N-hyperideals and related extensions

Mahdi Anbarloei[a]

[a]Department of Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran.

Abstract. Various classes of hyperideals have been studied in many papers in order to let us fully understand the structures of hyperrings in general. The purpose of this paper is the study of some hyperideals whose concept is created on the basis of the intersection of all n-ary prime hyperideals in category of Krasner (m, n)-hyperrings. In this regard we introduce notions of N-hyperideals, δ-N-hyperideals, (k, n)-absorbing δ-N-hyperideals and S-N-hyperideals. The overall framework of these structures is then explained, providing a number of major conclusions.

1. Introduction

The notion of prime ideal as an extension of the notion of prime number in the ring of integers plays a highly important role in the theory of rings. In 2017, Terkir et al. [13] proposed a new class of ideals called n-ideals and investigated some properties of them analogous with prime ideals. Let I be an ideal of ring R. I refers to an n-ideal if the condition $xy \in I$ with $x \notin \sqrt{0}$ implies $y \in I$ for all $x, y \in I$. Afterward, the concept of $(2, n)$-ideals in a commutative ring was defined by Tamekhante and Bouba [14]. They gave many results to show the relations between this new notion and others that already exist. Indeed, the notion is a generalization of n-ideals. A proper ideal I of R is said to be a $(2, n)$-ideal if $xyz \in I$ for $x, y, z \in R$ implies that $xy \in I$ or $yz \in \sqrt{0}$ or $xz \in \sqrt{0}$.

Algebraic hyperstructures arise as natural extensions of classical algebraic structures when the composition operator is multivalued. The pioneer of this theory was the French mathematician F. Marty [8], who introduced the concept of hypergroup in 1934 on the occasion of the 8th Congress of Scandinavian Mathematicians in Stockholm. Since the 1970s, hyperstructure theory has experienced a surge of interest, when its research field was greatly extended by the introduction of other useful notions. Nowadays, hypercompositional algebra has a variety of relationships with other areas of mathematics. One significant type of hyperrings was introduced by Krasner, when the addition is a hyperoperation, while the multiplication is an ordinary binary operation, which is said to be Krasner hyperring. An extension of the Krasner hyperrings, which is a subclass of (m, n)-hyperrings, was presented by Mirvakili and Davvaz which is called Krasner (m, n)-hyperring, in [9]. Ameri and Norouzi defined some substancial classes of hyperideals in Krasner (m, n)-hyperrings [1]. Later, in [7] the concepts of (k, n)-absorbing hyperideals and (k, n)-absorbing primary hyperideals were studied by Hila et al.. Norouzi and his colleagues illustrated a new defnition for normal hyperideals in Krasner (m, n)-hyperrings, with respect to that one given in [9] and they proposed in

[10] that these hyperideals correspond to strongly regular relations. In [5], Asadi and Ameri showed direct limit of a direct system in the category of Krasner (m, n)-hyperrigs. Ozel Ay et al. presented the idea of δ-primary on Krasner hyperrings [12]. The notion of δ-primary hyperideals in Krasner (m, n)-hyperrings, which unifies the prime and primary hyperideals under one frame, was defined in [3]. For a Krasner hyperring, the concept of n-hyperideals was briefly introduced by Omidi et al. in [11].

For a commutative Krasner (m, n)-hyperring H, $\sqrt{0}^{(m,n)}$ is the intersection is taken over all n-ary prime hyperideals.

$$\sqrt{0}^{(m,n)} = \left\{ a \in H \mid \left\{ \begin{array}{ll} g(a^{(s)}, 1_H^{(n-s)}) = 0, & s \leq n \\ g_{(l)}(a^{(s)}) = 0 & s > n, s = l(n-1) + 1 \end{array} \right\} \right\}$$

is an alternative definition of $\sqrt{0}^{(m,n)}$. In this paper, we aim to analyze some notions of hyperideals established on basis of the intersection of all n-ary prime hyperideals in a commutative Krasner (m, n)-hyperring. The paper is orgnized as follows. In Section 2, we have given some basic definitions and results of commutative Krasner (m, n)-hyperrings which we need to develop our paper. In Section 3, we introduce the idea of n-ary N-hyperiudeals in a commutative Krasner (m, n)-hyperring and give several characterizations of them. In Section 4, we extend the notion of n-ary N-hyperideals to n-ary δ-N-hyperideals. We obtain many specific results explaining the structures. Afterward, in Section 5, we study an expansion of the previous concept called (k, n)-absorbing δ-N-hyperideals. Some properties of them are provided. The last section is devoted for introducing the n-ary S-N-hyperideals.

2. Preliminaries

For a non-empty set H, the mapping $f : H^n \longrightarrow P^*(H)$ is called an n-ary hyperoperation where $P^*(H)$ is the family of all the nonempty subsets of H. An n-ary hypergroupoid is an algebraic system (H, f). Let $H_1, \cdots, H_n$ be non-empty subsets of H. Then $f(H_1^n) = f(H_1, \cdots, H_n)$ is defined by $\bigcup \left\{ f(a_1^n) \mid a_i \in H_i, 1 \leq i \leq n \right\}$. The sequence $a_i, a_{i+1}, \cdots, a_j$ is denoted by a_i^j and so we write

$$f(a_1, \cdots, a_i, b_{i+1}, \cdots, b_j, c_{j+1}, \cdots, c_n) = f(a_1^i, b_{i+1}^j, c_{j+1}^n).$$

Moreover, if $b_{i+1} = \cdots = b_j = b$, then we write $f(a_1^i, b^{(j-i)}, c_{j+1}^n)$. a_i^j is the empty symbol if $j < i$. For an n-ary hyperoperation f, $(l(n - 1) + 1)$-ary hyperoperation $f_{(l)}$ is given by

$$f_{(l)}(a_1^{l(n-1)+1}) = f\Big(f\big(\cdots, f(f(a_1^n), a_{n+1}^{2n-1}), \cdots\big), a_{(l-1)(n-1)+1}^{l(n-1)+1}\Big).$$

Recall from [9] that an algebraic hyperstructure (H, f, g), or simply H, is called a commutative Krasner (m, n)-hyperring if

(1) (H, f) is a canonical m-ary hypergroup, i.e., for $1 \leq i < j \leq n$ and $a_1^{2n-1} \in H$, $f\Big(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}\Big) = f\Big(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1}\Big)$ and $^{-1} : H \longrightarrow H$ is a unitary operation such that

     (i)   there exists a unique $e \in H$, such that $f(a, e^{(m-1)}) = a$ for every $a \in H$ and $e^{-1} = e$;

     (ii)   for each $a \in H$ there exists a unique $a^{-1} \in H$ with $e \in f(a, a^{-1}, e^{(m-2)})$;

     (iii)   if $a \in f(a_1^m)$, then $a_i \in f(a, a^{-1}, \cdots, a_{i-1}^{-1}, a_{i+1}^{-1}, \cdots, a_m^{-1})$ for all i,

     (iv) $f(a_1^m) = f(a_{\sigma(1)}^{\sigma(m)})$ for all $\sigma \in \mathbb{S}_n$ and for all $a_1^m \in H$,

(2) (H, g) is a n-ary semigroup,

(3) $g\Big(a_1^{i-1}, f(x_1^m), a_{i+1}^n\Big) = f\Big(g(a_1^{i-1}, x_1, a_{i+1}^n), \cdots, g(a_1^{i-1}, x_m, a_{i+1}^n)\Big)$ for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in H$, and $1 \leq i \leq n$;

(4) $g(0, a_2^n) = g(a_2, 0, a_3^n) = \cdots = g(a_2^n, 0) = 0$ for every $a_2^n \in H$,

(5) $g(a_1^n) = g(a_{\sigma(1)}^{\sigma(n)})$ for all $\sigma \in \mathbb{S}_n$ and for all $a_1^n \in H$.

For a non-empty subset S of H, if (S, f, g) is a Krasner (m, n)-hyperrin, then S is called a subhyperring of H. Let I be a non-empty subset of H. I refers to a hyperideal of (R, f, g) if (I, f) is an m-ary subhypergroup of (H, f) and for every $x_1^n \in R$ and $1 \le i \le n$, $g(a_1^{i-1}, I, a_{i+1}^n) \subseteq I$.

Definition 2.1. *[9] Suppose that (H_1, f_1, g_1) and (H_2, f_2, g_2) are two Krasner (m, n)-hyperrings. A mapping $d : H_1 \longrightarrow H_2$ is called a homomorphism if*

(i) $d(1_{R_1}) = 1_{R_2}$,

(ii) $d\big(f_1(a_1, \cdots, a_m)\big) = f_2\big(d(a_1), \cdots, d(a_m)\big)$,

(iii) $d\big(g_1(b_1, \cdots, b_n)\big) = g_2\big(d(b_1), \cdots, d(b_n)\big)$

for all $a_1^m \in H_1$ and $b_1^n \in H_1$.

Now, We recall some definitions from [1]. Let I be a hyperideal in a commutative Krasner (m, n)-hyperring R with scalar identity. $\sqrt{I}^{(m,n)}$ is the intersection is taken over all prime hyperideals of H which contain I. If the set of all prime hyperideals containing I is empty, then $\sqrt{I}^{(m,n)} = H$. Theorem 4.23 in [1] shows that if $a \in \sqrt{I}^{(m,n)}$ then there exists $t \in \mathbb{N}$ such that $g(a^{(t)}, 1_H^{(n-t)}) \in I$ for $t \le n$, or $g_{(l)}(a^{(t)}) \in I$ for $t = l(n-1) + 1$.

Definition 2.2. *[1] Let I be a proper hyperideal of a commutative Krasner (m, n)-hyperring H with the scalar identity 1_R. I refers to a*

(1) *maximal hyperideal of H if for every hyperideal I' of H, $I \subseteq I' \subseteq H$ implies that $I' = M$ or $I' = H$. The intersection of all maximal hyperideals of H is called the Jacobson radical of H which is denoted by $J_{(m,n)}(H)$. If H does not have any maximal hyperideal, we define $J_{(m,n)}(H) = H$.*

(2) *n-ary prime hyperideal if for hyperideals $I_1, \cdots, I_n$ of H, $g(I_1^n) \subseteq I$ implies that $I_i \subseteq I$ for some $1 \le i \le n$. Lemma 4.5 in [1] shows that if for all $a_1^n \in H$, $g(a_1^n) \in I$ implies that $a_i \in I$ for some $1 \le i \le n$, then I is a prime hyperideal.*

(3) *n-ary primary hyperideal if $g(a_1^n) \in I$ and $a_i \notin I$, then $g(a_1^{i-1}, 1_H, a_{i+1}^n) \in \sqrt{I}^{(m,n)}$ for some $1 \le i \le n$. Theorem 4.28 in [1] shows that if I is a primary hyperideal of H, then $\sqrt{I}^{(m,n)}$ is prime.*

Definition 2.3. *[1] Let a be an element in a commutative Krasner (m, n)-hyperring H. The hyperideal generated by a is denoted by $< a >$ and defined by*

$$g(H, a, 1^{(n-2)}) = \Big\{ g(r, a, 1^{(n-2)}) \mid r \in H \Big\}.$$

Definition 2.4. *[1] Let a be an element in a commutative Krasner (m, n)-hyperring H. It is invertible if there exists $b \in R$ with $1_H = g(a, b, 1_H^{(n-2)})$. Let U be a subset of H. Then U is invertible if and only if every element of U is invertible.*

3. n-ary N-hyperideals

In this section, we present the concept of n-ary N-hyperideals and study many properties of them with similar n-ary prime hyperideals.

Definition 3.1. *Let I be a proper hyperideal of a commutative Krasner (m, n)-hyperring H. I refers to an n-ary N-hyperideal if whenever $x_1^n \in H$ with $g(x_1^n) \in I$ and $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$ imply that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in I$.*

Example 3.2. *Let $x \geq 1$. Consider n-ary hyperintegral domain $([x, \infty) \cup \{0\}, \boxplus, \cdot)$ where $\boxplus$ is defined by*

$$a \boxplus b = \begin{cases} b \boxplus a = \{a\} & \text{if } b = 0, \\ \{min\{a, b\}\} & \text{if } a \neq 0, b \neq 0 \text{ and } a \neq b, \\ [a, \infty) \cup \{0\} & \text{if } a = b \neq 0. \end{cases}$$

and " $\cdot$ " is the usual multiplication. The hyperideal 0 is the only n-ary N-hyperideal of $([x, \infty) \cup \{0\}, \boxplus, \cdot)$.

Theorem 3.3. *For an n-ary N-hyperideal I of a commutative Krasner (m, n)-hyperring H, $I \subseteq \sqrt{0}^{(m,n)}$.*

Proof. Let I be an n-ary N-hyperideal of a commutative Krasner (m, n)-hyperring H however $I \nsubseteq \sqrt{0}^{(m,n)}$. Assume that $x \in I$ but $x \notin \sqrt{0}^{(m,n)}$. Since I is an n-ary N-hyperideal of H and $g(x, 1_H^{(n-1)}) \in I$, we get $g(1_H^{(n)}) \in I$, yielding a contradiction. Consequently, $I \subseteq \sqrt{0}^{(m,n)}$. $\square$

The next Theorem gives a characterization of n-ary N-hyperideals.

Theorem 3.4. *Let I be an n-ary prime hyperideal of a commutative Krasner (m, n)-hyperring H. Then I is an n-ary N-hyperideal if and only if $I = \sqrt{0}^{(m,n)}$.*

Proof. $\Longrightarrow$ Let I be an n-ary prime hyperideal of H. Clearly, $\sqrt{0}^{(m,n)} \subseteq I$. Suppose that I is an n-ary N-hyperideal of H. Hence we obtain $I \subseteq \sqrt{0}^{(m,n)}$, by Theorem 3.3. Then we conclude that $I = \sqrt{0}^{(m,n)}$.
$\Longleftarrow$ Let $I = \sqrt{0}^{(m,n)}$. We presume $x_1^n \in H$ with $g(x_1^n) \in I$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \leq i \leq n$. Since I is an n-ary prime hyperideal of H and $x_i \notin I$, then there exists $1 \leq j \leq i - 1$ or $i + 1 \leq j \leq n$ such that $x_j \in I$. This implies that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in I$. Thus, I is an n-ary N-hyperideal of H. $\square$

In view of Theorem 3.4, we have the following result.

Corollary 3.5. *Let H be a commutative Krasner (m, n)-hyperring. Then $\sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H if and only if it is an n-ary N-hyperideal of H.*

Proof. $\Longrightarrow$ Suppose that $\sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H. By Theorem 3.4, $\sqrt{0}^{(m,n)}$ is an n-ary N-hyperideal of H.
$\Longleftarrow$ Let $\sqrt{0}^{(m,n)}$ be an n-ary N-hyperideal of H. Assume that $g(x_1^n) \in \sqrt{0}^{(m,n)}$ for some $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \leq i \leq n$. Then we get $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \sqrt{0}^{(m,n)}$. Since $\sqrt{0}^{(m,n)}$ is an n-ary N-hyperideal of H, we can continue the process and obtain $x_j \in \sqrt{0}^{(m,n)}$ for some $1 \leq j \leq n$. This means that $\sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H. $\square$

Theorem 3.6. *Let I be a proper hyperideal of a commutative Krasner (m, n)-hyperring H. If every proper principal hyperideal is an n-ary N-hyperideal, then so is I.*

Proof. Assume that I is a proper hyperideal of H. Let $g(x_1^n) \in I$ for some $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \leq i \leq n$. Since every proper principal hyperideal is an n-ary N-hyperideal and $g(x_1^n) \in \langle g(x_1^n) \rangle$, we get $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \langle g(x_1^n) \rangle \subseteq I$. This means that I is a an n-ary N-hyperideal of H. $\square$

Theorem 3.7. *Let I be a proper hyperideal of a commutative Krasner (m, n)-hyperring H. Then the following statements are equivalent:*

(1) *I is an n-ary N-hyperideal of H.*

(2) $I = E_x$ *where* $E_x = \left\{ y \in H \mid g(x, y, 1_H^{(n-2)}) \in I \right\}$ *for every* $x \notin \sqrt{0}^{(m,n)}$.

(3) $g(I_1^n) \subseteq I$ *for some hyperideals* I_1^n *of* H *such that* $I_i \cap (H - \sqrt{0}^{(m,n)}) \neq \varnothing$ *for some* $1 \leq i \leq n$ *imply that* $g(I_1^{i-1}, 1_H, I_{i+1}^n) \subseteq I$.

Proof. (1) $\Longrightarrow$ (2) Let I be an n-ary N-hyperideal of H. We always have $I \subseteq E_x$ for every $x \in H$. Assume that $y \in E_x$ but $x \notin \sqrt{0}^{(m,n)}$. This implies that $g(x, y, 1_H^{(n-2)}) \in I$. Since I is an n-ary N-hyperideal of H and $x \notin \sqrt{0}^{(m,n)}$, then $y = g(y, 1_H^{(n-2)}) \in I$. Therefore, we have $I = E_x$.

    (2) $\Longrightarrow$ (3) Let $g(I_1^n) \subseteq I$ for some hyperideals I_1^n of H such that $I_i \cap (H - \sqrt{0}^{(m,n)}) \neq \varnothing$ for some $1 \leq i \leq n$. Then we get $x_i \in I_i$ such that $x_i \notin \sqrt{0}^{(m,n)}$. Hence, $g(I_1^{i-1}, x_i, I_{I+1}^n) \subseteq I$ which implies $g(I_1^{i-1}, 1_H, I_{i+1}^n) \subseteq E_{x_i}$. Since $I = E_{x_i}$ for every $x_i \notin \sqrt{0}^{(m,n)}$, then we get $g(I_1^{i-1}, 1_H, I_{i+1}^n) \subseteq I$.

    (3) $\Longrightarrow$ (1) Assume that $g(x_1^n) \in I$ for some $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \leq i \leq n$. We get $g\left(\langle x_1 \rangle, \cdots, \langle x_n \rangle\right) \subseteq \langle g(x_1^n) \rangle \subseteq I$ and $\langle x_i \rangle \not\subseteq \sqrt{0}^{(m,n)}$. Then we conclude that $g\left(\langle x_1 \rangle, \cdots, \langle x_{i-1} \rangle, 1_H, \langle x_{i+1} \rangle, \cdots, \langle x_n \rangle\right) \subseteq I$ which means $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in I$. Thus, I is an n-ary N-hyperideal of H. $\quad\square$

Theorem 3.8. *Let T be a non-empty subset of a commutative Krasner (m, n)-hyperring H. If I is an n-ary N-hyperideal of H such that $T \not\subseteq I$, then $E_T = \left\{ x \in H \mid g(x, T, 1_H^{(n-2)}) \subseteq I \right\}$ is an n-ary N-hyperideal of H.*

Proof. Clealy, $E_T \neq H$. Assume that $g(x_1^n) \in E_T$ for some $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \leq i \leq n$. This means that $g\left(g(x_1^n), t, 1_H^{(n-2)}\right) \in I$ for each $t \in T$ and so $g\left(x_i, g(x_1^{i-1}, t, x_{i+1}^n), 1_H^{(n-2)}\right) \in I$. Since I is an n-ary N-hyperideal of H and $x_i \notin \sqrt{0}^{(m,n)}$, we get $g\left(g(x_1^{i-1}, 1_H, x_{i+1}^n), t, 1_H^{(n-2)}\right) = g(x_1^{i-1}, t, x_{i+1}^n) \in I$ which means $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in E_T$. Consequently, E_T is an n-ary N-hyperideal of H. $\quad\square$

Theorem 3.9. *For a maximal N-hyperideal I of a commutative Krasner (m, n)-hyperring H, $I = \sqrt{0}^{(m,n)}$.*

Proof. Suppose that I is a maximal N-hyperideal of H. Let us consider $g(x_1^n) \in I$ for some $x_1^n \in H$. We may assume that $x_n \notin I$. By Theorem 3.8, we conclude that E_{x_n} is an n-ary N-hyperideal of H with $I \subseteq E_{x_n}$. Since I is a maximal N-hyperideal of H, we have $E_{x_n} = I$ and so $g(x_1^{n-1}, 1_H) \in I$. Now, we assume that $x_{n-1} \notin I$. In a similar way, we get $g(x_1^{n-2}, 1_H^{(2)}) \in I$. By continuing the argument, we obtain $x_1 \in I$ which implies I is an n-ary prime hyperideal of H. Then, we conclude that $I = \sqrt{0}^{(m,n)}$, by Theorem 3.4. $\quad\square$

The next result characterizes hyperrings admitting n-ary N-hyperideals.

Theorem 3.10. *Let H be a commutative Krasner (m, n)-hyperring. Then H admits an n-ary N-hyperideal if and only if $\sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H.*

Proof. $\Longrightarrow$ Let I be an n-ary N-hyperideal of H and let Σ be the set of all n-ary N-hyperideals of H. Then $\Sigma \neq \varnothing$, since $I \in \Sigma$. So Σ is a partially ordered set with respect to set inclusion relation. Now, we take the chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ in Σ. Put $P = \bigcup_{i=1}^{\infty} I_i$. We prove P is an n-ary N-hyperideal of H. Assume that $g(x_1^n) \in P$ for some $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$. Then there exists $s \in \mathbb{N}$ such that $g(x_1^n) \in I_s$. Then we get $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in I_s \subseteq P$, as I_s is an n-ary N-hyperideal of H and $x_i \notin \sqrt{0}^{(m,n)}$. This means that P is a upper bound of the mentioned chain. By Zorn's lemma, there is a hyperideal Q which is maximal in Σ. Hence $Q = \sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H, by Theorem 3.9.

$\Longleftarrow$ Let $\sqrt{0}^{(m,n)}$ be an n-ary prime hyperideal of H. Then it is an n-ary N-hyperideal of H, by Corollary 3.5. $\quad\square$

Corollary 3.11. *Let I be a hyperideal of a commutative Krasner (m, n)-hyperring H with $I \subseteq \sqrt{0}^{(m,n)}$. Then I is an n-ary N-hyperideal of H if and only if I is an n-ary primary hyperideal of H.*

Proof. Straightforward. □

Now, we are interested in the hyperrings over which 0 is the only n-ary N-hyperideal of H.

Theorem 3.12. *Let H be a commutative Krasner (m, n)-hyperring. Then 0 is the only n-ary N-hyperideal of H if and only if H is an n-ary hyperintegral domain.*

Proof. $\Longrightarrow$ Let 0 be the only n-ary N-hyperideal of H. Then by Theorem 3.10, $\sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H. Therefore $\sqrt{0}^{(m,n)}$ is an n-ary N-hyperideal of H, by Corollary 3.5. Thus $0 = \sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H which means H is an n-ary hyperintegral domain.

$\Longleftarrow$ Let H be an n-ary hyperintegral domain. Suppose that I is an n-ary N-hyperideal of H. Then $I \subseteq \sqrt{0}^{(m,n)}$, by Theorem 3.3. Since H is an n-ary hyperintegral domain, we get $0 = \sqrt{0}^{(m,n)}$. Hence $I = 0$ is the only n-ary N-hyperideal of H. □

Let (H_1, f_1, g_1) and (H_2, f_2, g_2) be two commutative Krasner (m, n)-hyperrings such that 1_{H_1} and 1_{H_2} be scalar identitis of H_1 and H_2, respectively. Then the (m, n)-hyperring $(H_1 \times H_2, f_1 \times f_2, g_1 \times g_2)$ is defined by m-ary hyperoperation $f = f_1 \times f_2$ and n-ary operation $g = g_1 \times g_2$, as follows:

$$f_1 \times f_2\big((x_1, y_1), \cdots, (x_m, y_m)\big) = \big\{(x, y) \mid x \in f_1(x_1^m), y \in f_2(y_1^m)\big\}$$

$$g_1 \times g_2\big((a_1, b_1), \cdots, (a_n, b_n)\big) = \big(g_1(a_1^n), g_2(b_1^n)\big),$$

for all $x_1^m, a_1^n \in H_1$ and $y_1^m, b_1^n \in H_2$ [2].

Theorem 3.13. *Let (H_1, f_1, g_1) and (H_2, f_2, g_2) be two commutative Krasner (m, n)-hyperrings such that 1_{H_1} and 1_{H_2} be scalar identitis of H_1 and H_2, respectively. Then $H_1 \times H_2$ has no n-ary N-hyperideals.*

Proof. Let I_1 and I_2 be hyperideals of H_1 and H_2, respectively, and let $I_1 \times I_2$ be an n-ary N-hyperideal of $H_1 \times H_2$. It is clear that $(1_{H_1}, 0_{H_2}), (0_{H_1}, 1_{H_2}), (1_{H_1}, 1_{H_2}) \notin \sqrt{0_{H_1 \times H_2}}^{(m,n)}$ but $g_1 \times g_2\big((1_{H_1}, 0_{H_2}), (0_{H_1}, 1_{H_2}), (1_{H_1}, 1_{H_2})^{(n-2)}\big) \in I_1 \times I_2$. Therefore we obtain $(1_{H_1}, 0_{H_2}) = g_1 \times g_2\big((1_{H_1}, 0_{H_2}), (1_{H_1}, 1_{H_2})^{(n-1)}\big) \in I_1 \times I_2$ and $(0_{H_1}, 1_{H_2}) = g_1 \times g_2\big((1_{H_1}, 1_{H_2}), (0_{H_1}, 1_{H_2}), (1_{H_1}, 1_{H_2})^{(n-2)}\big) \in I_1 \times I_2$. This implies that

$$(1_{H_1}, 1_{H_2}) = f_1 \times f_2\big((1_{H_1}, 0_{H_2}), (0_{H_1}, 1_{H_2}), (0_{H_1}, 0_{H_2})^{(m-2)}\big) \in I_1 \times I_2$$

which means $I_1 \times I_2 = H_1 \times H_2$. □

Theorem 3.14. *Suppose that (H_1, f_1, g_1) and (H_2, f_2, g_2) are two commutative Krasner (m, n)-hyperrings and $h : H_1 \longrightarrow H_2$ is a homomorphism. Then:*
(1) If h is a monomorphism and I_2 is an n-ary N-hyperideal of H_2, then $h^{-1}(I_2)$ is an n-ary N-hyperideal of H_1.
(2) Let h be an epimorphism and I_1 be a hyperideal of H_1 with $Ker(h) \subseteq I_1$. If I_1 is an n-ary N-hyperideal of H_1, then $h(I_1)$ is an n-ary N-hyperideal of H_2.

Proof. (1) Let $g_1(x_1^n) \in h^{-1}(I_2)$ for $x_1^n \in H_1$ such that $x_i \notin \sqrt{0_{H_1}}^{(m,n)}$ for some $1 \le i \le n$. Then we have $h\big(g_1(x_1^n)\big) = g_2\big(h(x_1^n)\big) \in I_2$. Since h is a monomorphism and $x_i \notin \sqrt{0_{H_1}}^{(m,n)}$, then $h(x_i) \notin \sqrt{0_{H_2}}^{(m,n)}$. Since I_2 is a N-hyperideal of H_2, we get the result that

$$g_2\big(h(x_1), ..., h(x_{i-1}), 1_{H_2}, h(x_{i+1}), ..., h(x_n)\big)$$

$$= h\big(g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n)\big)$$
$$\in I_2$$

which follows $g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n) \in h^{-1}(I_2)$. Thus $h^{-1}(I_2)$ is an n-ary N-hyperideal of H_1.

(2) Let $g_2(y_1^n) \in h(I_1)$ for some $y_1^n \in H_2$ such that $y_i \notin \sqrt{0_{H_2}}^{(m,n)}$. Since h is an epimorphism, then there exist $x_1^n \in R_1$ with $h(x_1) = y_1, \cdots, h(x_n) = y_n$. Therefore $h\big(g_1(x_1^n)\big) = g_2\big(h(x_1), ..., h(x_n)\big) = g_2(y_1^n) \in h(I_1)$. Since $Ker(h) \subseteq I_1$, then we have $g_1(x_1^n) \in I_1$. Since $y_i \notin \sqrt{0_{H_2}}^{(m,n)}$, then $x_i \notin \sqrt{0_{H_1}}^{(m,n)}$. Since I_1 is an n-ary N-hyperideal of R_1 and $x_i \notin \sqrt{0_{H_1}}^{(m,n)}$, we obtain $g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n) \in I_1$ which means

$$h\big(g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n)\big) = g_2\big(h(x_1), ..., h(x_{i-1}), 1_{H_2}, h(x_{i+1}), ..., h(x_n)\big)$$

$$= g_2(y_1^{i-1}, 1_{H_2}, y_{i+1}^n) \in h(I_1)$$

Consequently, $h(I_1)$ is an n-ary N-hyperideal of H_2. $\square$

Corollary 3.15. *Let H' be a subhyperring of a commutative Krasner (m,n)-hyperring H. If I is an n-ary N-hyperideal of H such that $H' \nsubseteq I$, then $H' \cap I$ is an n-ary N-hyperideal of H'.*

Proof. Let us consider the identity map j from H' into H. We conclude that $H' \cap I$ is an n-ary N-hyperideal of H', by Theorem 3.14 (2). $\square$

Let J be a hyperideal of a commutative Krasner (m,n)-hyperring (H, f, g). Then the set

$$H/J = \left\{ f(a_1^{i-1}, J, a_{i+1}^m) \mid a_1^{i-1}, a_{i+1}^m \in H \right\}$$

endowed with m-ary hyperoperation f which for all $a_{11}^{1m}, \cdots, a_{m1}^{mm} \in H$

$$f\big(f(a_{11}^{1(i-1)}, J, a_{1(i+1)}^{1m}), \cdots, f(a_{m1}^{m(i-1)}, J, a_{m(i+1)}^{mm})\big)$$

$$= f\big(f(a_{11}^{m1}), \cdots, f(a_{1(i-1)}^{m(i-1)}), J, f(a_{1(i+1)}^{m(i+1)}), \cdots, f(a_{1m}^{mm})\big)$$

and with n-ary hyperoperation g which for all $a_{11}^{1m}, \cdots, a_{n1}^{nm} \in H$

$$g\big(f(a_{11}^{1(i-1)}, J, a_{1(i+1)}^{1m}), \cdots, f(a_{n1}^{n(i-1)}, J, a_{n(i+1)}^{nm})\big)$$

$$= f\big(g(a_{11}^{n1}), \cdots, g(a_{1(i-1)}^{n(i-1)}), J, g(a_{1(i+1)}^{n(i+1)}), \cdots, f(a_{1m}^{nm})\big)$$

construct a commutative Krasner (m,n)-hyperring, and $(H/J, f, g)$ is said to be the quotient Krasner (m,n)-hyperring of H by J [1].

Now, we determine when the hyperideal I/J is n-ary N-hyperideal in H/J.

Theorem 3.16. *Let I and J be two hyperideals of a commutative Krasner (m,n)-hyperring H with $J \subseteq I$. If I is an n-ary N-hyperideal of H, then I/J is an n-ary N-hyperideal of H/I.*

Proof. Consider the projection map of H of J, that is, $\theta : H \longrightarrow H/J$, defined by $a \longrightarrow f(a, I, 0^{(m-2)})$. By using Theorem 3.14 (2), we are done. $\square$

Theorem 3.17. *Let I and J be two hyperideals of a commutative Krasner (m,n)-hyperring H with $J \subseteq I$. If I/J is an n-ary N-hyperideal of H/J with $J \subseteq \sqrt{0_H}^{(m,n)}$, then I is an n-ary N-hyperideal of H.*

Proof. Let $g(x_1^n) \in I$ for $x_1^n \in H$ such that $x_i \notin \sqrt{0_H}^{(m,n)}$ for some $1 \le i \le n$. Then we get

$$g\big(f(x_1, J, 0_H^{(m-2)}), \cdots, f(x_n, J, 0_H^{(m-2)})\big) = f\big(g(x_1^n), J, 0_{H/J}^{(m-2)}\big) \in I/J.$$

Note that $f(x_i, J, 0_H^{(m-2)}) \notin \sqrt{0_{H/J}}^{(m,n)}$. Since I/J is an n-ary N-hyperideal of H/J and $f(x_i, J, 0_H^{(m-2)}) \notin \sqrt{0_{H/J}}^{(m,n)}$, we get the result that

$$g\Big(f(x_1, J, 0_H^{(m-2)}), \cdots, f(x_{i-1}, J, 0_H^{(m-2)}), 1_{H/J}, f(x_{i+1}, J, 0_H^{(m-2)}),$$

$$\cdots, f(x_n, J, 0_H^{(m-2)}) \Big) \in I/J$$

which implies $f\Big(g(x_1^{i-1}, 1_H, x_{i+1}^n), J, 0_{H/J}^{(m-2)} \Big) \in I/J$. Therefore $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in I$. Thus I is an n-ary N-hyperideal of H. $\square$

Using Theorem 3.3 and Theorem 3.17, we have the next corollary.

Corollary 3.18. *Let I and J be two hyperideals of a commutative Krasner (m, n)-hyperring H with $J \subseteq I$. If I/J is an n-ary N-hyperideal of H/J such that J is an n-ary N-hyperideal of H, then I is an n-ary N-hyperideal of H.*

Proof. The proof follows from Theorem 3.3 and Theorem 3.17. $\square$

4. *n*-ary *δ*-*N*-hyperideals

Recall from [3] that a function δ is a hyperideal expansion of a commutative Krasner (m, n)-hyperring H if it assigns to each hyperideal I of H a hyperideal $\delta(I)$ of H such that $I \subseteq \delta(I)$ and if $I \subseteq J$ for any hyperideals I, J of H, then $\delta(I) \subseteq \delta(J)$. For instance, $\delta_0(I) = I$, $\delta_1(I) = \sqrt{I}^{(m,n)}$ and $\delta_H(I) = H$ for all hyperideals I of H are hyperideal expansions of H. Moreover, $\delta_q(I/J) = \delta(I)/J$ for expansion function δ of H and for all hyperideals I of H containing hyperideal J is an hyperideal expansion of H. By using a hyperideal expansion δ of H, we present the following definition.

Definition 4.1. *Assume that δ is a hyperideal expansion of a commutative Krasner (m, n)-hyperring H. A proper hyperideal I of H is said to be an n-ary δ-N-hyperideal if $g(x_1^n) \in I$ for $x_1^n \in H$ and $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$ imply that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(I)$.*

Example 4.2. *Assume that $\mathbb{Z}_{12} = \{0, 1, 2, 3, \cdots, 11\}$ is the set of all congruence classes of integers modulo 12 and $\mathbb{Z}_{12}^\star = \{1, 5, 7, 11\}$ is multiplicative subgroup of units $\mathbb{Z}_{12}$. Construct H as $\mathbb{Z}_{12}/\mathbb{Z}_{12}^\star$. Then we have $H = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{6}\}$ in which $\bar{0} = \{0\}$, $\bar{1} = \{1, 5, 7, 11\}$, $\bar{2} = \overline{10} = \{2, 10\}$, $\bar{3} = \bar{9} = \{3, 9\}$, $\bar{4} = \bar{8} = \{4, 8\}$, $\bar{6} = \{6\}$. Consider Krasner hyperring $(H, \boxplus, \circ)$ that for all $\bar{x}, \bar{y} \in H$, $\bar{x} \circ \bar{y} = \overline{xy}$ and 2-ary hyperoperation $\boxplus$ is defined as follows*

$\boxplus$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{6}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{6}$
$\bar{1}$	$\bar{1}$	$\bar{0}, \bar{2}, \bar{4}, \bar{6}$	$\bar{1}, \bar{3}$	$\bar{2}, \bar{4}$	$\bar{1}, \bar{3}$	$\bar{1}$
$\bar{2}$	$\bar{2}$	$\bar{1}, \bar{3}$	$\bar{0}, \bar{4}$	$\bar{1}$	$\bar{2}, \bar{6}$	$\bar{4}$
$\bar{3}$	$\bar{3}$	$\bar{2}, \bar{4}$	$\bar{1}$	$\bar{0}, \bar{6}$	$\bar{1}$	$\bar{3}$
$\bar{4}$	$\bar{4}$	$\bar{1}, \bar{3}$	$\bar{2}, \bar{6}$	$\bar{1}$	$\bar{0}, \bar{4}$	$\bar{2}$
$\bar{6}$	$\bar{6}$	$\bar{1}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{0}$

In the hyperring, $\sqrt{0}^{(2,2)} = \{\bar{0}, \bar{6}\}$ and $I = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is a 2-ary δ_1-N-hyperring.

Theorem 4.3. *Let I be an n-ary δ-primary hyperideal of a commutative Krasner (m, n)-hyperring H such that $I \subseteq \sqrt{0}^{(m,n)}$. Then I is an n-ary δ-N-hyperideal of H.*

Proof. Let $g(x_1^n) \in I$ for $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$. By the assumption, $x_i \notin I$. Since I is an n-ary δ-primary hyperideal of H, we conclude that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(I)$. Thus I is an n-ary δ-N-hyperideal of H. $\square$

The next Theorem shows that the inverse of Theorem 4.3 is true if $I = \sqrt{0}^{(m,n)}$.

Theorem 4.4. *Let $\sqrt{0}^{(m,n)}$ be an n-ary δ-N-hyperideal of a commutative Krasner (m,n)-hyperring H. Then $\sqrt{0}^{(m,n)}$ is an n-ary δ-primary hyperideal of H.*

Proof. Let $g(x_1^n) \in \sqrt{0}^{(m,n)}$ for some $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$. Since $\sqrt{0}^{(m,n)}$ is an n-ary δ-N-hyperideal of H, we get the result that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(\sqrt{0}^{(m,n)})$, as needed. $\square$

Theorem 4.5. *Let I be a proper hyperideal of a commutative Krasner (m,n)-hyperring H. Then the followings are equivalent:*

(1) *I is an n-ary δ-N-hyperideal of H.*

(2) *$E_x \subseteq \sqrt{0}^{(m,n)}$ for all $x \notin \delta(I)$ where $E_x = \left\{ y \in H \mid g(x, y, 1_H^{(n-2)}) \in I \right\}$.*

(3) *If $g(x, I_1^{n-1}) \subseteq I$ for some hyperideals I_1^{n-1} of H and for some $x \in H$ implies $x \in \sqrt{0}^{(m,n)}$ or $g(1_H, I_1^{n-1}) \subseteq \delta(I)$.*

(4) *If $g(I_1^n) \subseteq I$ for some hyperideals I_1^{n-1} of H, then $I_i \cap (H - \sqrt{0}^{(m,n)}) = \varnothing$ for some $1 \le i \le n$ or $g(I_1^{i-1}, 1_H, I_{i+1}^n) \subseteq \delta(I)$.*

Proof. (1) $\implies$ (2) Assume that $y \in E_x$. So $g(y, x, 1_H^{(n-2)}) \in I$. Since I is an n-ary δ-N-hyperideal of H and $x = g(x, 1_H^{(n-2)}) \notin \delta(I)$, we obtain $y \in \sqrt{0}^{(m,n)}$ which means $E_x \subseteq \sqrt{0}^{(m,n)}$.
(2) $\implies$ (3) Let $g(x, I_1^{n-1}) \subseteq I$ for some hyperideals I_1^{n-1} of H and for some $x \in H$ such that $g(1_H, I_1^{n-1}) \not\subseteq \delta(I)$. Hence there exist $a_1 \in I_1, \cdots, a_{n-1} \in I_{n-1}$ such that $g(1_H, a_1^{n-1}) \notin \delta(I)$. Since $x \in E_{g(1_H, a_1^{n-1})}$, we conclude that $x \in \sqrt{0}^{(m,n)}$.
(3) $\implies$ (4) Let $g(I_1^n) \subseteq I$ for some hyperideals I_1^{n-1} of H such that $I_i \cap (H - \sqrt{0}^{(m,n)}) \ne \varnothing$ for some $1 \le i \le n$. Therefore we have some $x \in I_i \cap (H - \sqrt{0}^{(m,n)})$. Since $g(I_1^{i-1}, x, I_{i+1}^n) \subseteq I$, we get $g(I_1^{i-1}, 1_H, I_{i+1}^n) \subseteq \delta(I)$, as $x \notin \sqrt{0}^{(m,n)}$.
(4) $\implies$ (1) Suppose that $g(x_1^n) \in I$ for $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$. Let us consider $I_1 = \langle x_1 \rangle, \cdots, I_n = \langle x_n \rangle$. Since $I_i \cap (H - \sqrt{0}^{(m,n)}) \ne \varnothing$, we conclude that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in g(I_1^{i-1}, 1_H, I_{i+1}^n) \subseteq \delta(I)$. Thus I is an n-ary δ-N-hyperideal of H. $\square$

Theorem 4.6. *Let I be an n-ary δ-N-hyperideal of a commutative Krasner (m,n)-hyperring H and let $x \notin \delta(I)$. If $F_x \subseteq \delta(E_x)$ where $F_x = \left\{ y \in H \mid g(y, x, 1_H^{(n-2)}) \in \delta(I) \right\}$ and $E_x = \left\{ y \in H \mid g(y, x, 1_H^{(n-2)}) \in I \right\}$, then E_x is an n-ary δ-N-hyperideal of H.*

Proof. Let $g(x_1^n) \in E_x$ for $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$. Hence $g\big(g(x_1^n), x, 1_H^{(n-2)}\big) = g\big(x_i, g(x_1^{i-1}, x, x_{i+1}^n), 1_H^{(n-2)}\big) \in I$. Since I is an n-ary δ-N-hyperideal of H, we have $g(x_1^{i-1}, x, x_{i+1}^n) \in \delta(I)$. Hence $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in F_x \subseteq \delta(E_x)$, as needed. $\square$

Recall that a hyperideal expansion δ of H is intersection preserving if it satisfies $\delta(I_1 \cap I_2) = \delta(I_1) \cap \delta(I_2)$ for each hyperideals I_1 and I_2 of a commutative Krasner (m,n)-hyperring H. For instance, the hyperideal expansion δ_1 of H is intersection preserving.

Theorem 4.7. *Let I_1^n be some n-ary δ-N-hyperideals of a Krasner (m,n)-hyperring H and let the hyperideal expansion δ of H be intersection preserving. Then $\bigcap_{i=1}^n I_i$ is an n-ary δ-N-hyperideal of H.*

Proof. Put $I = \bigcap_{i=1}^n I_i$. Suppose that $g(x_1^n) \in I$ for $x_1^n \in H$ with $g(x_1^{i-1}, 1_H, x_{i+1}^n) \notin \delta(I)$. Since the hyperideal expansion δ of H is intersection preserving, then there exists $1 \le t \le n$ with $g(x_1^{i-1}, 1_H, x_{i+1}^n) \notin \delta(I_t)$. As I_t ia an n-ary δ-N-hyperideal of H, we have $x_i \in \sqrt{0}^{(m,n)}$. Therefore $I = \bigcap_{i=1}^n I_i$ is an n-ary δ-N-hyperideal of H. $\square$

Theorem 4.8. *Suppose that I_1, I_2 and I_3 are proper hyperideals of a commutative Krasner (m,n)-hyperring H with $I_1 \subseteq I_2 \subseteq I_3$ and $\delta(I_1) = \delta(I_3)$. If I_3 is an n-ary δ-N-hyperideal of H, then I_2 is an n-ary δ-N-hyperideal of H.*

Proof. Let $g(x_1^n) \in I_2$ for some $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$. Since I_3 is an n-ary δ-N-hyperideal of H and $I_2 \subseteq I_3$, then we conclude that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(I_3)$. Then we get $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(I_1)$. Therefore $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(I_2)$, as $I_1 \subseteq I_2$. Thus I_2 is an n-ary δ-N-hyperideal of H. $\square$

Theorem 4.9. *Suppose that I is a proper hyperideal of a commutative Krasner (m,n)-hyperring H such that $\delta(I)$ is an n-ary N-hyperideal of H. Then I is an n-ary δ-N-hyperideal of H.*

Proof. Let $\delta(I)$ be an n-ary N-hyperideal of H. Let $g(x_1^n) \in I$ for $x_1^n \in H$ with $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \leq i \leq n$. Since $I \subseteq \delta(I)$ and $\delta(I)$ is an n-ary N-hyperideal of H, then $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(I)$. It follows that I is an n-ary δ-N-hyperideal of H. $\square$

The inverse of the previous theorem is true if $\delta = \delta_1$. See the next Theorem.

Theorem 4.10. *If I is an n-ary δ_1-N-hyperideal of a commutative Krasner (m,n)-hyperring H, then $\delta_1(I)$ is an n-ary N-hyperideal of H.*

Proof. Let $g(x_1^n) \in \delta_1(I)$ for $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$. It implies that there exists $t \in \mathbb{N}$ such that if $t \leq n$, then $g\big(g(x_1^n)^{(t)}, 1_H^{(n-t)}\big) \in I$. Therefore we get

$$g\Big(x_i^{(t)}, g(x_i^{i-1}, 1_H, x_{i+1}^n)^{(t)}, 1_H^{(n-2t)}\Big)$$

$$= g\Big(x_i^{(t)}, g(x_i^{i-1}, 1_H, x_{i+1}^n)^{(t)}, g(1_R^{(n)}), 1_H^{(n-2t-1)}\Big)$$
$$= g\Big(g(x_i^{(t)}, 1_H^{(n-t)}), g(x_i^{i-1}, 1_H, x_{i+1}^n)^{(t)}, 1_H^{(n-t-1)}\Big)$$
$$\subseteq I.$$

Since I is an n-ary δ_1-N-hyperideal of H and $g(x_i^{(t)}, 1^{(n-t)}) \notin \sqrt{0}^{(m,n)}$, we get the result that $g\big(g(x_i^{i-1}, 1_H, x_{i+1}^n)^{(t)}, 1_H^{(n-t)}\big) \in \delta_1(I)$ and so $g(x_i^{i-1}, 1_H, x_{i+1}^n) \in \delta_1(I)$. It follows that $\delta_1(I)$ is an n-ary N-hyperideal of H. By using a similar argument, one can easily complete the proof where $t = l(n-1) + 1$. $\square$

Theorem 4.11. *Suppose that $\sqrt{0}^{(m,n)}$ is the only maximal hyperideal of a commutative Krasner (m,n)-hyperring H. Then for all $a \in H$, $\langle a \rangle$ is an n-ary hyperideal of H.*

Proof. Assume that $\sqrt{0}^{(m,n)}$ is the only maximal hyperideal of a commutative Krasner (m,n)-hyperring H. Let $a \in H$ and $g(a_1^n) \in \langle a \rangle$ for $a_1^n \in H$ such that $a_i \notin \langle a \rangle$ for some $1 \leq i \leq n$. From $a_i \notin \langle a \rangle$, it follows that a_i is an invertible element. Then we have $g(a_1^{i-1}, 1_H, a_{i+1}^n) \in \langle a \rangle \subseteq \delta\big(\langle a \rangle\big)$ which means that $\langle a \rangle$ is an n-ary hyperideal of H. $\square$

Theorem 4.12. *Let I be a hyperideal of a commutative Krasner (m,n)-hyperring H and δ be a a hyperideal expansion of H such that $\delta\big(\delta(I)\big) = \delta(I)$. If I is an n-ary δ-N-hyperideal of H and $x \notin \sqrt{0}^{(m,n)}$, then $\delta(E_x) = \delta(I)$ where $E_x = \Big\{ y \in H \mid g(y, x, 1_H^{(n-2)}) \in I \Big\}$.*

Proof. Assume that I is an n-ary δ-N-hyperideal of H and $x \notin \sqrt{0}^{(m,n)}$. Let $y \in E_x$. This means $g(y, x, 1_H^{(n-2)}) \in I$. Since I is an n-ary hyperideal of H and $x \notin \sqrt{0}^{(m,n)}$, we get $y = (y, 1^{(n-1)}) \in \delta(I)$ which implies $E_x \subseteq \delta(I)$ and so $\delta(E_x) \subseteq \delta\big(\delta(I)\big)$. By the assumption we have $\delta(E_x) \subseteq \delta(I)$. On the other hand, we have $I \subseteq E_x$ and so $\delta(I) \subseteq \delta(E_x)$. Consequently, $\delta(E_x) \subseteq \delta(I)$. $\square$

Theorem 4.13. *Suppose that I is a maximal n-ary δ-N-hyperideal of a commutative Krasner (m,n)-hyperring H and $x \notin \delta(I)$. If $F_x \subseteq \delta(E_x)$ where $F_x = \left\{ y \in H \mid g(y, x, 1_H^{(n-2)}) \in \delta(I) \right\}$ and $E_x = \left\{ y \in H \mid g(y, x, 1_H^{(n-2)}) \in I \right\}$, then $I = \sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H.*

Proof. Let I be a maximal n-ary δ-N-hyperideal of H. Assume that $g(a_1^n) \in I$ for $a_1^n \in H$ such that $a_i \notin I$ for some $1 \leq i \leq n$. Then, by Theorem 4.6, E_x is an n-ary δ-N-hyperideal of H. Therefore $E_x = I$ by the maximality of I. Thus we have $g(a_1^{i-1} 1_H, a_{i+1}^n) \in I$. Since I is a maximal n-ary δ-N-hyperideal of H, we can continue the process and so get $a_j \in I$ for some $1 \leq j \leq n$ which implies I is an n-ary prime hyperideal of H. This means that $\sqrt{0}^{(m,n)} \subseteq I$. Now, suppose that $I \nsubseteq \sqrt{0}^{(m,n)}$. Then there exist some $x \in I$ such that $x \notin \sqrt{0}^{(m,n)}$. Since $g(x, 1_H^{(n-1)}) \in I$ and I is an n-ary δ-N-hyperideal of H, then we get the result that $g(1_H^{(n)}) \in I$ which is a contradiction. Then we conclude that $I = \sqrt{0}^{(m,n)}$. $\square$

Theorem 4.14. *Let γ and δ be two hyperideal expansions of a commutative Krasner (m,n)-hyperring H. If I is an n-ary γ-N-hyperideal of H and $\gamma(J) \subseteq \delta(J)$ for all hyperideals J of H, then I is an n-ary δ-N-hyperideal of H. Moreover, if $\delta(I)$ is an n-ary γ-N-hyperideal of H, then I is an n-ary $\gamma \circ \delta$-N-hyperideal of H.*

Proof. The proof of the first assertion is straightforward. For the second assertion, suppose that $g(x_1^n) \in I$ for $x_1^n \in H$ such that $x_i \notin \sqrt{0}^{(m,n)}$. Therefore we get $g(x_1^n) \in \delta(I)$, as $I \subseteq \delta(I)$. By the assumption, we conclude that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \gamma\big(\delta(I)\big) = \gamma \circ \delta(I)$ which means I is an n-ary $\gamma \circ \delta$-N-hyperideal of H. $\square$

Assume that (H_1, f_1, g_1) and (H_2, f_2, g_2) are two commutative Krasner (m,n)-hyperrings and $h : H_1 \longrightarrow H_2$ a hyperring homomorphism. Let δ and γ be two hyperideal expansions of H_1 and H_2, respectively. Recall from [3] that h is a $\delta\gamma$-homomorphism if $\delta\big(h^{-1}(I_2)\big) = h^{-1}\big(\gamma(I_2)\big)$ for the hyperideal I_2 of H_2. Note that $\gamma\big(h(I_1)\big) = h\big(\delta(I_1)\big)$ for $\delta\gamma$-epimorphism h and for hyperideal I_1 of H_1 such that $Ker(h) \subseteq I_1$.

For instance, suppose that (H_1, f_1, g_1) and (H_2, f_2, g_2) are two Krasner (m,n)-hyperrings. If δ_1 of H_1 and γ_1 of H_2 are the hyperideal expansions defined in Example 3.2 of [3], then all homomorphism $h : H_1 \longrightarrow H_2$ is a $\delta_1\gamma_1$-homomorphism.

Theorem 4.15. *Let δ and γ be two hyperideal expansions of commutative Krasner (m,n)-hyperrings (H_1, f_1, g_1) and (H_2, f_2, g_2), respectively, and let $h : H_1 \longrightarrow H_2$ be a $\delta\gamma$-homomorphism. Then the followings hold :*

(1) *If I_2 is an n-ary γ-N-hyperideal of H_2 and h is a monomorphism, then $h^{-1}(I_2)$ is an n-ary δ-N-hyperideal of H_1.*

(2) *If h is an epimorphism and I_1 is an n-ary δ-N-hyperideal of H_1 containing $Ker(h)$, then $h(I_1)$ is an n-ary γ-N-hyperideal of H_2.*

Proof. (1) Let $g_1(x_1^n) \in h^{-1}(I_2)$ for $x_1^n \in H_1$. It follows that $g_2\big(h(x_1), \cdots, h(x_n)\big) = h\big(g_1(x_1^n)\big) \in I_2$. Since I_2 is an n-ary γ-N-hyperideal of H_2, then $h(x_i) \in \sqrt{0_{H_2}}^{(m,n)}$ for some $1 \leq i \leq n$ or

$$g_2\big(h(x_1), ..., h(x_{i-1}), 1_{H_2}, h(x_{i+1}), ..., h(x_n)\big)$$

$$= h\big(g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n)\big)$$
$$\in \gamma(I_2).$$

In the first possibility, since $Ker(h) = \left\{ 0_{H_1} \right\}$, we obtain $x_i \in \sqrt{0_{H_1}}^{(m,n)}$. In the second possibility, we get the result that $g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n) \in h^{-1}\big(\gamma(I_2)\big) = \delta\big(h^{-1}(I_2)\big)$. Consequently, $h^{-1}(I_2)$ is a δ-N-hyperideal of H_1.

(2) Assume that $g_2(y_1^n) \in h(I_1)$ for $y_1^n \in H_2$ with $y_i \notin \sqrt{0_{H_2}}^{(m,n)}$ for some $1 \leq i \leq n$. Then there exist $x_1^n \in H_1$ such that $h(x_1) = y_1, \cdots, h(x_n) = y_n$ as h is an epimorphism. Therefor $h\big(g_1(x_1^n)\big) = g_2\big(h(x_1), \cdots, h(x_n)\big) =$

$g_2(y_1^n) \in h(I_1)$.

Since I_1 contains $Ker(h)$, we have $g_1(x_1^n) \in I_1$. Since $y_i \notin \sqrt{0_{H_2}}^{(m,n)}$, then $x_i \notin \sqrt{0_{H_1}}^{(m,n)}$. Since I_1 is a δ-N-hyperideal of H_1 and $x_i \notin \sqrt{0_{H_1}}^{(m,n)}$, it follows that $g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n) \in \delta(I_1)$ which implies

$$h\Big(g_1(x_1^{i-1}, 1_{H_1}, x_{i+1}^n)\Big) = g_2\Big(h(x_1), ..., h(x_{i-1}), 1_{H_2}, h(x_{i+1}), ..., h(x_n)\Big)$$

$$= g_2(y_1^{i-1}, 1_{H_2}, y_{i+1}^n)$$
$$\in h\big(\delta(I_1)\big).$$

By the assumption, we have $h\big(\delta(I_1)\big) = \gamma\big(h(I_1)\big)$. So $g_2(y_1^{i-1}, 1_{H_2}, y_{i+1}^n) \in \gamma\big(h(I_1)\big)$. Hence $h(I_1)$ is an n-ary γ-N-hyperideal of H_2. $\quad\square$

Corollary 4.16. *Suppose that I and J are two hyperideals of a commutative Krasner (m,n)-hyperring H with $J \subseteq I$. If I is an n-ary δ-N-hyperideal of H, then I/J is an n-ary δ_q-J-hyperideal of H/J.*

Proof. The claim follows by using Theorem 4.15 (2) and by a similar argument to that of 4.18. $\quad\square$

The next theorem shows that for an n-ary δ-N-hyperideal I of H if $\delta(\sqrt{I}^{(m,n)})$ contains $\sqrt{\delta(I)}^{(m,n)}$, then $\sqrt{I}^{(m,n)}$ is an n-ary δ-N-hyperideal of H.

Theorem 4.17. *Suppose that I is an n-ary δ-N-hyperideal of a commutative Krasner (m,n)-hyperring H with $\sqrt{\delta(I)}^{(m,n)} \subseteq \delta(\sqrt{I}^{(m,n)})$. Then $\sqrt{I}^{(m,n)}$ is an n-ary δ-N-hyperideal of H.*

Proof. Assume that $g(x_1^n) \in \sqrt{I}^{(m,n)}$ for $x_1^n \in H$ with $x_i \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$. Then there exists $t \in \mathbb{N}$ such that if $t \le n$ implies that $g\big(g(x_1^n)^{(t)}, 1_H^{(n-t)}\big) \in I$. Then we get the result that

$$g\Big(x_i^{(t)}, g(x_1^{i-1}, 1_H, x_{i+1}^n)^{(t)}, 1_H^{(n-2t)}\Big) = g\Big(g(x_i^{(t)}, 1_H^{(n-t)}), g(x_1^{i-1}, 1_H, x_{i+1}^n)^{(t)},$$
$$1_H^{(n-t-1)}\Big) \subseteq I.$$

Since $g(x_i^{(t)}, 1^{(n-t)}) \notin \sqrt{0}^{(m,n)}$ and I is an n-ary δ-N-hyperideal of H, then we have $g\big(g(x_1^{i-1}, 1_H, x_{i+1}^n)^{(t)}, 1_H^{(n-t)}\big) \in \delta(I)$. It follows that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \sqrt{\delta(I)}^{(m,n)}$. Since $\sqrt{\delta(I)}^{(m,n)} \subseteq \delta(\sqrt{I}^{(m,n)})$, then $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in \delta(\sqrt{I}^{(m,n)})$, as needed. A similar argument will show that if $t = l(n-1) + 1$, then $\sqrt{I}^{(m,n)}$ is an n-ary δ-N-hyperideal of H. $\quad\square$

Recall from [1] that a non-empty subset S of a Krasner (m,n)-hyperring H is said to be an n-ary multiplicative, if $g(t_1^n) \in S$ for $t_1, \cdots, t_n \in S$. The notion of Krasner (m,n)-hyperring of fractions was introduced in [4]. Assume that δ is a hyperideal expansion of a commutative Krasner (m,n)-hyperring H and S is an n-ary multiplicative subset of H with $1 \in S$. Then δ_S is a hyperideal expansion of $S^{-1}H$ with $\delta_S(S^{-1}I) = S^{-1}\big(\delta(I)\big)$.

Theorem 4.18. *Suppose that S is an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H with $1 \in S$. If I is an n-ary δ-N-hyperideal of H and $I \cap S = \varnothing$, then $S^{-1}I$ is an n-ary δ_S-N-hyperideal of $S^{-1}H$.*

Proof. Let I be an n-ary δ-N-hyperideal of H. Assume that $G(\frac{x_1}{s_1}, ..., \frac{x_n}{s_n}) \in S^{-1}I$ for $\frac{x_1}{s_1}, ..., \frac{x_n}{s_n} \in S^{-1}H$ such that $\frac{x_i}{s_i} \notin \sqrt{0_{S^{-1}H}}^{(m,n)}$ for some $1 \le i \le n$. Hence $\frac{g(x_1^n)}{g(s_1^n)} \in S^{-1}I$. This means that there exists $t \in S$ such that $g\big(t, g(x_1^n), 1_H^{(n-2)}\big) \in I$ and then $g\big(x_i, g(x_1^{i-1}, t, x_{i+1}^n), 1_H^{(n-2)}\big) \in I$. Since $x_i \notin \sqrt{0}^{(m,n)}$ and I is an n-ary δ-N-hyperideal of H, then we get the result that $g(x_1^{i-1}, t, x_{i+1}^n) \in \delta(I)$. Therefore $G\Big(\frac{x_1}{s_1}, \cdots, \frac{x_{i-1}}{s_{i-1}}, \frac{1_H}{1_H}, \frac{x_{i+1}}{s_{i+1}}, \cdots, \frac{x_n}{s_n}\Big) = \frac{g(x_1^{i-1}, 1_H, x_{i+1}^n)}{g(s_1^{i-1}, 1_H, s_{i+1}^n)} = \frac{g(x_1^{i-1}, t, x_{i+1}^n)}{g(s_1^{i-1}, t, s_{i+1}^n)} \in S^{-1}\big(\delta(I)\big) = \delta_S\big(S^{-1}(I)\big)$. Consequently, $S^{-1}I$ is an n-ary δ_S-N-hyperideal of $S^{-1}H$. $\quad\square$

5. (k, n)-absorbing δ-N-hyperideals

In his paper [14], Tamekkante et al. introduced a generalization of n-ideals of commutative rings, which they defined as $(2, n)$-ideals. In this section, we generalize this concept for hyperrings by using the notion given in previous section.

Definition 5.1. *Given a hyperideal expansion δ, a proper hyperideal I of a commutative Krasner (m, n)-hyperring H is called (k, n)-absorbing N-hyperideal if $g(a_1^{kn-k+1}) \in I$ for $a_1^{kn-k+1} \in H$ implies that $g(a_1^{(k-1)n-k+2}) \in \sqrt{0}^{(m,n)}$ or a g-product of $(k-1)n - k + 2$ of $a_i's$ except $g(a_1^{(k-1)n-k+2})$ is in $\delta(I)$.*

Example 5.2. *If we continue with Example 4.2 and use its notation, then $I = \{0, 3, 6\}$ is a $(2, 2)$-absorbing δ_1-N-hyperring of $\mathbb{Z}_{12}/\mathbb{Z}_{12}^{\star}$.*

Theorem 5.3. *Let I be a (k, n)-absorbing N-hyperideal of a commutative Krasner (m, n)-hyperring H. Then $\sqrt{I}^{(m,n)}$ is a (k, n)-absorbing δ-N-hyperideal.*

Proof. Assume that $g(a_1^{kn-k+1}) \in \sqrt{I}^{(m,n)}$ for $a_1^{kn-k+1} \in H$. We presume none of the g-products of $(k-1)n - k + 2$ of the $a_i's$ other than $g(a_1^{(k-1)n-k+2})$ are in $\delta(\sqrt{I}^{(m,n)})$. Since $g(a_1^{kn-k+1}) \in \sqrt{I}^{(m,n)}$, then for some $t \in \mathbb{N}$ we have for $t \leq n$, $g\left(g(a_1^{kn-k+1})^{(t)}, 1_H^{(n-t)}\right) \in I$ or for $t > n$ with $t = l(n-1) + 1$, $g_{(l)}\left(g(a_1^{kn-k+1})^{(t)}\right) \in I$. In the first possibilty, since all g-products of the $a_i's$ other than $g(a_1^{(k-1)n-k+2})$ are not in $\delta(\sqrt{I}^{(m,n)})$, then they are not in I. Since I is a (k, n)-absorbing J-hyperideal of H, then we have $g\left(g(x_1^{(k-1)n-k+2})^{(l)}, 1_H^{(n-t)}\right) \in \sqrt{0}^{(m,n)}$ which means $g(x_1^{(k-1)n-k+2}) \in \sqrt{0}^{(m,n)}$. In the second possibilty, the claim follows by using a similar argument. $\square$

Theorem 5.4. *Suppose that I is a hyperideal of a commutative Krasner (m, n)-hyperring H such that $\delta(I)$ is a $(2, n)$-absorbing N-hyperideal. Then I is a $(3, n)$-absorbing δ-N-hyperideal of H.*

Proof. Let $g(a_1^{3n-2}) \in I$ for $a_1^{3n-2} \in H$ but $g(a_1^{2n-1}) \notin \sqrt{0}^{(m,n)}$. This means that $g\left(g(a_1, a_{2n}^{3n-2}), a_2^{2n-1}\right) \in I \subseteq \delta(I)$. Since $\delta(I)$ is a $(2, n)$-absorbing N-hyperideal of H and $g(a_2^{2n-1}) \notin \sqrt{0}^{(m,n)}$, then we get the result that $g(a_1^n, a_{2n}^{3n-2}) \in \delta(I)$ or $g(a_1, a_{n+1}^{2n-1}, a_{2n}^{3n-2}) \in \delta(I)$. Consequently, I is a $(3, n)$-absorbing δ-N-hyperideal of H. $\square$

Theorem 5.5. *Assume that I is a hyperideal of a commutative Krasner (m, n)-hyperring H such that $\delta(I)$ is a $(k + 1, n)$-absorbing δ-N-hyperideal of H. Then I is a $(k + 1, n)$-absorbing δ-N-hyperideal of H.*

Proof. Suppose that $g(a_1^{(k+1)n-(k+1)+1}) \in I$ for $a_1^{(k+1)n-(k+1)+1} \in H$ such that $g(a_1^{kn-k+1}) \notin \sqrt{0}^{(m,n)}$. So $g(a_1^{(k+1)n-(k+1)+1}) = g\left(a_1^{kn-k}, g(a_{kn-k+1}^{(k+1)n-(k+1)+1})\right) \in I \subseteq \delta(I)$. Since $\delta(I)$ is a $(k + 1, n)$-absorbing δ-N-hyperideal and $g(a_1^{kn-k+1}) \notin \sqrt{0}^{(m,n)}$, we get the result that $g\left(a_1^{i-1}, a_{i+1}^{kn-k}, g(a_{kn-k+1}^{(k+1)n-(k+1)+1})\right) \in \delta(I)$ for $1 \leq i \leq n$. Thus I is a $(k + 1, n)$-absorbing δ-N-hyperideal of H. $\square$

Theorem 5.6. *Let I be a δ-N-hyperideal of a commutative Krasner (m, n)-hyperring H. Then I is a $(2, n)$-absorbing δ-N-hyperideal of H.*

Proof. Let $g(a_1^{2n-1}) \in I$ for $a_1^{2n-1} \in H$. Since I is a δ-N-hyperideal of H, we get the result that $g(a_1^n) \in \sqrt{0}^{(m,n)}$ or $g(a_{n+1}^{2n-1}) \in \delta(I)$. Therefore we have $g(a_i, x_{n+1}^{2n-1}) \in \delta(I)$, for $1 \leq i \leq n$, as $\delta(I)$ is a hyperideal of H. Thus I is $(2, n)$-absorbing δ-primary. $\square$

Next, we determine all integers $k > n$.

Theorem 5.7. *Let I is a (k, n)-absorbing δ-N-hyperideal of a commutative Krasner (m, n)-hyperring H. Then I is (s, n)-absorbing δ-N-hyperideal for $s > n$.*

Proof. Let $g(a_1^{(k+1)n-(k+1)+1}) \in I$ for $a_1^{(k+1)n-(k+1)+1} \in H$. Put $g(a_1^{n+2}) = a$. Since I is (k,n)-absorbing δ-N-hyperideal, then we obtain $g(a, \cdots, a_{(k+1)n-(k+1)+1}) \in \sqrt{0}^{(m,n)}$ or a g-product of $kn-k+1$ of the a_i's except $g(a, \cdots, a_{(k+1)n-(k+1)+1})$ is in $\delta(I)$. This implies that $g(a_i, a_{n+3}^{(k+1)n-(k+1)+1}) \in \delta(I)$ for all $1 \le i \le n+2$ which implies I is a $(k+1, n)$-absorbing δ-N-hyperideal. Consequently, I is an (s,n)-absorbing δ-N-hyperideal for $s > n$. $\square$

6. *n*-ary *S*-*N*-hyperideals

In this section, we study the notion of n-ary S-N-hyperideals of a Krasner (m,n)-hyperring H, where S is an n-ary multiplicative subset of H. The following definition constitutes the S-version of the class of N-hyperideals.

Definition 6.1. *Assume that S is an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H. A hyperideal I of R with $I \cap S = \varnothing$ is said to be an n-ary S-N-hyperideal if there exists an $s \in S$ such that for all $x_1^n \in H$ if $g(x_1^n) \in I$ with $g(s, x_i, 1_H^{(n-2)}) \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$, then $g(x_1^{i-1}, s, x_{i+1}^n) \in I$. This element s in S is called an S-element of I.*

Now we give a charactrization of an n-ary S-N-hyperideal.

Theorem 6.2. *Assume that S is an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H and I is a hyperideal of H disjoint with S. Then I is an n-ary S-N-hyperideal of H if and only if there exists $s \in S$, for all hyperideals I_1^n of H, if $g(I_1^n) \subseteq I$, then $g(s, I_i, 1^{(n-1)}) \subseteq \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$ or $g(I_1^{i-1}, s, I_{i+1}^n) \subseteq I$.*

Proof. $\Longrightarrow$ Let I be an n-ary S-N-hyperideal of H. Suppose that $g(I_1^n) \in I$ for some hyperideals I_1^n of H such that $g(s, I_i, 1_H^{(n-2)}) \nsubseteq \sqrt{0}^{(m,n)}$ and $g(I_1^{i-1}, s, I_{i+1}^n) \nsubseteq I$ for all $s \in S$. Then there exists $a_i \in I_i$ for each $1 \le i \le n$ such that $g(a_1^n) \in I$ but $g(s, a_i, 1_H^{(n-2)}) \notin \sqrt{0}^{(m,n)}$ and $g(a_1^{i-1}, s, a_{i+1}^n) \notin I$, a contradiction.
$\Longleftarrow$ Suppose that $g(x_1^n) \in I$ for some $x_1^n \in H$. Then $g(\langle x_1 \rangle, \cdots, \langle x_n \rangle) \subseteq I$. By the assumption, we are done. $\square$

Theorem 6.3. *Let S be an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H. If I_1^n are some n-ary S-N-hyperideals of H, then $\bigcap_{t=1}^n I_t$ is an n-ary S-N-hyperideal of H.*

Proof. Let I_1^n be n-ary S-N-hyperideals of H. Suppose that for each $1 \le t \le n$, there exists $s_t \in S$ such that if $g(a_1^n) \in I_t$ for some $a_1^n \in H$, then $g(s_t, a_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$ or $g(a_1^{i-1}, s_t, a_{i+1}^n) \in I_t$. Now, assume that $g(a_1^n) \in \bigcap_{t=1}^n I_t$ for some $a_1^n \in H$. This means that $g(a_1^n) \in I_t$ for each $1 \le t \le n$. Put $\prod_{t=1}^n s_t \in S$. Then we get the result thet $g(s, a_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$ or $g(a_1^{i-1}, s, a_{i+1}^n) \in \bigcap_{t=1}^n I_t$, as claimed. $\square$

Theorem 6.4. *Assume that S is an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H and I is a hyperideal of H such that $I \cap S = \varnothing$. If $E_s = \left\{ y \in H \mid g(y, s, 1_H^{(n-2)}) \in I \right\}$ is an n-ary N-hyperideal of H for some $s \in S$, then I is an n-ary S-N-hyperideal of H.*

Proof. Let E_s be an n-ary N-hyperideal of H for some $s \in S$. Suppose that $g(x_1^n) \in I$ for $x_1^n \in H$ such that $g(s, x_i, 1_H^{(n-2)}) \notin \sqrt{0}^{(m,n)}$ for some $1 \le i \le n$. Therefore $g\left(g(x_1^n), s, 1_H^{(n-2)}\right) \in I$ and so $g(x_1^n) \in E_s$. Since E_s is an n-ary N-hyperideal of H for some $s \in S$ and $x_i \notin \sqrt{0}^{(m,n)}$, we get the result that $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in E_s$ which means $g\left(g(x_1^{i-1}, 1_H, x_{i+1}^n), s, 1_H^{(n-2)}\right) = g(x_1^{i-1}, s, x_{i+1}^n) \in I$. Consequently, I is an n-ary S-N-hyperideal of H. $\square$

In the following theorem, we determine a condition on I when the converse holds.

Theorem 6.5. *Assume that S is an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H and I is a hyperideal of H such that $I \cap S = \varnothing$. If I is an n-ary S-N-hyperideal of H and $F_s = \left\{ y \in H \mid g(y, s, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)} \right\}$ is an n-ary N-hyperideal for an S-element $s \in S$ of I, then $E_s = \left\{ y \in H \mid g(y, s, 1_H^{(n-2)}) \in I \right\}$ is an n-ary N-hyperideal of H.*

Proof. Let $g(x_1^n) \in E_s$ for $x_1^n \in H$. Then we have $g\big(g(x_1^n), s, 1_H^{(n-2)}\big) \in I$ and so $g\big(x_i, g(x_1^{i-1}, s, x_{i+1}^n), 1_H^{(n-2)}\big) \in I$. Since I is an n-ary S-N-hyperideal of H, we get the result that either $g(s, x_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$ or $g\big(g(x_1^{i-1}, s, x_{i+1}^n), s, 1_H^{(n-2)}\big) = g\big(g(x_1^{i-1}, 1_H, x_{i+1}^n), g(s^{(2)}, 1_H^{(n-2)}), 1_H^{(n-2)}\big) \in I$. In the former case, by Theorem 3.3 we conclude that $F_s = \sqrt{0}^{(m,n)}$, as F_s is an n-ary N-hyperideal. Therefore $x_i \in \sqrt{0}^{(m,n)}$. In the second case, suppose that $g(x_1^{i-1}, s, x_{i+1}^n) \notin I$. Then we obtain $g\big(g(s^{(2}, 1_H^{(n-2)}), s, 1_H^{(n-2)}\big) = g(s^{(3)}, 1_H^{(n-3)}) \in \sqrt{0}^{(m,n)}$. It implies that $s \in \sqrt{0}^{(m,n)}$, a contradiction. Then we conclude that $g(x_1^{i-1}, s, x_{i+1}^n) \in I$ which means $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in E_s$ which implies E_s is an n-ary N-hyperideal of H. $\square$

Theorem 6.6. *Let $S \subseteq S'$ be two n-ary multiplicative subsets of a commutative Krasner (m,n)-hyperring H and I be an n-ary S'-N-hyperideal of H. If for each $s \in S'$, there is an element $s' \in S'$ with $g(s, s', 1_H^{(n-2)}) \in S$, then I is an n-ary S-N-hyperideal of H.*

Proof. Let $g(a_1^n) \in I$. Since I is an n-ary S'-N-hyperideal of H, we have either $g(s, a_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$ or $g(a_1^{i-1}, s, a_{i+1}^n) \in I$ for a S'-element $s \in S'$ of I. By the assumption, there exists $s' \in S'$ such that $s'' = g(s, s', 1_H^{(n-2)}) \in S$. From $g(s, a_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$, it follows that $g\big(s', g(s, a_i, 1_H^{(n-2)}), 1_H^{(n-2)}\big) \in \sqrt{0}^{(m,n)}$. Also, from $g(a_1^{i-1}, s, a_{i+1}^n) \in I$ it follows that $g\big(s', g(a_1^{i-1}, s, a_{i+1}^n), 1_H^{(n-2)}\big) \in I$. Consequently, we get the result that $g(s'', a_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$ or $g(a_1^{i-1}, s'', a_{i+1}^n) \in I$, as needed. $\square$

Theorem 6.7. *Let $S \subseteq S'$ be two n-ary multiplicative subsets of a commutative Krasner (m,n)-hyperring H such that $1_H \in S$ and I be a hyperideal of H with $I \cap S' = \varnothing$. If I is an n-ary S-N-hyperideal of H, then $S'^{-1}I$ is an n-ary $S'^{-1}S$-N-hyperideal of $S'^{-1}H$ and $S'^{-1}I \cap H = E_s$ where $E_s = \big\{ y \in H \mid g(y, s, 1^{(n-2)}) \in I \big\}$ and s is an S-element of I.*

Proof. Let I be an n-ary S-N-hyperideal of H. It is easy to see that $S'^{-1}S \cap S'^{-1}I = \varnothing$. Assume that $\frac{s}{1_H} \in S'^{-1}S$ for some S-element s of I. Let $G(\frac{a_1}{s_1}, \cdots, \frac{a_n}{s_n}) \in S'^{-1}I$ for $a_1^n \in H$ and $s_1^n \in S'$ such that $G(\frac{s}{1_H}, \frac{a_i}{s_i}, \frac{1_H}{1_H}^{(n-2)}) \notin \sqrt{0_{S'}^{-1}H}^{(m,n)}$ for some $1 \le i \le n$. So $\frac{g(a_1^n)}{g(s_1^n)} \in S'^{-1}I$ which follows there exists $t \in S'$ such that $g(t, g(a_1^n), 1_H^{(n-2)}) \in I$ and then $g\big(a_i, g(a_1^{i-1}, t, a_{i+1}^n), 1_H^{(n-2)}\big) \in I$. Since I is an n-ary S-N-hyperideal of H and $g(s, a_i, 1_H^{(n-2)}) \notin \sqrt{0_H}^{(m,n)}$, we get the result that $g\big(s, g(a_1^{i-1}, t, a_{i+1}^n), 1_H^{(n-2)}\big) \in I$ which means $G\big(\frac{a_1}{s_1}, \cdots, \frac{a_{i-1}}{s_{i-1}}, \frac{s}{1_H}, \frac{a_{i+1}}{s_{i+1}}, \cdots, \frac{a_n}{s_n}\big) = \frac{g(a_1^{i-1}, s, a_{i+1}^n)}{g(s_1^{i-1}, 1_H, s_{i+1}^n)} = \frac{g(s, g(a_1^{i-1}, t, a_{i+1}^n), 1^{(n-2)})}{g(s_1^{i-1}, t, s_{i+1}^n)} \in S'^{-1}I$. Thus $S'^{-1}I$ is an n-ary $S'^{-1}S$-N-hyperideal of $S'^{-1}H$. For the second assertion, suppose that $x \in S'^{-1}I \cap H$. Then there exists $a \in I$ such that $\frac{x}{1_H} = \frac{a}{t}$ for some $t \in S'$. Therefore there exists $u \in S'$ such that $g(u, a, 1_H^{(n-2)}) \in I$. Since I is an n-ary S-N-hyperideal of H, then there exists $s \in S \subseteq S'$ such that we have $g(s, u, 1_H^{(n-2)}) \in \sqrt{0_H}^{(m,n)}$ or $g(s, a, 1_H^{(n-2)}) \in I$. In the former case, we have a contradiction since $S' \cap \sqrt{0_H}^{(m,n)} = \varnothing$. Hence $g(s, a, 1_H^{(n-2)}) \in I$ which implies $a \in E_s$ which means $S'^{-1}I \cap H \subseteq E_s$. Since the inclusion $E_s \subseteq S'^{-1}I \cap H$ holds, we get $S'^{-1}I \cap H = E_s$. $\square$

We have the following corollary of Theorem 6.7.

Corollary 6.8. *Let S be an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H such that $1_H \in S$ and I be a hyperideal of H with $I \cap S = \varnothing$. If I is an n-ary S-N-hyperideal of H, then $S^{-1}I$ is an n-ary N-hyperideal of $S^{-1}H$ and $S^{-1}I \cap H = E_s$ where $E_s = \big\{ y \in H \mid g(y, s, 1^{(n-2)}) \in I \big\}$ and s is an S-element of I.*

Proof. Let I be an n-ary S-N-hyperideal of H. Then, by Theorem 6.7, $S^{-1}I$ is an n-ary $S^{-1}S$-N-hyperideal of $S^{-1}H$. Suppose that $G(\frac{a_1}{s_1}, \cdots, \frac{a_n}{s_n}) \in S^{-1}I$ for $a_1^n \in H$ and $s_1^n \in S$. There there exist an $S^{-1}S$-element $\frac{u}{v}$ of $S^{-1}I$ such that $G(\frac{u}{v}, \frac{a_i}{s_i}, \frac{1_H}{1_H}^{(n-2)}) \in \sqrt{0_{S^{-1}H}}^{(m,n)}$ or $G\big(\frac{a_1}{s_1}, \cdots, \frac{a_{i-1}}{s_{i-1}}, \frac{u}{v}, \frac{a_{i+1}}{s_{i+1}}, \cdots, \frac{a_n}{s_n}\big) \in S^{-1}I$ for some $1 \le i \le n$. Then we get

the result that $S^{-1}I$ is an n-ary N-hyperideal of $S^{-1}H$, as $\frac{u}{v}$ is invertible. For the second assertion, we use a similar argument to that of Theorem 6.7. $\square$

From Theorem 6.7 and Corollary 6.8, we can conclude the following.

Theorem 6.9. *Let S be an n-ary multiplicative subset of a commutative Krasner (m,n)-hyperring H such that $1_H \in S$ and I be a hyperideal of H with $I \cap S = \varnothing$. Then I is an n-ary S-N-hyperideal of H if and only if $S^{-1}I$ is an n-ary N-hyperideal of $S^{-1}H$ and $S^{-1}I \cap H = E_s$ and $S^{-1}\sqrt{0}^{(m,n)} \cap H = F_{s'}$ where $E_s = \left\{ y \in H \mid g(y, s, 1^{(n-2)}) \in I \right\}$ and $F_{s'} = \left\{ z \in H \mid g(z, s', 1^{(n-2)}) \in \sqrt{0}^{(m,n)} \right\}$ for some $s, s' \in S$.*

Proof. $\Longrightarrow$ Let I be an n-ary S-N-hyperideal of H. Then, by Corollary 6.8 we conclude that $S^{-1}I$ is an n-ary N-hyperideal of $S^{-1}H$. The rest of the claim follows by a similar argument to that of Theorem 6.7.

$\Longleftarrow$ Assume that $S^{-1}I$ is an n-ary N-hyperideal of $S^{-1}H$, $S^{-1}I \cap H = E_s$ and $S^{-1}\sqrt{0}^{(m,n)} \cap H = F_{s'}$ where $E_s = \left\{ y \in H \mid g(y, s, 1_H^{(n-2)}) \in I \right\}$ and $F_{s'} = \left\{ z \in H \mid g(z, s', 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)} \right\}$ for some $s, s' \in S$. Put $g(s, s', 1_H^{(n-2)}) = s''$. Assume that $g(a_1^n) \in I$ for $a_1^n \in H$. This implies that $G(\frac{a_1}{1_H}, \cdots, \frac{a_n}{1_H}) \in S^{-1}I$. Then we get the result that either $\frac{a_i}{1_H} \in \sqrt{S^{-1}0}^{(m,n)} = S^{-1}\sqrt{0}^{(m,n)}$ or $G\left(\frac{a_1}{1_H}, \cdots, \frac{a_{i-1}}{1_H}, \frac{1_H}{1_H}, \frac{a_{i+1}}{1_H}, \cdots, \frac{a_n}{1_H} \right) \in S^{-1}I$ for some $1 \le i \le n$. In the first case, we get $g(t, a_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$ for some $t \in S$ which implies $a_i = \frac{g(t, a_i, 1_H^{(n-2)})}{g(t, 1_H^{(n-1)})} \in S^{-1}\sqrt{0}^{(m,n)} \cap H = F_{s'}$. This means that $g(s', a_i, 1_H^{(n-2)}) \in \sqrt{0}^{(m,n)}$. Then we obtain $g(s'', a_i, 1_H^{(n-2)}) = g\left(g(s, s', 1_H^{(n-2)}), a_i, 1_H^{(n-2)} \right) \in \sqrt{0}^{(m,n)}$. In the second case, we get $g(a_1^{i-1}, t', a_{i+1}^n) \in I$ for some $t' \in S$. This implies that $g(a_1^{i-1}, 1_H, a_{i+1}^n) = \frac{g(a_1^{i-1}, t', a_{i+1}^n)}{g(t', 1_H^{(n-1)})} \in S^{-1}I \cap H = E_s$. Then we conclude that $g(a_1^{i-1}, s'', a_{i+1}^n) = g\left(a_1^{i-1}, g(s, s', 1_H^{(n-2)}), a_{i+1}^n \right) \in I$. Consequently, I is an n-ary S-N-hyperideal of H. $\square$

Theorem 6.10. *Let S_1 and S_2 be n-ary multiplicative subsets of commutative Krasner (m,n)-hyperrings (H_1, f_1, g_1) and (H_2, f_2, g_2), respectively, and let 1_{H_1} and 1_{H_2} be scalar identitis of H_1 and H_2, respectively. Then the following statements hold.*

(1) *$I_1 \times H_2$ is an n-ary S-N-hyperideal of $H_1 \times H_2$ where $S = S_1 \times S_2$ if and only if I_1 is an n-ary S_1-N-hyperideal of H_1 and $S_2 \cap \sqrt{0_{H_2}} \neq \varnothing$.*

(2) *$H_1 \times I_2$ is an n-ary S-N-hyperideal of $H_1 \times H_2$ where $S = S_1 \times S_2$ if and only if I_2 is an n-ary S_2-N-hyperideal of H_2 and $S_1 \cap \sqrt{0_{H_1}} \neq \varnothing$.*

Proof. $\Longrightarrow$ Let $I_1 \times H_2$ be an n-ary S-N-hyperideal of $H_1 \times H_2$ where $S = S_1 \times S_2$ and let (s_1, s_2) be an S-element of $I_1 \times H_2$. Suppose that $g_1(a_1^n) \in I_1$ for some $a_1^n \in H_1$. So we consider the following cases.
Case 1: Let $g_1(a_1^{i-1}, s_1, a_{i+1}^n) \notin I_1$. Then we get

$$g_1 \times g_2\left((a_1, 1_{H_2}), \cdots, (a_n, 1_{H_2}) \right) \in I_1 \times H_2$$

but

$$g_1 \times g_2\left((a_1, 1_{H_2}), \cdots, (a_{i-1}, 1_{H_2}), (s_1, s_2), (a_{i+1}, 1_{H_2}), \cdots (a_n, 1_{H_2}) \right) \notin I_1 \times H_2.$$

Then we get the result that $g_1 \times g_2\left((s_1, s_2), (a_i, 1_{H_2}), (1_{H_1}, 1_{H_2})^{(n-2)} \right) \in \sqrt{0_{H_1 \times H_2}} = \sqrt{0_{H_1}} \times \sqrt{0_{H_2}}$, as $I_1 \times H_2$ is an n-ary S-N-hyperideal of $H_1 \times H_2$. Hence $g(s_1, a_i, 1_{H_1}^{(n-2)}) \in \sqrt{0_{H_1}}$ and $s_2 \in S_2 \cap \sqrt{0_{H_2}}$, as needed.
Case 2: In this case we suppose that $g_1(a_1^{i-1}, s_1, a_{i+1}^n) \in I_1$. Then we have

$$g_1 \times g_2\left((a_1, 1_{H_2}), \cdots, (a_{i-1}, 1_{H_2}), (s_1, s_2), (a_{i+1}, 1_{H_2}), \cdots (a_n, 1_{H_2}) \right) \in I_1 \times H_2.$$

Since $g_1 \times g_2\big((s_1,s_2)^{(2)}, (1_{H_1},1_{H_2})^{(n-2)}\big) \notin I_1 \times H_2$, we get the result that

$$g_1 \times g_2\big((a_1,1_{H_2}),\cdots,(a_{i-1},1_{H_2}),(s_1,s_2),(a_{i+1},1_{H_2}),\cdots(a_n,1_{H_2})\big)$$
$$\in \sqrt{0_{H_1 \times H_2}}$$
$$= \sqrt{0_{H_1}} \times \sqrt{0_{H_2}}.$$

$\Longleftarrow$ Let I_1 be an n-ary S_1-N-hyperideal of H_1 and let s_1 be an S_1-element of I_1 and $s_2 \in S_2 \cap \sqrt{0_{H_2}}$. Now, assume that $g_1 \times g_2\big((a_1,b_1),\cdots,(a_n,b_n)\big) \in I_1 \times H_2$ for some $a_1^n \in H_1$ and $b_1^n \in H_2$. Then we have $g_1(a_1^n) \in I_1$. Hence we get either $g_1(s_1,a_i,1_H^{(n-2)}) \in \sqrt{0_{H_1}}$ for some $1 \le i \le n$ or $g(a_1^{i-1},s_1,a_{i+1}^n) \in I_1$. Therefore $g_1 \times g_2\big((s_1,1_{H_2}),(a_i,b_i),(1_{H_1},1_{H_2})^{(n-2)}\big) \in \sqrt{0_{H_1}} \times \sqrt{0_{H_2}}$ or $g_1 \times g_2\big((a_1,b_1),\cdots,(a_{i-1},b_{i-1}),(s_1,s_2),(a_{i+1},b_{i+1}),\cdots,(a_n,b_n)\big) \in I_1 \times H_2$. So (s_1,s_2) is an S-element of $I_1 \times H_2$.

(2) The proof is similar to (1). $\square$

Theorem 6.11. *Let S_1 and S_2 be n-ary multiplicative subsets of commutative Krasner (m,n)-hyperrings (H_1,f_1,g_1) and (H_2,f_2,g_2), respectively, and let I_1 and I_2 be proper hyperideals of H_1 and H_2, respectively. If one of the following cases holds,*

(1) $\quad \sqrt{0_{H_2}}^{(m,n)} \cap S_2 \ne \emptyset$ *and I_1 is an n-ary S_1-N-hyperideal of H_1.*

(2) $\quad \sqrt{0_{H_1}}^{(m,n)} \cap S_1 \ne \emptyset$ *and I_2 is an n-ary S_2-N-hyperideal of H_2.*

then $I_1 \times I_2$ is an n-ary S-N-hyperideal of $H_1 \times H_2$ where $S = S_1 \times S_2$.

Proof. Let $\sqrt{0_{H_2}}^{(m,n)} \cap S_2 \ne \emptyset$ and I_1 be an n-ary S_1-N-hyperideal of H_1. So $0_{H_2} \in I_2 \cap S_2 \ne \emptyset$ and $I_1 \cap S_1 = \emptyset$. Suppose that $g_1 \times g_2\big((a_1,b_1),\cdots,(a_n,b_n)\big) \in I_1 \times I_2$ for some $a_1^n \in H_1$ and $b_1^n \in H_2$ and s_1 is an S_1-element of I_1. Hence we have $g_1(a_1^n) \in I_1$. Then either $g_1(s_1,a_i,1_{H_1}^{(n-2)}) \in \sqrt{0_{H_1}}^{(m,n)}$ for some $1 \le i \le n$ or $g_1(a_1^{i-1},s_1,a_{i+1}^n) \in I_1$. So

$$g_1 \times g_2\big((s_1,0),(a_i,b_i),(1_{H_1},1_{H_2})^{(n-2)}\big) \in \sqrt{0_{H_1}}^{(m,n)} \times \sqrt{0_{H_2}}^{(m,n)}$$

or

$$g_1 \times g_2\big((a_1,b_1),\cdots,(a_{i-1},b_{i-1}),(s_1,0),(a_{i+1},b_{i+1}),\cdots,(a_n,b_n)\big) \in I_1 \times I_2.$$

Thus $(s_1,0_{H_2})$ is an S-element of $I_1 \times I_2$.
Also, if Case 2 holds, then by using a similar argument, we can show that $I_1 \times I_2$ is an n-ary S-N-hyperideal of $H_1 \times H_2$ where $S = S_1 \times S_2$. $\square$

7. Conclusion

In this paper, our purpose was to introduce and study the concept of n-ary N-hyperideals and some of it′s generalizations such as n-ary δ-N-hyperideals, (k,n)-absorbing δ-N-hyperideals and n-ary S-N-hyperideals over a commutative Krasner (m,n)-hyperring. Some results and characterizations with a number of supporting examples were given to explain the structures of these concepts. For instance, we indicated that a commutative Krasner (m,n)-hyperring H admits an n-ary N-hyperideal if and only if $\sqrt{0}^{(m,n)}$ is an n-ary prime hyperideal of H. We proved that if $\delta(I)$ is an n-ary N-hyperideal of H, then I is an n-ary δ-N-hyperideal of H. We showed that the inverse of the theorem is true if $\delta = \delta_1$. We characterized n-ary N-hyperideals and also n-ary S-N-hyperideals of cartesian product of Krasner (m,n)-hyperrings. As future work, we will study n-ary N-subhypermodules.

References

[1] R. Ameri, M. Norouzi, *Prime and primary hyperideals in Krasner (m,n)-hyperrings*, European Journal Of Combinatorics (2013) 379-390.

[2] M. Anbarloei, *n-ary 2-absorbing and 2-absorbing primary hyperideals in Krasner (m,n)-hyperrings*, Matematicki Vesnik **71** (3) (2019) 250-262.

[3] M. Anbarloei, *Unifing the prime and primary hyperideals under one frame in a Krasner (m,n)-hyperring*, Comm. Algebra **49** (2021) 3432-3446.

[4] M. Anbarloei, *Krasner (m,n)-hyperring of fractions*, Jordan Journal of Mathematics and Statistic **16** (1) (2023) 165-185.

[5] A. Asadi, R. Ameri, *Direct limit of Krasner (m,n)-hyperrings*, Journal of Sciences **31** (1) (2020) 75-83.

[6] B. Fahid, Z. Dongsheng, *2-Absorbing δ-primary ideals of commutative rings*, Kyungpook Mathematical Journal **57** (2017) 193-198.

[7] K. Hila, K. Naka, B. Davvaz, *On (k,n)-absorbing hyperideals in Krasner (m,n)-hyperrings*, Quarterly Journal of Mathematics **69** (2018) 1035-1046.

[8] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandenaves, Stockholm, (1934) 45-49.

[9] S. Mirvakili, B. Davvaz, *Relations on Krasner (m,n)-hyperrings*, European J. Combin. **31**(2010) 790-802.

[10] M. Norouzi, R.Ameri, V. Leoreanu-Fotea, *Normal hyperideals in Krasner (m,n)-hyperrings*, An. St. Univ. Ovidius Constanta **26** (3) (2018) 197-211.

[11] S. Omidi, B. Davvaz, J. Zhan, *Some properties of n-hyperideals in commutative hyperrings*, Journal of Algebraic Hyperstructures and Logical Algebras **1** (2) (2020) 23-30.

[12] E. Ozel Ay, G. Yesilot, D. Sonmez, *δ-Primary Hyperideals on Commutative Hyperrings*, Int. J. Math. and Mathematical Sciences (2017) Article ID 5428160, 4 pages.

[13] U. Tekir, S. Koc, K. H. Oral, *n-ideals of commutative rings*, Filomat **31**(10) (2017) 2933-2941.

[14] M. Tamekkante, E. M. Bouba, *(2,n)-ideals of commutative rings*, Journal of Algebra and Its Applications **18** (6) (2019). DOI: 10.1142/S0219498819501032.

[15] M.M. Zahedi, R. Ameri, *On the prime, primary and maximal subhypermodules*, Ital. J. Pure Appl. Math. **5** (1999) 61-80.