt{2})/4$. *Moreover,* $Im[\langle T\mathbf{x}, \mathbf{x}\rangle_J]$ *is also zero, so that* $W_{-J}^+(T) = \{z \in \mathbb{C} : Re(z) \geq -\frac{\sqrt{2}}{4}\}$. *Therefore, we have that*

$$W_J(T) = \{z \in \mathbb{C} : Re(z) \geq -\frac{\sqrt{2}}{4}\}.$$

□

Remark 3.7. *Sometimes, it is better to compute the Krein space J-numerical ranges directly than to compute the joint numerical range. Now, we try to compute the joint numerical ranges of Example 3.6. For* $\mathbf{x} = (x, y) \in \mathbb{C}^2$ *with* $\langle J\mathbf{x}, \mathbf{x}\rangle = 1$, *we have that*

$$Re(\overline{x}y) = \frac{1}{\sqrt{2}} - \frac{1}{2}(|x|^2 - |y|^2), \tag{7}$$

so that we obtain that

$$\langle H\mathbf{x}, \mathbf{x}\rangle = \frac{1}{2} - \frac{1}{2\sqrt{2}}|x|^2 + \frac{3}{2\sqrt{2}}|y|^2,$$
$$\langle G\mathbf{x}, \mathbf{x}\rangle = -\frac{1}{2\sqrt{2}i}\overline{x}y + \frac{1}{2\sqrt{2}i}x\overline{y} = \frac{1}{\sqrt{2}}Im(\overline{x}y).$$

However, we cannot compute $\langle H\mathbf{x}, \mathbf{x}\rangle$ *and* $\langle G\mathbf{x}, \mathbf{x}\rangle$ *under the condition* $\langle J\mathbf{x}, \mathbf{x}\rangle = 1$. □

4. Krein space J-numerical ranges of upper triangular 2×2 matrices

Proposition 4.1. *Let* $T = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ *and* $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *be a fundamental symmetry in* $M_2(\mathbb{C})$ *where* α, β *and* γ *are complex numbers.*

(1) If $\gamma \neq 0$ *and* $\alpha = \beta \neq 0$, *then* $W_J(T) = \{\pm\alpha + t\gamma : t > 0\}$.

(2) If $\gamma = 0$ *and* $\alpha \neq \beta$, *then* $W_J(T)$ *is the straight line obtained by translating by* $\frac{\alpha+\beta}{2}$ *after rotating by* $\pi/2$ *in the counterclockwise direction the straight line passing through the origin and* $\frac{\alpha-\beta}{2}$.

(3) If $\gamma \neq 0$ *and* $\alpha \neq \beta$, *then* $W_J(T)$ *is the half plane containing* γ, *which is determined by the straight line obtained by rotating by* $\pi/2$ *in the counterclockwise direction the straight line passing through the origin and* α.

Proof. For any vector $\mathbf{x} = (x, y)$ in $\mathbb{C}^2$, we have that

$$\langle \mathbf{x}, \mathbf{x}\rangle_J = 2\mathrm{Re}(\overline{x}y), \quad \langle T\mathbf{x}, \mathbf{x}\rangle_J = \beta\overline{x}y + \alpha x\overline{y} + \gamma|y|^2. \tag{8}$$

(1) Suppose that $\gamma \neq 0$ and $\alpha = \beta \neq 0$. If $\langle \mathbf{x}, \mathbf{x}\rangle_J = 1$, then we have $\mathrm{Re}(\overline{x}y) = 1/2$ and we obtain from (8) that

$$\langle T\mathbf{x}, \mathbf{x}\rangle_J = 2\alpha\mathrm{Re}(\overline{x}y) + \gamma|y|^2 = \alpha + \gamma|y|^2.$$

If $\langle T\mathbf{x}, \mathbf{x}\rangle_J = \alpha$, then we see that $\gamma|y|^2 = 0$, so that $y = 0$, which contradicts to $\mathrm{Re}(\overline{x}y) = 1/2$. Hence, we have that

$$W_J^+(T) = \{\alpha + t\gamma : t > 0\}.$$

Similarly, if $\langle \mathbf{x}, \mathbf{x}\rangle_J = -1$, then we also have $\mathrm{Re}(\overline{x}y) = -1/2$, so that $\langle T\mathbf{x}, \mathbf{x}\rangle_J = -\alpha + \gamma|y|^2$. If $\langle T\mathbf{x}, \mathbf{x}\rangle_J = -\alpha$, then this contradicts to $\mathrm{Re}(\overline{x}y) = -1/2$. Thus, we see that

$$W_J^-(T) = \{-\alpha + t\gamma : t > 0\},$$

which implies that

$$W_J(T) = \{\pm\alpha + t\gamma : t > 0\}.$$

(2) We first assume that $\gamma = 0$ and $\beta = -\alpha \neq 0$. For $\mathbf{x} \in \mathbb{C}^2$ with $\langle \mathbf{x}, \mathbf{x}\rangle_J = 1$, we have that

$$\langle T\mathbf{x}, \mathbf{x}\rangle_J = -\alpha\overline{x}y + \alpha x\overline{y} = i[2\alpha\mathrm{Im}(x\overline{y})].$$

This means that $W_J^+(T)$ is the straight line obtained by rotating by $\pi/2$ in the counterclockwise direction the straight line passing through the origin and α in the counterclockwise direction. Similarly, we see that $W_J^-(T)$ is also the same as $W_J^+(T)$. Therefore, $W_J(T)$ is the straight line obtained by rotating by $\pi/2$ in the counterclockwise direction the straight line passing through the origin and α.

Now, we consider the matrix

$$T - \frac{\alpha+\beta}{2}I = \begin{pmatrix} \frac{\alpha-\beta}{2} & \gamma \\ 0 & -\frac{\alpha-\beta}{2} \end{pmatrix}.$$

By Proposition 2.1, we have that

$$W_J(T) = W_J(T - \frac{\alpha+\beta}{2}I) + \frac{\alpha+\beta}{2}.$$

The above argument says that $W_J(T)$ is the straight line obtained by translating by $(\alpha+\beta)/2$ after rotating by $\pi/2$ in the counterclockwise direction the straight line passing through the origin and $(\alpha-\beta)/2$.

(3) We first assume that $\gamma \neq 0$ and $\beta = -\alpha \neq 0$. For $\mathbf{x} \in \mathbb{C}^2$ with $\langle \mathbf{x}, \mathbf{x}\rangle_J = 1$, we have that

$$\langle T\mathbf{x}, \mathbf{x}\rangle_J = -2\alpha\mathrm{Im}(\overline{x}y)i + \gamma|y|^2.$$

It follows from (ii) that $W_J^+(T)$ is the half plane containing γ, which is determined by the straight line obtained by rotating by $\pi/2$ in the counterclockwise direction the straight line passing through the origin and α.

For $\mathbf{x} \in \mathbb{C}^2$ with $\langle \mathbf{x}, \mathbf{x} \rangle_J = -1$, we also have that $\langle T\mathbf{x}, \mathbf{x} \rangle_J = -2\alpha \mathrm{Im}(\overline{x}y)i + \gamma|y|^2$, so that $W_J^-(T)$ is also the same as $W_J^+(T)$. Therefore, $W_J(T)$ is the half plane containing γ, which is determined by the straight line obtained by rotating by $\pi/2$ in the counterclockwise direction the straight line passing through the origin and α. □

Now, we compare the computation of the joint numerical ranges with the computation of the Krein space J-numerical ranges in Proposition 4.1.

Remark 4.2. *Let T and J be same as in Proposition 4.1. Since $\langle \mathbf{x}, \mathbf{x} \rangle_J = 1$ gives the equation $Re(\overline{x}y) = 1/2$, we have that for any vector $\mathbf{x} = (x, y)$ in $\mathbb{C}^2$*

$$\langle H\mathbf{x}, \mathbf{x} \rangle = Re((\overline{\alpha} + \beta)\overline{x}y) + Re(\gamma)|y|^2,$$
$$\langle G\mathbf{x}, \mathbf{x} \rangle = \frac{1}{2i}\{(-\overline{\alpha} + \beta)\overline{x}y + (\alpha - \overline{\beta})x\overline{y} + 2iIm(\gamma)|y|^2\}.$$

If $\alpha = \beta$, then we obtain that $\langle H\mathbf{x}, \mathbf{x} \rangle = Re(\alpha) + Re(\gamma)|y|^2$ and $\langle G\mathbf{x}, \mathbf{x} \rangle = Im(\alpha) + Im(\gamma)|y|^2$. As in the proof of Proposition 4.1, we observe that $\alpha \notin W_J^+(T)$ and it follows from Lemma 3.1 that

$$W_J^+(T) = \{\alpha + t\gamma : t > 0\}.$$

Since the relation $W_J^-(T) = W_{-J}^+(T)$ holds, we obtain from the computation of $-\langle Hx, x \rangle$ and $-\langle Gx, x \rangle$ that

$$W_J^-(T) = \{-\alpha + t\gamma : t > 0\}.$$

Thus, we have $W_J(T) = \{\pm\alpha + t\gamma : t > 0\}$ as in Proposition 4.1.

On the other hand, in the case when $\gamma = 0$ and $\alpha \neq \beta$, we have

$$\langle Hx, x \rangle = Re[(\overline{\alpha} + \beta)\overline{x}y] \text{ and } \langle Gx, x \rangle = -Im[(\overline{\alpha} - \beta)\overline{x}y].$$

However, these don't give any information of the joint numerical ranges of H and G. □

Recall that if $T \in M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices, and $S \in M_n(\mathbb{C})$ is a self-adjoint matrix, then the *S-numerical range* $V_S(T)$ of T is defined by

$$V_S(T) = \left\{\frac{\langle Tx, x \rangle}{\langle Sx, x \rangle} : x \in \mathbb{C}^n, \langle Sx, x \rangle \neq 0\right\}.$$

The following theorem is known as the hyperbolical range theorem for S-numerical ranges of 2×2 complex matrices for $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In the following theorem, we will replace S by J.

Theorem 4.3. *[5, Theorem 3.2] Let T be a 2×2 complex matrix and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be a fundamental symmetry in $M_2(\mathbb{C})$. Then the J-numerical range $V_J(T)$ is bounded by a degenerate hyperbola, with the eigenvalues λ_1 and λ_2 of JT as foci and with nontransverse axis of length $\sqrt{|\lambda_1|^2 + |\lambda_2|^2 - tr(T^*JTJ)}$. For the degenerate cases, $V_J(T)$ is a singleton, a line, a subset of a line, the whole complex plane, or the complex plane except a line.*

It is obvious that $W_J(T) = V_J(JT)$ for any $T \in M_n(\mathbb{C})$. So, by observing the proof of Theorem 4.3 it is not difficult to compute the Krein space J-numerical ranges of 2×2 upper triangular complex matrices for $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Recall that $U \in \mathcal{L}(\mathcal{K})$ is called a pseudo-unitary operator if there exists a linear, invertible, self-adjoint operator $S \in \mathcal{L}(\mathcal{K})$ such that $U^*S = SU^{-1}$.

Remark 4.4. *Let* $T = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ *and* $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *be a fundamental symmetry in* $M_2(\mathbb{C})$*. Then it follows from [5, Lemma 3.2] that we can write* $U^*(JT)U = \begin{pmatrix} \alpha & \gamma e^{i\theta} \\ 0 & -\beta \end{pmatrix}$ *for some pseudo-unitary matrix* U *where* $2\theta = \arg(\gamma)$*.*

If $\alpha = \beta$*, then we have that*

$$W_J(T) = V_J(JT) = V_J(U^*(JT)U) = \alpha + V_J(M_0) \quad \text{where } M_0 = \begin{pmatrix} 0 & \gamma e^{i\theta} \\ 0 & 0 \end{pmatrix}.$$

So we observe that $W_J(T)$ *is the singleton* $\{\alpha\}$ *if* $\gamma = 0$*, and the whole complex plane* $\mathbb{C}$ *if* $\gamma \neq 0$*.*

If $\alpha \neq \beta$*, then we have that*

$$W_J(T) = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} V_J(M_1) \quad \text{where } M_1 = \begin{pmatrix} 1 & \dfrac{2\gamma e^{i\theta}}{\alpha-\beta} \\ 0 & 1 \end{pmatrix}.$$

(1) If $\gamma \neq 0$*, then the following statements hold.*

  (i) If $|\alpha-\beta|^2 > |\gamma|^2$*, then* $W_J(T)$ *is bounded by a hyperbola centered at* $(\alpha+\beta)/2$ *with the eigenvalues* α *and* β *of* T *as foci and with a transverse and non-transverse axis of lengths* $\sqrt{|\alpha-\beta|^2 - |\gamma|^2}$ *and* $\sqrt{|\gamma|^2}$*, respectively.*

  (ii) If $|\alpha-\beta|^2 = |\gamma|^2$*, then* $W_J(T) = \mathbb{C}$*, except the line passing through the point* $(\alpha+\beta)/2$ *and perpendicular to the line segment joining* α *and* β*. In particular,* $W_J^+(T)$ *is one of the open half planes determined by that line.*

  (iii) If $|\alpha-\beta|^2 < |\gamma|^2$*, then* $W_J(T)$ *is the whole complex plane.*

(2) If $\gamma = 0$*, then* T *is a diagonal matrix with* $|\alpha-\beta| > 0$*, and hence* $W_J(T)$ *is the straight line determined by* α *and* β*, except the open line segment with* α *and* β *as endpoints.* □

Remark 4.5. *(1) If* $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$*, then* $\alpha = 1$ *and* $\beta = 0$*, hence* $|\alpha-\beta| > 0$*. From the last statement of Remark 4.4,* $W_J(T)$ *is the straight line through* 1 *and* 0*, except the open line segment whose endpoints are* 1 *and* 0*. In fact, we can obtain from (i) of Proposition 3.2 that*

$$W_J(T) = (-\infty, 0] \cup [1, \infty).$$

(2) If $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$*, then* $\alpha = \beta$ *and* $\gamma \neq 0$*. Thus it follows from Remark 4.4 that* $W_J(T)$ *is the whole complex plane* $\mathbb{C}$*. Indeed, we can see that this is the same as (i) of Proposition 3.5.* □

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