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General (k, p)-Riemann-Liouville fractional integrals

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Abstract. The main motivation of this study is to establish a general version of the Riemann-Liouville fractional integrals with two exponential parameters k and p which is determined over the (k, p)-gamma function. In particular, we present the harmonic, geometric and arithmetic (k, p)- Riemann-Liouville fractional integrals. When $p = k$, these integrals reduce to k-Riemann-Liouville fractional integrals. Some formulas relating to general (k, p)-Riemann-Liouville fraction integrals are also given.

1. Introduction

In 2007, Diaz and Pariguan [2] have defined new functions called k -gamma and k -beta functions.

Definition 1.1. *Given $\alpha \in \mathbb{C}/k\mathbb{Z}^-; k \in \mathbb{R}^+ - \{0\}$ and $Re(\alpha) > 0$, then the integral representation of k-Gamma Function is given by*

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1}$$

Some basic equalities satisfied by the k-gamma function are given bellow [2].

Properties 1.2. *for all α, $k > 0$ and $n \in \mathbb{N}$, the fundamental formulae satisfied by k-gamma function are ,*

$$\Gamma_k(\alpha + nk) = k^n \left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k} + 1\right)\ldots\left(\frac{\alpha}{k} + (n-1)\right) \Gamma_k(\alpha), \tag{2}$$

$$\Gamma_k(\alpha + k) = \alpha\, \Gamma_k(\alpha), \tag{3}$$

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k}). \tag{4}$$

Remark 1.3. *The above definition and properties reduce to the gamma function and its properties when $k \longrightarrow 1$.*

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The k-beta function satisfies the following identities.

$$\beta_k(x;y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\,dt. \tag{5}$$

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x>0, \quad y>0. \tag{6}$$

In 2017, Kuldeep [4] defined the two-parameter gamma function called (k,p) gamma function which is a generalization of k-gamma.

Definition 1.4. *Given $\alpha \in \mathbb{C}/k\mathbb{Z}^-; k,p \in \mathbb{R}^+ - \{0\}$ and $Re(\alpha) > 0$, then the integral representation of (k,p)-Gamma Function is given by*

$$\Gamma_{(k,p)}(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{p}}\,dt. \tag{7}$$

We present certain base formulas related to the (k,p)-gamma function are mentioned in [4], [3].

Properties 1.5. *For all α, $k,p>0$ and $n\in\mathbb{N}$, the fundamental formulae satisfied by (k,p)-gamma function are ,*

$$\Gamma_{(k,p)}(\alpha+nk) = p^n\left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k}+1\right)...\left(\frac{\alpha}{k}+(n-1)\right)\Gamma_{(k,p)}(\alpha), \tag{8}$$

$$\Gamma_{(k,p)}(\alpha+k) = \frac{p\alpha}{k}\,\Gamma_{(k,p)}(\alpha), \tag{9}$$

$$\Gamma_{(k,p)}(\alpha) = \left(\frac{p}{k}\right)^{\frac{\alpha}{k}}\Gamma_k(\alpha) = \left(\frac{p^{\frac{\alpha}{k}}}{k}\right)\Gamma\left(\frac{\alpha}{k}\right). \tag{10}$$

We deduce that

$$\Gamma_{(k,p)}(1) = \left(\frac{p^{\frac{1}{k}}}{k}\right)\Gamma\left(\frac{1}{k}\right), \quad \Gamma_{(k,p)}(k) = \left(\frac{p}{k}\right), \quad \Gamma_{(k,p)}(p) = \left(\frac{p^{\frac{p}{k}}}{k}\right)\Gamma\left(\frac{p}{k}\right). \tag{11}$$

Remark 1.6. *The above definition and properties reduce to the k-gamma function and its properties when $p=k$.*

By using the formula (10), we get

$$\beta_k(x,y) = \frac{\Gamma_{(k,p)}(x)\Gamma_{(k,p)}(y)}{\Gamma_{(k,p)}(x+y)} = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x>0, \quad y>0. \tag{12}$$

In another way, the fractional calculus theory has been used as a mathematical tool in a variety of pure and practical fields. This approach has been used in different scientific fields. In applied mathematics, various fractional operators have been used to show a set of integral inequalities and their generalizations. One among the vital applications of fractional integrals is the k-Riemann-Liouville fractional integral operator which is an important tool and a source of many research works in field science such as the theory of inequalities, differential equations, integral inequalities. see for example [1], [5], [6], [8], [9]. The right and left-sided k-Riemann-Liouville fractional integrals of order $\alpha>0$, for a continuous function f on $[a,b]$ are defined as

$$\mathrm{J}^{\alpha}_{a^+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \quad a<x\le b, \tag{13}$$

$$\mathrm{J}^{\alpha}_{b^-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt, \quad a\le x<b, \tag{14}$$

where the k-gamma function verified

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \qquad \Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha) \text{ for all } \alpha, k > 0.$$

Motivated by studies from above, we present in the following section the general (k, p)-Riemann-Liouville fractional integrals of order α with two exponential parameters k and p which generalize the k-Riemann-Liouville fractional integrals. As part of the study, we will follow in the footsteps of the authors in [7].

2. The general (k, p)-Riemann-Liouville fractional integrals

Let $\varphi :]0, +\infty[\times]0, +\infty[\longrightarrow]0, +\infty[$ be a map satisfying the condition $\varphi(k, k) = k$. For example

1. the arithmetic mean $\varphi_1(k, p) = \dfrac{k+p}{2}$,
2. the geometric mean $\varphi_2(k, p) = \sqrt{kp}$,
3. $\varphi_3(k, p) = \dfrac{k^2}{p}$, called the inverse harmonic case.

Definition 2.1. *Let $[a, b] \subseteq [0, +\infty]$, where $a < b$, $f \in L_1[a, b]$ and $k, p > 0$. The right and the left-sided general (k, p)-Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined as*

$$_{a^+}J^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x (x-t)^{\frac{\alpha}{\varphi(k,p)}-1} f(t)dt, \quad a < x \le b. \tag{15}$$

$$_{b^-}J^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b (t-x)^{\frac{\alpha}{\varphi(k,p)}-1} f(t)dt, \quad a \le x < b. \tag{16}$$

Here, $\Gamma_{(k,p)}$ is the (k, p)-Gamma Function defined as in (7).

Remark 2.2. *By using $p = k$, the general (k, p)-Riemann-Liouville fractional integrals (15) and (16) reduce to k-Riemann-Liouville fractional integrals (13) and (14). For $p = k = 1$ we get the classical Riemann-Liouville fractional integrals.*

In the following Theorem we show that the general (k, p)-Riemann-Liouville fractional integrals are clearly defined.

Theorem 2.3. *The fractional integrals (15), (16) are defined for functions $f \in L_1[a, b]$, existing almost everywhere and*

$$_{a^+}J^{\alpha}_{\varphi(k,p)} f(x),\ _{b^-}J^{\alpha}_{\varphi(k,p)} f(x) \in L_1[a, b]. \tag{17}$$

Moreover

$$\left\| J^{\alpha}_{\varphi(k,p)} f(x) \right\|_{L_1[a,b]} \le C \left\| f(t) \right\|_{L_1[a,b]}, \tag{18}$$

where

$$C = \max\left(\frac{(b-a)^{\frac{\alpha_1}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}, \frac{(b-a)^{\frac{\alpha_2}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k(\frac{\alpha_2}{\varphi(k,p)}+1)\right)} \right), \quad 0 < \alpha_2 < \varphi(k, p) < \alpha_1.$$

Proof. Let $f(x) \in L_1[a,b]$.

- Let $\frac{\alpha}{\varphi(k,p)} = 1$, is obvious.
- Let $\frac{\alpha}{\varphi(k,p)} > 1$. Let $\Omega = [a,b] \times [a,b]$, we pose for all $(x,t) \in \Omega$, posing

$$F_1(x,t) = \begin{cases} (x-t)^{\frac{\alpha}{\varphi(k,p)}-1} & , a \leq t \leq x, \\ 0 & , x \leq t \leq b, \end{cases}$$

and

$$F_2(x,t) = \begin{cases} 0 & , a \leq t \leq x, \\ (t-x)^{\frac{\alpha}{\varphi(k,p)}-1} & , x \leq t \leq b. \end{cases}$$

For $i = 1, 2$ we have

$$\int_a^b F_i(x,t)dx \leq \int_a^b (b-a)^{\frac{\alpha}{\varphi(k,p)}-1} dx = (b-a)^{\frac{\alpha}{\varphi(k,p)}},$$

therefore

$$\begin{aligned} \int_a^b \int_a^b F_i(x,t) \left| f(t) \right| dxdt &\leq \int_a^b (b-a)^{\frac{\alpha}{\varphi(k,p)}} \left| f(t) \right| dt \\ &= (b-a)^{\frac{\alpha}{\varphi(k,p)}} \left\| f(t) \right\|_{L_1[a,b]} < \infty. \end{aligned}$$

Hence, by Tonelli's theorem we deduce that the function $F_i(x,t) \left| f(t) \right|$ is integrable over Ω. Using now Fubini's theorem, we get

$$\int_a^b \left(\int_a^b F_i(x,t) \left| f(t) \right| dx \right) dt = \int_a^b \left(\int_a^b F_i(x,t) \left| f(t) \right| dt \right) dx,$$

and

$$\int_a^b F_i(x,t) \left| f(t) \right| dt \in L_1[a,b],$$

this gives us (17) for $i = 1, 2$.

- Let $0 < \frac{\alpha}{\varphi(k,p)} < 1$, put $\lambda = 1 - \frac{\alpha}{\varphi(k,p)}$ hence $0 < \lambda < 1$. By Fubini's theorem we get

$$\begin{aligned} \int_a^b \left| {}_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)} f(x) \right| dx &= \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b \left| \int_a^x (x-t)^{\frac{\alpha}{\varphi(k,p)}-1} f(t)dt \right| dx \\ &\leq \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b \int_a^x \frac{\left| f(t) \right|}{(x-t)^{\lambda}} dtdx \\ &= \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b \left| f(t) \right| \left(\int_t^b \frac{1}{(x-t)^{\lambda}} dx \right) dt \\ &= \frac{1}{(1-\lambda)\, k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b \left| f(t) \right| (b-t)^{1-\lambda} dt \\ &\leq \frac{(b-a)^{1-\lambda}}{(1-\lambda)\, k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b \left| f(t) \right| dt. \end{aligned}$$

Therefore

$$\int_a^b \left|{}_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(x)\right|dx \le \frac{(b-a)^{\frac{\alpha}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k(\frac{\alpha}{\varphi(k,p)}+1)\right)}\left\|f(t)\right\|_{L_1[a,b]} < +\infty.$$

Similarly, we have

$$\int_a^b \left|{}_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(x)\right|dx \le \frac{(b-a)^{\frac{\alpha}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k(\frac{\alpha}{\varphi(k,p)}+1)\right)}\left\|f(t)\right\|_{L_1[a,b]} < +\infty.$$

This gives us our desired formula (17).

□

Theorem 2.4. *Let* $\frac{\alpha}{\varphi(k,p)} > 1$ *and* $f \in L_1[a,b]$, *then the fractional integrals (15), (16) are*

$$_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(x),\ {}_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(x) \in C[a,b]. \tag{19}$$

Proof. Let $y, z \in [a,b]$, $y \le z$ and $y \to z$, we have

$$\begin{aligned} &k\Gamma_{(k,p)}(\tfrac{k\alpha}{\varphi(k,p)})\left|{}_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(z) - {}_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(y)\right| \\ &= \left|\left(\int_z^b\left[(t-z)^{\frac{\alpha}{\varphi(k,p)}-1} - (t-y)^{\frac{\alpha}{\varphi(k,p)}-1}\right]f(t)dt\right) - \int_y^z (t-y)^{\frac{\alpha}{\varphi(k,p)}-1}f(t)dt\right| \\ &\le \int_z^b \left|(t-z)^{\frac{\alpha}{\varphi(k,p)}-1} - (t-y)^{\frac{\alpha}{\varphi(k,p)}-1}\right|\left|f(t)\right|dt + (z-y)^{\frac{\alpha}{\varphi(k,p)}-1}\left\|f(t)\right\|_{L_1[a,b]}, \end{aligned}$$

thus

$$\left|{}_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(z) - {}_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(y)\right| \to 0 \quad \text{as } y \to z.$$

Similarly

$$\left|{}_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(z) - {}_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(y)\right| \to 0 \quad \text{as } y \to z.$$

This gives us (19). □

Now, we demonstrate the commutativity and the semigroup properties of the general (k,p)-Riemann-Liouville fractional integrals.

Theorem 2.5. *Let* $\alpha, \beta > 0$.

$$_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)}\left({}_{a^+}\mathrm{J}^{\beta}_{\varphi(k,p)}f(x)\right) = {}_{a^+}\mathrm{J}^{\alpha+\beta}_{\varphi(k,p)}f(x) = {}_{a^+}\mathrm{J}^{\beta}_{\varphi(k,p)}\left({}_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(x)\right). \tag{20}$$

$$_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}\left({}_{b^-}\mathrm{J}^{\beta}_{\varphi(k,p)}f(x)\right) = {}_{b^-}\mathrm{J}^{\alpha+\beta}_{\varphi(k,p)}f(x) = {}_{b^-}\mathrm{J}^{\beta}_{\varphi(k,p)}\left({}_{b^-}\mathrm{J}^{\alpha}_{\varphi(k,p)}f(x)\right). \tag{21}$$

Equations (20) and (21) are satisfied in any point for $f(t) \in C([a,b])$ *and in almost every point for* $f(t) \in L_1[a,b]$.

Proof. Using Fubini's theorem, we get

$$\begin{aligned} &\left[k^2\,\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)\Gamma_k\left(\frac{k\beta}{\varphi(k,p)}\right)\right]{}_{a^+}\mathrm{J}^{\alpha}_{\varphi(k,p)}\left({}_{a^+}\mathrm{J}^{\beta}_{\varphi(k,p)}f(x)\right) \\ &= \int_a^x (x-t)^{\frac{\alpha}{\varphi(k,p)}-1}\left(\int_a^t (t-s)^{\frac{\beta}{\varphi(k,p)}-1}f(s)ds\right)dt \\ &= \int_a^x f(s)\left(\int_s^x (x-t)^{\frac{\alpha}{\varphi(k,p)}-1}(t-s)^{\frac{\beta}{\varphi(k,p)}-1}dt\right)ds. \end{aligned} \tag{22}$$

The inner integral in (22) is evaluated by the change of variable $t = y(x-s) + s$, we have

$$\begin{aligned}
&\int_s^x (x-t)^{\frac{\alpha}{\varphi(k,p)}-1}(t-s)^{\frac{\beta}{\varphi(k,p)}-1}dt \\
&= (x-s)^{\frac{\alpha+\beta}{\varphi(k,p)}-1}\int_0^1 (1-y)^{\frac{\alpha}{\varphi(k,p)}-1}(y)^{\frac{\beta}{\varphi(k,p)}-1}dy \\
&= k\,(x-s)^{\frac{\alpha+\beta}{\varphi(k,p)}-1}\beta_k\left(\frac{k\alpha}{\varphi(k,p)}, \frac{k\beta}{\varphi(k,p)}\right) \quad , \qquad (23)
\end{aligned}$$

Using k-beta propriety (12) and (23) in (22), we deduce that

$$_{a^+}\mathrm{J}^{\,\alpha}_{\varphi(k,p)}\left(_{a^+}\mathrm{J}^{\,\beta}_{\varphi(k,p)}f(x)\right) = \frac{1}{k\,\Gamma_{(k,p)}\left(\frac{k(\alpha+\beta)}{\varphi(k,p)}\right)}\int_a^x (x-s)^{\frac{\alpha+\beta}{\varphi(k,p)}-1}f(s)ds,$$

then, we get the desired equality (20), similarly to the equality (21). □

Lemma 2.6. *Let $k, p, \alpha, \lambda > 0$, then*

$$_{a^+}\mathrm{J}^{\,\alpha}_{\varphi(k,p)}\left((x-a)^{\frac{\lambda}{\varphi(k,p)}-1}\right) = \frac{\Gamma_{(k,p)}\left(\frac{k\lambda}{\varphi(k,p)}\right)}{\Gamma_{(k,p)}\left(\frac{k(\alpha+\lambda)}{\varphi(k,p)}\right)}(x-a)^{\frac{(\alpha+\lambda)}{\varphi(k,p)}-1}, \qquad (24)$$

$$_{b^-}\mathrm{J}^{\,\alpha}_{\varphi(k,p)}\left((b-x)^{\frac{\lambda}{\varphi(k,p)}-1}\right) = \frac{\Gamma_{(k,p)}\left(\frac{k\lambda}{\varphi(k,p)}\right)}{\Gamma_{(k,p)}\left(\frac{k(\alpha+\lambda)}{\varphi(k,p)}\right)}(b-x)^{\frac{(\alpha+\lambda)}{\varphi(k,p)}-1}. \qquad (25)$$

Proof. Using the change of variable $t = y(x-a) + a$, we deduce

$$\begin{aligned}
&_{a^+}\mathrm{J}^{\,\alpha}_{\varphi(k,p)}\left((x-a)^{\frac{\lambda}{\varphi(k,p)}-1}\right) \\
&= \frac{1}{k\,\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^x (x-t)^{\frac{\alpha}{\varphi(k,p)}-1}(t-a)^{\frac{\lambda}{\varphi(k,p)}-1}dt \\
&= \frac{(x-a)^{\frac{\alpha+\lambda}{\varphi(k,p)}-1}}{k\,\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_0^1 (1-y)^{\frac{\alpha}{\varphi(k,p)}-1}(y)^{\frac{\lambda}{\varphi(k,p)}-1}dy \\
&= \frac{(x-a)^{\frac{\alpha+\lambda}{\varphi(k,p)}-1}}{\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\beta_k\left(\frac{k\alpha}{\varphi(k,p)}, \frac{k\lambda}{\varphi(k,p)}\right).
\end{aligned}$$

This completes the proof of (24). By taking $t = b - y(b-x)$ the proof of equality (25) is similar to (24). □

Some special results of the equalities (24) and (25) above are given in the next Corollary..

Corollary 2.7. 1. *Take $p = k$, we get*

$$_{a^+}\mathrm{J}^{\,\alpha}_{k}\left((x-a)^{\frac{\lambda}{k}-1}\right) = \frac{\Gamma_k(\lambda)}{\Gamma_k(\alpha+\lambda)}(x-a)^{\frac{(\alpha+\lambda)}{k}-1}, \qquad (26)$$

$$_{b^-}\mathrm{J}^{\,\alpha}_{k}\left((b-x)^{\frac{\lambda}{k}-1}\right) = \frac{\Gamma_k(\lambda)}{\Gamma_k(\alpha+\lambda)}(b-x)^{\frac{(\alpha+\lambda)}{k}-1}. \qquad (27)$$

2. *Choose $p = k$ and $\lambda = 1$, thus*

$$_{a^+}\mathrm{J}^{\,\alpha}_{k}\left((x-a)^{\frac{1}{k}-1}\right) = \frac{\Gamma_k(1)}{\Gamma_k(\alpha+1)}(x-a)^{\frac{(\alpha+1)}{k}-1}, \qquad (28)$$

$$_{b^-}\mathrm{J}^{\,\alpha}_{k}\left((b-x)^{\frac{1}{k}-1}\right) = \frac{\Gamma_k(1)}{\Gamma_k(\alpha+1)}(b-x)^{\frac{(\alpha+1)}{k}-1}. \qquad (29)$$

3. Special cases of general (k, p)-Riemann-Liouville fractional integrals

In this section, we give three interesting cases of general (k, p) -Riemann-Liouville fractional integrals according to the choosing of the map $\varphi(k, p)$.

3.1. Geometric (k, p)-Riemann-Liouville fractional integrals

$$\text{Taking}, \qquad \varphi(k, p) = \sqrt{kp}.$$

Definition 3.1. *Let $[a, b] \subseteq [0, +\infty]$, where $a < b$, $f \in L_1[a, b]$ and $k, p > 0$. The right and the left-sided geometric (k, p) -Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined as*

$$_{a^+}\mathrm{G}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\,\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\,\alpha\right)} \int_a^x (x - t)^{\frac{\alpha}{\sqrt{kp}} - 1} f(t)dt, \quad a < x \leq b. \tag{30}$$

$$_{b^-}\mathrm{G}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\,\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\,\alpha\right)} \int_x^b (t - x)^{\frac{\alpha}{\sqrt{kp}} - 1} f(t)dt, \quad a \leq x < b. \tag{31}$$

Here, $\Gamma_{(k,p)}$ denote the (k, p)-Gamma Function given as in (7).

3.2. Arithmetic (k, p)-Riemann-Liouville fractional integrals

$$\text{Putting}, \qquad \varphi(k, p) = \frac{k + p}{2}.$$

Definition 3.2. *Let $[a, b] \subseteq [0, +\infty]$, where $a < b$, $f \in L_1[a, b]$ and $k, p > 0$. The right and the left-sided arithmetic (k, p) -Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined as*

$$_{a^+}\mathrm{A}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\,\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x (x - t)^{\frac{2\alpha}{k+p} - 1} f(t)dt, \quad a < x \leq b. \tag{32}$$

$$_{b^-}\mathrm{A}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\,\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b (t - x)^{\frac{2\alpha}{k+p} - 1} f(t)dt, \quad a \leq x < b. \tag{33}$$

Here, $\Gamma_{(k,p)}$ denote the (k, p)-Gamma Function given as in (7).

3.3. Harmonic (k, p)-Riemann-Liouville fractional integrals

$$\text{Chosing}, \qquad \varphi(k, p) = \frac{k^2}{p}.$$

Definition 3.3. *Let $[a, b] \subseteq [0, +\infty]$, where $a < b$, $f \in L_1[a, b]$ and $k, p > 0$. The right and the left-sided harmonic (k, p) -Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined as*

$$_{a^+}\mathrm{H}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\,\Gamma_{(k,p)}\left(\frac{p}{k}\,\alpha\right)} \int_a^x (x - t)^{\frac{p\alpha}{k^2} - 1} f(t)dt, \quad a < x \leq b. \tag{34}$$

$$_{b^-}\mathrm{H}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\,\Gamma_{(k,p)}\left(\frac{p}{k}\,\alpha\right)} \int_x^b (t - x)^{\frac{p\alpha}{k^2} - 1} f(t)dt, \quad a \leq x < b. \tag{35}$$

Here, $\Gamma_{(k,p)}$ denote the (k, p)-Gamma Function given (7) by

$$\Gamma_{(k,p)}(\alpha) = \int_0^{\infty} t^{\alpha - 1} e^{-\frac{t^k}{p}} dt.$$

4. Conclusion

In this work, we introduce a general version of the Riemann-Liouville fractional integrals which is called (k,p)- Riemann-Liouville fractional integrals. By special choice of the parameters, we present some new definitions which are called geometric and arithmetic (k,p)- Riemann-Liouville fractional integrals. In the future works, the author can prove some important integral inequalities based on integrals defined in this paper.

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