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Statistical convergence in intuitionistic fuzzy $\mathcal{G}$–metric spaces with order n

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Abstract. Mohiudddine and Alotaibi [25] introduced the notion of intuitionistic generalized fuzzy metric space to extend the generalized fuzzy metric space. Choi et al.[Structure for g-Metric Spaces and Related Fixed Point Theorems, arXiv preprint arXiv:1804.03651, (2018)] has recently proposed the notion of g–metric as a generalized notion of the distance function. Employing the idea in this paper, we first put forth the notion of Intuitionistic fuzzy G–metric space with order n as a generalization of intuitionistic fuzzy metric space. We describe some properties of this novel space and construct examples based on it. Then, we propose the concepts of statistical convergence, statistical limit points and statistical cluster points of sequences in this space and establish theorems in their regard by providing appropriate examples in support of them.

1. Introduction and preliminaries

The area of fuzzy theory has grown substantially in the fields of both pure and applied mathematics. In 1965, Zadeh [34] proposed the theory of fuzzy sets, which is a generalization of crisp set theory. Based on this theory, several applications can be found in [3, 4, 7, 8, 13, 16]. In 1975, to generalize the usual notion of metric, Kramosil and Michálek [24] put forward the notion of fuzzy metric space. Also, a number of authors [14, 20, 23] presented the notion of fuzzy metric in various ways. George and Veeramani [19] and Deng [11] modified the definition of fuzzy metric space introduced by Kramosil and Michálek [24] and Grabiec [20], respectively, and defined a Hausdorff topology on the modified fuzzy metric space. On the other hand, Atanassov [5] introduced an extended notion of fuzzy sets, referred to as intuitionistic fuzzy sets. Over the years, various authors [6, 10, 11] have made significant progress toward understanding intuitionistic fuzzy sets. As an extension of the fuzzy metric space proposed by George and Veeramani [19], Park [30] developed the concept of intuitionistic fuzzy metric space. Alaca et al. [2] proposed the same as an extension of fuzzy metric space due to Kramosil and Michálek [24].

Across many scientific disciplines, the distance function is a fundamental concept. Despite the current large and complicated data sets, it is necessary to generalize the definition of a distance function. At first, it was Gahler[18] who developed the notion of 2–metric space as a nonlinear generalization of ordinary metric space, having the area of a triangle in $\mathbb{R}^2$ as an example. Later, it was found that there is no easy relation between the 2–metric space and the ordinary metric space. Ha et a1.[21] have demonstrated that

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while an ordinary metric is a continuous function of its variables, a 2–metric need not be. This led Dhage [12] to develop a new type of generalized metric space known as D–metric space. It was later verified in [28] that the topological structure of D–metric spaces was incorrect. Finally, to overcome this problem, Mustafa and Sims [29] introduced a more appropriate generalized metric space called G–metric space, having the perimeter of a triangle in $\mathbb{R}^2$ as the best possible example. This new approach put an impressive impact on the field of metric space. Motivated by this, authors in [33] and [25] developed the ideas of the generalized fuzzy metric space and intuitionistic generalized fuzzy metric space, respectively. In the same way, Zhou et al.[35] proposed the probabilistic version of G–metric space known as Menger probabilistic G–metric space.

On the other hand, Choi et al.[9] generalized the idea of the ordinary distance between two points by taking $n+1$ points in place of two points and introduced the notion of g–metric with order n. By utilizing this distance notion, Abazari [1] introduced the Menger probabilistic g–metric space as the generalization of Menger probabilistic G–metric space, and explored the statistical convergence with respect to the strong topology by employing the concept of n–dimensional asymptotic density of subsets of $\mathbb{N}$. In recent times, there have been significant applications of statistical convergence, as evidenced in [22, 26, 27]. These developments prompt the exploration of generalizing the concept of the g–metric to intuitionistic fuzzy settings. Moreover, within this generalized framework, one may fairly question the validity of the concept of statistical convergence by incorporating the n–dimensional asymptotic density.

Motivated by the above, in this paper, we develop an idea of the intuitionistic fuzzy version of g–metric space and we call it intuitionistic fuzzy G–metric space with order n that will also be a generalization of the intuitionistic generalized fuzzy metric space introduced in [25]. Employing the notion of n–dimensional asymptotic density [1], we also explore the statistical convergence and statistical Cauchy criteria of sequences in the above well–defined space. Moreover, we discuss the statistical limit points and the statistical cluster points of sequences in this space.

Now let us review a few definitions and notations we will use in this paper. Throughout this study, $\mathbb{R}^+$ stands for the set of non–negative real numbers.

Definition 1.1. ([29]) Let $\mathfrak{X}$ be an arbitrary non–empty set and $G : \mathfrak{X}^3 \longrightarrow \mathbb{R}^+$ be a mapping. Then $(\mathfrak{X}, G)$ is called G–metric space if, for all $x, y, z, w \in \mathfrak{X}$, the following conditions hold:

(G-1) $G(x, y, z) = 0$ if $x = y = z$,

(G-2) $G(x, x, y) > 0$ if $x \neq y$,

(G-3) $G(x, x, y) \leq G(x, y, z)$ if $y \neq z$,

(G-4) $G(x, y, z) = G(y, z, x) = G(x, z, y) = ...$ (symmetry in all three variables),

(G-5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$.

In such a case, the function G is known as a G–metric on the set $\mathfrak{X}$.

Example 1.2. Suppose $(\mathfrak{X}, d)$ is an ordinary metric space. Define $G : \mathfrak{X}^3 \longrightarrow \mathbb{R}^+$ by

$$G(x, y, z) = \frac{1}{2}\Big(d(x, y) + d(y, z) + d(x, z)\Big).$$

Then $(\mathfrak{X}, G)$ is a G–metric space.

Remark 1.3. Let $(\mathfrak{X}, G)$ be a G–metric space. Define $d_G : \mathfrak{X}^2 \longrightarrow \mathbb{R}^+$ by

$$d_G(x, y) = \frac{1}{3}\Big(G(x, y, y) + G(x, x, y)\Big).$$

Then $(\mathfrak{X}, d_G)$ is an ordinary metric space.

Definition 1.4. ([31]) A function $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is said to be continuous t–norm if, (1) $*$ is commutative and associative, (2) $\eta=\eta*1$ for any $0\leq\eta\leq 1$, (3) for each $0\leq\eta_1,\eta_2,\eta_3,\eta_4\leq 1$, if $\eta_1\leq\eta_3$ and $\eta_2\leq\eta_4$ then $\eta_1*\eta_2\leq\eta_3*\eta_4$, and (4) $*$ is continuous.

A function $\diamond:[0,1]\times[0,1]\longrightarrow[0,1]$ is said to be continuous t–conorm if, (1′) $\diamond$ is commutative and associative, (2′) $\eta=\eta*0$ for any $0\leq\eta\leq 1$, (3′) for each $0\leq\eta_1,\eta_2,\eta_3,\eta_4\leq 1$, if $\eta_1\leq\eta_3$ and $\eta_2\leq\eta_4$ then $\eta_1\diamond\eta_2\leq\eta_3\diamond\eta_4$, and (4′) $\diamond$ is continuous.

Example 1.5. Let $\eta_1,\eta_2\in[0,1]$. Then

(1) $\eta_1*\eta_2=\min\{\eta_1,\eta_2\}$ and $\eta_1*\eta_2=\eta_1\cdot\eta_2$ are continuous t–norms.

(2) $\eta_1\diamond\eta_2=\max\{\eta_1,\eta_2\}$ and $\eta_1\diamond\eta_2=\min\{\eta_1+\eta_2,1\}$ are continuous t–conorms on $[0,1]$.

Remark 1.6. The notions of t–norms and t–conorm are the axiomatic skeletons used to characterize fuzzy intersections and unions, respectively.

By utilizing the concepts of continuous t–norms and continuous t–conorms, Park [30] developed the notion of intuitionistic fuzzy metric space as follows:

Definition 1.7. ([30]) Let $\mathfrak{X}$ be a non–empty set, $*$ and $\diamond$ be continuous t–norm and continuous t–conorm, respectively, and $\mathtt{M},\mathtt{N}$ be fuzzy sets on $\mathfrak{X}^2\times(0,\infty)$. The five–tuple $(\mathfrak{X},\mathtt{M},\mathtt{N},*,\diamond)$ is called an intuitionistic fuzzy metric space (in short, *IFM*–space) if, for all $x,y,z\in\mathfrak{X}$ and $p,q>0$, the following conditions hold:

(IF–1) $\mathtt{M}(x,y,p)+\mathtt{N}(x,y,p)\leq 1$,

(IF–2) $\mathtt{M}(x,y,p)>0$,

(IF–3) $\mathtt{M}(x,y,p)=1\iff x=y$,

(IF–4) $\mathtt{M}(x,y,p)=\mathtt{M}(y,x,p)$,

(IF–5) $\mathtt{M}(x,z,p)*\mathtt{M}(z,y,q)\leq\mathtt{M}(x,y,p+q)$,

(IF–6) $\mathtt{M}(x,y,.):(0,\infty)\longrightarrow(0,1]$ is continuous,

(IF–7) $\mathtt{N}(x,y,p)<1$,

(IF–8) $\mathtt{N}(x,y,p)=0\iff x=y$,

(IF–9) $\mathtt{N}(x,y,p)=\mathtt{N}(y,x,p)$

(IF–10) $\mathtt{N}(x,z,p)\diamond\mathtt{N}(z,y,q)\geq\mathtt{N}(x,y,p+q)$,

(IF–11) $\mathtt{N}(x,y,.):(0,\infty)\longrightarrow(0,1]$ is continuous.

In such a case, the tuple $(\mathtt{M},\mathtt{N})$ is called an intuitionistic fuzzy metric (in short, *IFM*) on $\mathfrak{X}$.

Definition 1.8. ([25]) Let $\mathfrak{X}$ be a non–empty set, $*$ and $\diamond$ be continuous t–norm and continuous t–conorm, respectively, and M,N be fuzzy sets on $\mathfrak{X}^3\times(0,\infty)$. The five–tuple $(\mathfrak{X},\mathrm{M},\mathrm{N},*,\diamond)$ is called an intuitionistic generalized fuzzy metric space (in short, *IGFM*–space) if, for all $x,y,z,w\in\mathfrak{X}$ and $p,q>0$, the following conditions hold:

(a) $\mathrm{M}(x,y,z,p)+\mathrm{N}(x,y,z,p)\leq 1$,

(b) $\mathrm{M}(x,x,y,p)>0$ for $x\neq y$,

(c) $\mathrm{M}(x,x,y,p)\geq\mathrm{M}(x,y,z,p)$ for $y\neq z$,

(d) $\mathrm{M}(x,y,z,p)=1\iff x=y=z$,

(e) $M(x, y, z, p) = M(\pi(x, y, z), p)$, where π is the permutation function,

(f) $M(x, w, w, p) * M(w, y, z, q) \leq M(x, y, z, p + q)$,

(g) $M(x, y, z, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous,

(h) M is non–decreasing on $(0, \infty)$, $\lim_{p\to\infty} M(x, y, z, p) = 1$ and $\lim_{p\to 0} M(x, y, z, p) = 0$,

(i) $N(x, x, y, p) < 1$ for $x \neq y$,

(j) $N(x, x, y, p) \leq N(x, y, z, p)$ for $y \neq z$,

(k) $N(x, y, z, p) = 0 \iff x = y = z$,

(l) $N(x, y, z, p) = N(\pi(x, y, z), p)$, where π is the permutation function,

(m) $N(x, w, w, p) \diamond N(w, y, z, q) \geq N(x, y, z, p + q)$,

(n) $N(x, y, z, .) : (0, \infty) \longrightarrow [0, 1]$ a continuous,

(o) N is non–increasing on $(0, \infty)$, $\lim_{p\to\infty} N(x, y, z, p) = 0$ and $\lim_{p\to 0} N(x, y, z, p) = 1$.

The tuple (M, N) is known as an intuitionistic generalized fuzzy metric (in short, *IGF*–metric) on $\mathfrak{X}$.

Definition 1.9. ([9]) Let $\mathfrak{X}$ be a non–empty set. A function $g : \mathfrak{X}^{n+1} \longrightarrow \mathbb{R}^+$, where $\mathfrak{X}^n = \prod_{i=1}^{n} \mathfrak{X}^i$, is called g–metric with order n on $\mathfrak{X}$ if the following conditions hold:

(g_1) $g(x_0, x_1, ..., x_n) = 0$ iff $x_0 = x_1 = ... = x_n$,

(g_2) $g(x_0, x_1, ..., x_n) = g(x_{\pi(0)}, x_{\pi(1)}, ..., x_{\pi(n)})$ for any permutation π on $\{0, 1, ..., n\}$,

(g_3) $g(x_0, x_1, ..., x_n) \leq g(y_0, y_1, ..., y_n)$ for any $(x_0, x_1, ..., x_n), (y_0, y_1, ..., y_n) \in \mathfrak{X}^{n+1}$ with $\{x_0, x_1, ..., x_n\} \subsetneq \{y_0, y_1, ..., y_n\}$,

(g_4) for all $x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, z \in \mathfrak{X}$ with $l + m + 1 = n$,
$g(x_0, x_1, ..., x_l, z, ..., z) + g(y_0, y_1, ..., y_m, z, ..., z) \geq g(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m)$.

The tuple $(\mathfrak{X}, g)$ is called g–metric space. For $n = 1$ and $n = 2$, a g–metric reduces to the ordinary metric and G–metric, respectively.

Example 1.10. ([9]) Let $(\mathfrak{X}, d)$ be an ordinary metric space. Define $g : \mathfrak{X}^{n+1} \longrightarrow \mathbb{R}^+$ by

$$g(x_0, x_1, ..., x_n) = \max_{0\leq i,j\leq n} \left\{|x_i - x_j|\right\}$$

for all $x_0, x_1, ..., x_n \in \mathfrak{X}$. Then $(\mathfrak{X}, g)$ is a g–metric space.

Definition 1.11. ([9]) A g–metric on $\mathfrak{X}$ is multiplicity independent, if

$$g(x_0, x_1, ..., x_n) = g(y_0, y_1, ..., y_n)$$

for all $(x_0, x_1, ..., x_n), (y_0, y_1, ..., y_n) \in \mathfrak{X}^{n+1}$ with $\{x_0, x_1, ..., x_n\} = \{y_0, y_1, ..., y_n\}$.

Definition 1.12. ([9]) Let $(\mathfrak{X}, g)$ be a g–metric space. The g–ball with center at $x \in \mathfrak{X}$ and radius $r > 0$ is defined as:

$$\mathbf{B}_g(x, r) = \left\{y \in \mathfrak{X} : g(x, y, ..., y) < r\right\}.$$

Definition 1.13. ([9]) Let $(\mathfrak{X}, g)$ be a g–metric space and (x_k) be a sequence in $\mathfrak{X}$. Then

(a) (x_k) is g–convergent to some $x \in \mathfrak{X}$ if, $\forall\ \xi > 0$, $\exists\ K \in \mathbb{N}$ such that

$$g(x_{i_1}, x_{i_2}, ..., x_{i_n}, x) < \xi, \forall\ i_1, i_2, ..., i_n \geq K.$$

(b) (x_k) is g–Cauchy if, $\forall\ \xi > 0$, $\exists\ K \in \mathbb{N}$ such that

$$g(x_{i_0}, x_{i_1}, ..., x_{i_n}) < \xi, \forall\ i_0, i_1, ..., i_n \geq K.$$

A g–metric space $(\mathfrak{X}, g)$ is complete if every g–Cauchy sequence is g–convergent in $\mathfrak{X}$.

2. Intuitionistic fuzzy generalized metric space (*IF𝒢M*– space)

In this section, by using the concepts of continuous t–norms, continuous t–conorms and intuitionistic fuzzy sets, we introduce the intuitionistic fuzzy version of g–metric space known as *IF𝒢M*– as follows:

Definition 2.1. Let $\mathfrak{X}$ be an arbitrary non–empty set, $*$ and $\diamond$ be continuous t–norm and continuous t–conorm, respectively, and $\mathfrak{M}, \mathfrak{N}$ be fuzzy sets on $\mathfrak{X}^{n+1} \times (0, \infty)$. The five–tuple $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is said to be an intuitionistic fuzzy generalized metric space (briefly, *IF𝒢M–space*) with order n if, for all $p, q \in (0, \infty)$, the following conditions hold:

(IFG–1) $\mathfrak{M}\big(x_0, x_1, ..., x_n, p\big) + \mathfrak{N}\big(x_0, x_1, ..., x_n, p\big) \leq 1$ for all $x_0, x_1, ..., x_n \in \mathfrak{X}$,

(IFG–2) $\mathfrak{M}\big(x_0, x_0, ..., x_0, x_1, p\big) > 0$ for $x_0 \neq x_1$, $\forall\ x_0, x_1 \in \mathfrak{X}$,

(IFG–3) $\mathfrak{M}\big(x_0, x_1, ..., x_n, p\big) \geq \mathfrak{M}\big(y_0, y_1, ..., y_n, p\big)$, $\forall\ (x_0, x_1, ..., x_n), (y_0, y_1, ..., y_n) \in \mathfrak{X}^{n+1}$ with $\{x_0, x_1, ..., x_n\} \subsetneq \{y_0, y_1, ..., y_n\}$,

(IFG–4) $\mathfrak{M}\big(x_0, x_1, ..., x_n, p\big) = 1 \iff x_0 = x_1 = ... = x_n$,

(IFG–5) $\mathfrak{M}\big(x_0, x_1, ..., x_n, p\big) = \mathfrak{M}\big(x_{\pi(0)}, x_{\pi(1)}, ..., x_{\pi(n)}, p\big)$ for any permutation π on $\{0, 1, ..., n\}$,

(IFG–6) for all $x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, z \in \mathfrak{X}$ with $l + m + 1 = n$,
$$\mathfrak{M}\big(x_0, x_1, ..., x_l, z, ..., z, p\big) * \mathfrak{M}\big(y_0, y_1, ..., y_m, z, ..., z, q\big) \leq \mathfrak{M}\big(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, p + q\big),$$

(IFG–7) $\mathfrak{M}\big(x_0, x_1, ..., x_n, .\big) : (0, \infty) \longrightarrow [0, 1]$ is a continuous function,

(IFG–8) $\lim_{p \to \infty} \mathfrak{M}\big(x_0, x_1, ..., x_n, p\big) = 1$ for all $x_0, x_1, ..., x_n \in \mathfrak{X}$,

(IFG–9) $\mathfrak{N}\big(x_0, x_0, ..., x_0, x_1, p\big) < 1$ for $x_0 \neq x_1$, $\forall\ x_0, x_1 \in \mathfrak{X}$,

(IFG–10) $\mathfrak{N}\big(x_0, x_1, ..., x_n, p\big) \leq \mathfrak{N}\big(y_0, y_1, ..., y_n, p\big)$, $\forall\ (x_0, x_1, ..., x_n), (y_0, y_1, ..., y_n) \in \mathfrak{X}^{n+1}$ with $\{x_0, x_1, ..., x_n\} \subsetneq \{y_0, y_1, ..., y_n\}$,

(IFG–11) $\mathfrak{N}\big(x_0, x_1, ..., x_n, p\big) = 0 \iff x_0 = x_1 = ... = x_n$,

(IFG–12) $\mathfrak{N}\big(x_0, x_1, ..., x_n, p\big) = \mathfrak{N}\big(x_{\pi(0)}, x_{\pi(1)}, ..., x_{\pi(n)}, p\big)$ for any permutation π on $\{0, 1, ..., n\}$,

(IFG–13) for all $x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, z \in \mathfrak{X}$ with $l + m + 1 = n$,
$$\mathfrak{N}\big(x_0, x_1, ..., x_l, z, ..., z, p\big) \diamond \mathfrak{N}\big(y_0, y_1, ..., y_m, z, ..., z, q\big) \geq \mathfrak{N}\big(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, p + q\big),$$

(IFG–14) $\mathfrak{N}\big(x_0, x_1, ..., x_n, .\big) : (0, \infty) \longrightarrow [0, 1]$ is a continuous function,

(IFG–15) $\lim_{p\to\infty} \mathfrak{N}\big(x_0, x_1, ..., x_n, p\big) = 0$ for all $x_0, x_1, ..., x_n \in \mathfrak{X}$.

Further, we call the tuple $(\mathfrak{M}, \mathfrak{N})$, the intuitionistic fuzzy generalized metric (in short, *IF$\mathcal{G}$M*) on $\mathfrak{X}$. In order to avoid any confusion, we refer to the five–tuple $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ as the *IF$\mathcal{G}$M*–space instead of *IF$\mathcal{G}$M*–space with order n.

Example 2.2. Let $(\mathfrak{X}, g)$ be a g–metric space with order n. For $p > 0$, define

$$\mathfrak{M}\big(x_0, x_1, ..., x_n, p\big) = \frac{p}{p + g\big(x_0, x_1, ..., x_n\big)}$$

and

$$\mathfrak{N}\big(x_0, x_1, ..., x_n, p\big) = \frac{g\big(x_0, x_1, ..., x_n\big)}{p + g\big(x_0, x_1, ..., x_n\big)},$$

where $\eta_1 * \eta_2 = \eta_1 \cdot \eta_2$ and $\eta_1 \diamond \eta_2 = \min\{\eta_1 + \eta_2, 1\}$ for all $\eta_1, \eta_2 \in [0, 1]$. Then $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is an *IF$\mathcal{G}$M*–space.

Proof. We only show that $(\mathfrak{M}, \mathfrak{N})$ satisfies the conditions (**IFG**–6) and (**IFG**–13) and the rest follows easily.

(**IFG**–6): Let $p, q > 0$ and $x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, z \in \mathfrak{X}$ with $l + m + 1 = n$. Then

$$g\big(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m\big) \leq g\big(x_0, x_1, ..., x_l, z, ..., z\big) + g\big(y_0, y_1, ..., y_m, z, ..., z\big). \tag{1}$$

Now

$$\begin{aligned}
\mathfrak{M}\big(x_0, x_1.., x_l, z, ..., z, p\big) * \mathfrak{M}\big(y_0, y_1.., y_m, z, ..., z, q\big) &= \frac{p}{p + g\big(x_0, x_1.., x_l, z, ..., z\big)} \cdot \frac{q}{q + g\big((y_0, y_1.., y_m, z, ..., z\big)} \\
&\leq \frac{pq}{pq + p.g\big(y_0, y_1.., y_m, z, ..., z\big) + q.g\big(x_0, x_1.., x_l, z, ..., z\big)} \\
&= \frac{1}{1 + \frac{g\left(y_0, y_1.., y_m, z, ..., z\right)}{q} + \frac{g\left(x_0, x_1.., x_l, z, ..., z\right)}{p}} \\
&\leq \frac{1}{1 + \frac{g\left(y_0, y_1.., y_m, z, ..., z\right)}{p+q} + \frac{g\left(x_0, x_1.., x_l, z, ..., z\right)}{p+q}} \\
&= \frac{p + q}{p + q + g\big(y_0, y_1.., y_m, z, ..., z\big) + g\big(x_0, x_1.., x_l, z, ..., z\big)}.
\end{aligned}$$

Therefore using the Equation 1, it follows that

$$\begin{aligned}
\mathfrak{M}\big(x_0, x_1.., x_l, z, ..., z, p\big) * \mathfrak{M}\big(y_0, y_1.., y_m, z, ..., z, q\big) &\leq \frac{p + q}{p + q + g\big(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m\big)} \\
&= \mathfrak{M}\big(x_0, x_1.., x_l, y_0, y_1, ..., y_m, p + q\big).
\end{aligned}$$

(**IFG**–13): As above, select p, q, l and m. Then

$$g\big(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m\big) \leq g\big(x_0, x_1, ..., x_l, z, ..., z\big) + g\big(y_0, y_1, ..., y_m, z, ..., z\big).$$

$$\implies 1+\frac{p+q}{g\Big(x_0,x_1,...,x_l,y_0,y_1,...,y_m\Big)} \geq 1+\frac{p+q}{g\Big(x_0,x_1,...,x_l,z,...,z\Big)+g\Big(y_0,y_1,...,y_m,z,...,z\Big)}.$$

$$\implies \frac{g\Big(x_0,x_1,...,x_l,y_0,y_1,...,y_m\Big)}{p+q+g\Big(x_0,x_1,...,x_l,y_0,y_1,...,y_m\Big)} \leq \frac{g\Big(x_0,x_1,...,x_l,z,...,z\Big)+g\Big(y_0,y_1,...,y_m,z,...,z\Big)}{p+g\Big(x_0,x_1,...,x_l,z,...,z\Big)+q+g\Big(y_0,y_1,...,y_m,z,...,z\Big)}$$

$$\leq \frac{g\Big(x_0,x_1,...,x_l,z,...,z\Big)}{p+g\Big(x_0,x_1,...,x_l,z,...,z\Big)}+\frac{g\Big(y_0,y_1,...,y_m,z,...,z\Big)}{q+g\Big(y_0,y_1,...,y_m,z,...,z\Big)}.$$

Therefore, $\mathfrak{N}\Big(x_0,x_1,...,x_l,y_0,y_1,...,y_m,p+q\Big) \leq \mathfrak{N}\Big(x_0,x_1,...,x_l,z,z,...,z,p\Big)+\mathfrak{N}\Big(y_0,y_1,...,y_m,z,z,...,z,q\Big)$. Since, $\mathfrak{N}\Big(x_0,x_1,...,x_l,y_0,y_1,...,y_m,p+q\Big) \leq 1$, we have

$$\mathfrak{N}\Big(x_0,x_1,...,x_l,y_0,y_1,...,y_m,p+q\Big) \leq \min\Big\{\mathfrak{N}\Big(x_0,x_1,...,x_l,z,z,...,z,p\Big)+\mathfrak{N}\Big(y_0,y_1,...,y_m,z,z,...,z,q\Big),1\Big\}$$
$$= \mathfrak{N}\Big(x_0,x_1,...,x_l,z,z,...,z,p\Big) \diamond \mathfrak{N}\Big(y_0,y_1,...,y_m,z,z,...,z,q\Big).$$

□

It can also be observed that the above example is also true for $\eta_1 * \eta_2 = \min\{\eta_1,\eta_2\}$ and $\eta_1 \diamond \eta_2 = \max\{\eta_1,\eta_2\}$, $\forall\ \eta_1,\eta_2 \in [0,1]$. Since the above metric space $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ is induced by the g–metric, known as standard *IF𝒢M*–space.

Definition 2.3. The tuple $(\mathfrak{M},\mathfrak{N})$ on *IF𝒢M*–space $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ is said to be multiplicity independent if, for every $p>0$,

$$\mathfrak{M}\Big(x_0,x_1,...,x_n,p\Big)=\mathfrak{M}\Big(y_0,y_1,...,y_n,p\Big)$$

and

$$\mathfrak{N}\Big(x_0,x_1,...,x_n,p\Big)=\mathfrak{N}\Big(y_0,y_1,...,y_n,p\Big)$$

hold for all $(x_0,x_1,...,x_n),(y_0,y_1,...,y_n) \in \mathfrak{X}^{n+1}$ such that $\{x_0,x_1,...,x_n\}=\{y_0,y_1,...,y_n\}$.

Example 2.4. Let $\mathfrak{X}$ be a non–empty set, $\eta_1 * \eta_2 = \eta_1 \cdot \eta_2$ and $\eta_1 \diamond \eta_2 = \eta_1+\eta_2-\eta_1\cdot\eta_2$ for all $\eta_1,\eta_2 \in [0,1]$. For every $x_0,x_1,...,x_n \in \mathfrak{X}$ and $p>0$, define $\mathfrak{M}$ and $\mathfrak{N}$ by

$$\mathfrak{M}\Big(x_0,x_1,...,x_n,p\Big)=\left\{\begin{array}{lr} 1, & \text{if } x_0=x_1=...=x_n, \\ \frac{p}{p+1}, & \text{Otherwise} \end{array}\right\}$$

$$\mathfrak{N}\Big(x_0,x_1,...,x_n,p\Big)=\left\{\begin{array}{lr} 0, & \text{if } x_0=x_1=...=x_n, \\ \frac{1}{p+1}, & \text{Otherwise.} \end{array}\right\}$$

Then $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ is an *IF𝒢M*–space such that $(\mathfrak{M},\mathfrak{N})$ is multiplicity independent.

Remark 2.5. (a) For $n=1$ and $n=2$, the *IF𝒢M*–space reduces to *IFM*–space and *IGFM*–space, respectively, and the multiplicity independence coincides with symmetries in the respective metrics.

(b) In Definition 2.1, if we allow the conditions (**IFG–3**) and (**IFG–10**) for $\{x_0,x_1,...,x_n\} \subseteq \{y_0,y_1,...,y_n\}$, then $(\mathfrak{M},\mathfrak{N})$ becomes multiplicity independent.

Lemma 2.6. *Consider* $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ *to be an IF𝒢M–space. Then* $\mathfrak{M}\Big(x_0,x_1,...,x_n,p\Big)$ *and* $\mathfrak{N}\Big(x_0,x_1,...,x_n,p\Big)$ *are non–decreasing and non–increasing functions with respect to p, respectively, for all* $x_0,x_1,...,x_n \in \mathfrak{X}$.

Proof. By using (**IFG**–6) and (**IFG**–13), we have

$$\begin{aligned}\mathfrak{M}\Big(x_0, x_1, ..., x_n, p+q\Big) &\geq \mathfrak{M}\Big(x_0, x_0, ..., x_0, p\Big) * \mathfrak{M}\Big(x_0, x_1, ..., x_n, q\Big)\\ &\geq 1 * \mathfrak{M}\Big(x_0, x_1, ..., x_n, q\Big)\\ &\geq \mathfrak{M}\Big(x_0, x_1, ..., x_n, q\Big)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{N}\Big(x_0, x_1, ..., x_n, p+q\Big) &\leq \mathfrak{N}\Big(x_0, x_0, ..., x_0, p\Big) \diamond \mathfrak{N}\Big(x_0, x_1, ..., x_n, q\Big)\\ &\leq 0 \diamond \mathfrak{N}\Big(x_0, x_1, ..., x_n, q\Big)\\ &\leq \mathfrak{N}\Big(x_0, x_1, ..., x_n, q\Big).\end{aligned}$$

Since $p+q > p$ and $p+q > q$, we are done. □

In the following Remark, we show that from any given IFM–space, one can generate an IFGM–space under certain restrictions.

Remark 2.7. Let $(\mathfrak{X}, \mathtt{M}, \mathtt{N}, *, \diamond)$ to be an *IFM*–space satisfying $\lim_{p\to\infty} \mathtt{M}(x, y, p) = 1$ and $\lim_{p\to\infty} \mathtt{N}(x, y, p) = 0, \forall\, x, y \in \mathfrak{X}$. Then $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is an *IF$\mathcal{G}$M*–space, where

$$\mathfrak{M}\Big(x_0, x_1, ..., x_n, p\Big) = \min_{0\leq i,j\leq n} \Big\{\mathtt{M}(x_i, x_j, p)\Big\} \text{ and}$$

$$\mathfrak{N}\Big(x_0, x_1, ..., x_n, p\Big) = \max_{0\leq i,j\leq n} \Big\{\mathtt{N}(x_i, x_j, p)\Big\}.$$

Proof. We only verify the conditions (**IFG**–1), (**IFG**–6) and (**IFG**–13). Let $p, q > 0$ be given.

(**IFG**–1): Suppose $\mathfrak{M}\Big(x_0, x_1, ..., x_n, p\Big) = \mathtt{M}(x_\alpha, x_\beta, p)$ and $\mathfrak{N}\Big(x_0, x_1, ..., x_n, p\Big) = \mathtt{N}(x_\gamma, x_\delta, p)$ for some $\alpha, \beta, \gamma, \delta \in \{0, 1, ..., n\}$, i.e., $\mathtt{M}(x_\alpha, x_\beta, p) \leq \mathtt{M}(x_i, x_j, p)$ and $\mathtt{N}(x_\gamma, x_\delta, p) \geq \mathtt{N}(x_i, x_j, p)$ for all $i, j \in \{0, 1, ..., n\}$. Therefore,

$$\begin{aligned}\mathfrak{M}\Big(x_0, x_1, ..., x_n, p\Big) + \mathfrak{N}\Big(x_0, x_1, ..., x_n, p\Big) &= \mathtt{M}(x_\alpha, x_\beta, p) + \mathtt{N}(x_\gamma, x_\delta, p)\\ &\leq \mathtt{M}(x_\gamma, x_\delta, p) + \mathtt{N}(x_\gamma, x_\delta, p)\\ &\leq 1.\end{aligned}$$

(**IFG**–6): Suppose $x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, z \in \mathfrak{X}$ such that $l + m + 1 = n$. Then

$$\begin{aligned}&\mathfrak{M}\Big(x_0, x_1, ..., x_l, z, ..., z, p\Big) * \mathfrak{M}\Big(y_0, y_1, ..., y_m, z, ..., z, q\Big)\\ &\quad= \min_{0\leq i,j\leq l} \Big\{\mathtt{M}(x_i, x_j, p), \mathtt{M}(x_i, z, p)\Big\} * \min_{0\leq r,s\leq m} \Big\{\mathtt{M}(y_r, y_s, q), \mathtt{M}(y_r, z, q)\Big\}\\ &\quad\leq \min_{\substack{0\leq i,j\leq 1\\ 0\leq r,s\leq m}} \Big\{\mathtt{M}(x_i, x_j, p) * \mathtt{M}(y_r, y_s, q), \mathtt{M}(x_i, x_j, p) * \mathtt{M}(y_r, z, q), \mathtt{M}(x_i, z, p) * \mathtt{M}(y_r, y_s, q), \mathtt{M}(x_i, z, p) * \mathtt{M}(y_r, z, q)\Big\}.\end{aligned}$$

By using the condition (3) of Definition 1.4 and the condition (**IF**–5), we obtain $\mathtt{M}(x_i, z, p) * \mathtt{M}(y_r, z, q) \leq$

$\mathtt{M}(x_i, y_r, p+q)$, $\mathtt{M}(x_i, x_j, p) * \mathtt{M}(y_r, y_s, q) \leq \mathtt{M}(x_i, x_j, p+q)$ and $\mathtt{M}(x_i, z, p) * \mathtt{M}(y_r, y_s, q) \leq \mathtt{M}(y_r, y_s, p+q)$. Consequently,

$$\begin{aligned}
&\mathfrak{M}\Big(x_0, x_1, ..., x_l, z, ..., z, p\Big) * \mathfrak{M}\Big(y_0, y_1, ..., y_m, z, ..., z, q\Big) \\
&\quad \leq \min_{\substack{0\leq i,j\leq 1 \\ 0\leq r,s\leq m}} \Big\{\mathtt{M}(x_i, x_j, p+q), \mathtt{M}(x_i, x_j, p) * \mathtt{M}(y_r, z, q), \mathtt{M}(y_r, y_s, p+q), \mathtt{M}(x_i, y_r, p+q)\Big\} \\
&\quad \leq \min_{\substack{0\leq i,j\leq 1 \\ 0\leq r,s\leq m}} \Big\{\mathtt{M}(x_i, x_j, p+q), \mathtt{M}(y_r, y_s, p+q), \mathtt{M}(x_i, y_r, p+q)\Big\} \\
&\quad = \mathfrak{M}\Big(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, p+q\Big).
\end{aligned}$$

(**IFG**–13):

$$\begin{aligned}
&\mathfrak{N}\Big(x_0, x_1, ..., x_l, z, ..., z, p\Big) \diamond \mathfrak{N}\Big(y_0, y_1, ..., y_m, z, ..., z, q\Big) \\
&\quad = \max_{0\leq i,j\leq l} \Big\{\mathtt{N}(x_i, x_j, p), \mathtt{N}(x_i, z, p)\Big\} \diamond \max_{0\leq r,s\leq m} \Big\{\mathtt{N}(y_r, y_s, q), \mathtt{N}(y_r, z, q)\Big\} \\
&\quad \geq \max_{\substack{0\leq i,j\leq 1 \\ 0\leq r,s\leq m}} \Big\{\mathtt{N}(x_i, x_j, p) \diamond \mathtt{N}(y_r, y_s, q), \mathtt{N}(x_i, x_j, p) \diamond \mathtt{N}(y_r, z, q), \mathtt{N}(x_i, z, p) \diamond \mathtt{N}(y_r, y_s, q), \mathtt{N}(x_i, z, p) \diamond \mathtt{N}(y_r, z, q)\Big\}.
\end{aligned}$$

Again, in the similar way by using the condition (3′) of the Definition 1.4 and the condition (**IF**–10), we have

$$\begin{aligned}
&\mathfrak{N}\Big(x_0, x_1, ..., x_l, z, ..., z, p\Big) \diamond \mathfrak{N}\Big(y_0, y_1, ..., y_m, z, ..., z, q\Big) \\
&\quad \geq \max_{\substack{0\leq i,j\leq 1 \\ 0\leq r,s\leq m}} \Big\{\mathtt{N}(x_i, x_j, p+q), \mathtt{N}(y_r, y_s, p+q), \mathtt{N}(x_i, y_r, p+q)\Big\} \\
&\quad = \mathfrak{N}\Big(x_0, x_1, ..., x_l, y_0, y_1, ..., y_m, p+q\Big\}.
\end{aligned}$$

Hence the result follows. □

We can see that $(\mathfrak{M}, \mathfrak{N})$ is multiplicity independent in this case.

Remark 2.8. Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an *IF𝒢M*–space. Then $(\mathfrak{X}, \mathtt{M}, \mathtt{N}, *, \diamond)$ is an *IFM*–space, where $(\mathtt{M}, \mathtt{N})$ is defined by

$$\mathtt{M}(x, y, p) = \min\Big\{\mathfrak{M}\Big(\alpha_0, \alpha_1, ..., \alpha_n, p\Big) : \alpha_i \in \{x, y\}, i = 0, 1, ..., n\Big\},$$

$$\mathtt{N}(x, y, p) = \max\Big\{\mathfrak{N}\Big(\alpha_0, \alpha_1, ..., \alpha_n, p\Big) : \alpha_i \in \{x, y\}, i = 0, 1, ..., n\Big\}.$$

Proof. We only show that the tuple $(\mathtt{M}, \mathtt{N})$ satisfies the conditions (**IF**–5) and (**IF**–10) and the rest follows easily. Let $p, q > 0$ and $x, y, z \in \mathfrak{X}$. Then

(**IF**–5): $\mathtt{M}(x, y, p+q) = \min\Big\{\mathfrak{M}\Big(\alpha_0, \alpha_1, ..., \alpha_n, p+q\Big) : \alpha_i \in \{x, y\}, i = 0, 1, ..., n\Big\}$ and $\mathtt{M}(x, z, p) * \mathtt{M}(z, y, q) = \min\Big\{\mathfrak{M}\Big(\beta_0, \beta_1, ..., \beta_n, p\Big) : \beta_i \in \{x, z\}, i = 0, 1, ..., n\Big\} * \min\Big\{\mathfrak{M}\Big(\gamma_0, \gamma_1, ..., \gamma_n, q\Big) : \gamma_i \in \{z, y\}, i = 0, 1, ..., n\Big\}$. We need to show that $\mathtt{M}(x, y, p+q) \geq \mathtt{M}(x, z, p) * \mathtt{M}\Big(z, y, q\Big)$. If $x = y$, the inequality hold. Suppose $x \neq y$. Then, there is $\delta_0, \delta_1, ..., \delta_n \in \{x, y\}$ in such a way that

$$\begin{aligned}
\mathtt{M}(x, y, p+q) &= \min\Big\{\mathfrak{M}\Big(\alpha_0, \alpha_1, ..., \alpha_n, p+q\Big) : \alpha_i \in \{x, y\}, i = 0, 1, ..., n\Big\} \\
&= \mathfrak{M}\Big(\delta_0, \delta_1, ..., \delta_n, p+q\Big).
\end{aligned}$$

Hence, $\mathfrak{M}\big(\delta_0, \delta_1, ..., \delta_n, p+q\big) \leq \mathfrak{M}\big(\alpha_0, \alpha_1, ..., \alpha_n, p+q\big)$, $\forall\ \alpha_0, \alpha_1, ..., \alpha_n \in \{x, y\}$. Suppose l is cardinality of the set $\big\{i : \delta_i = x, i \in \{0, 1, ..., n\}\big\}$. It is evident that $1 \leq l \leq n$. Without loss of generality, let us assume $\delta_0 = \delta_1 = ... = \delta_l = x$ and $\delta_{l+1} = ... = \delta_n = y$. Then

$$\begin{aligned} \mathtt{M}(x, y, p+q) &= \mathfrak{M}\big(\delta_0, \delta_1, ..., \delta_n, p+q\big) \\ &= \mathfrak{M}\big(\underbrace{x, x, ..., x}_{l\text{–times}}, y, y, ..., y, p+q\big). \end{aligned}$$

Now, by (**IFG**–6), we have

$$\mathfrak{M}\big(\underbrace{x, x, ..., x}_{l\text{–times}}, y, y, ..., y, p+q\big) \geq \mathfrak{M}\big(\underbrace{x, x, ..., x}_{l\text{–times}}, z, z, ..., z, p\big) * \mathfrak{M}\big(\underbrace{z, z, ..., z}_{l\text{–times}}, y, y, ..., y, q\big).$$

Also,

$$\begin{aligned} \mathfrak{M}\big(\underbrace{x, x, ..., x}_{l\text{–times}}, z, z, ..., z, p\big) &\geq \min\Big\{\mathfrak{M}\big(\beta_0, \beta_1, ..., \beta_n, p\big) : \beta_i \in \{x, z\}, i = 0, 1, ..., n\Big\} \\ &= \mathtt{M}(x, z, p), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{M}\big(\underbrace{z, z, ..., z}_{l\text{–times}}, y, y, ..., y, q\big) &\geq \min\Big\{\mathfrak{M}\big(\gamma_0, \gamma_1, ..., \gamma_n, q\big) : \gamma_i \in \{z, y\}, i = 0, 1, ..., n\Big\} \\ &= \mathfrak{M}(z, y, q). \end{aligned}$$

Using the monotonicity of $*$, we obtain

$$\begin{aligned} \mathfrak{M}\big(\underbrace{x, x, ..., x}_{l\text{–times}}, z, z, ..., z, p\big) * \mathfrak{M}\big(\underbrace{z, z, ..., z}_{l\text{–times}}, y, y, ..., y, q\big) &\geq \mathtt{M}(x, z, p) * \mathtt{M}(z, y, q). \\ \implies \mathtt{M}(x, y, p+q) &\geq \mathtt{M}(x, z, p) * \mathtt{M}(z, y, q). \end{aligned}$$

(**IF**–10): $\mathtt{N}(x, y, p+q) = \max\Big\{\mathfrak{N}\big(\alpha_0, \alpha_1, ..., \alpha_n, p+q\big) : \alpha_i \in \{x, y\}, i = 0, 1, ..., n\Big\}$ and $\mathtt{N}(x, z, p) \diamond \mathfrak{N}(z, y, q) = \max\Big\{\mathfrak{N}\big(\beta_0, \beta_1, ..., \beta_n, p\big) : \beta_i \in \{x, z\}, i = 0, 1, ..., n\Big\} \diamond \max\Big\{\mathfrak{N}\big(\gamma_0, \gamma_1, ..., \gamma_n, q\big) : \gamma_i \in \{z, y\}, i = 0, 1, ..., n\Big\}$. We show that $\mathtt{N}(x, y, p+q) \leq \mathtt{N}(x, z, p) \diamond \mathtt{N}\big(z, y, q\big)$. If $x = y$, the proof is trivial. Let $x \neq y$. Then, there exists $\eta_0, \eta_1, ..., \eta_n \in \{x, y\}$ so that

$$\begin{aligned} \mathtt{N}(x, y, p+q) &= \max\Big\{\mathfrak{N}\big(\alpha_0, \alpha_1, ..., \alpha_n, p+q\big) : \alpha_i \in \{x, y\}, i = 0, 1, ..., n\Big\} \\ &= \mathfrak{N}\big(\eta_0, \eta_1, ..., \eta_n, p+q\big). \end{aligned}$$

i.e., $\mathfrak{N}\big(\eta_0, \eta_1, ..., \eta_n, p+q\big) \geq \mathfrak{N}\big(\alpha_0, \alpha_1, ..., \alpha_n, p+q\big)$, $\forall\ \alpha_0, \alpha_1, ..., \alpha_n \in \{x, y\}$. Suppose m is cardinality of the set $\big\{i : \eta_i = x, i \in \{0, 1, ..., n\}\big\}$, so $1 \leq m \leq n$. Without loss of generality, assume $\eta_0 = \eta_1 = ... = \eta_m = x$ and $\eta_{m+1} = ... = \eta_n = y$. Then

$$\begin{aligned} \mathtt{N}(x, y, p+q) &= \mathfrak{N}\big(\eta_0, \eta_1, ..., \eta_n, p+q\big) \\ &= \mathfrak{N}\big(\underbrace{x, x, ..., x}_{m\text{–times}}, y, y, ..., y, p+q\big). \end{aligned}$$

Now, by (**IFG**–13), we have

$$\mathfrak{N}\Big(\underbrace{x, x, ..., x}_{m\text{–times}}, y, y, ..., y, p+q\Big) \leq \mathfrak{N}\Big(\underbrace{x, x, ..., x}_{m\text{–times}}, z, z, ..., z, p\Big) \diamond \mathfrak{N}\Big(\underbrace{z, z, ..., z}_{m\text{–times}}, y, y, ..., y, q\Big).$$

Also,

$$\mathfrak{N}\Big(\underbrace{x, x, ..., x}_{m\text{–times}}, z, z, ..., z, p\Big) \leq \max\Big\{\mathfrak{N}\Big(\beta_0, \beta_1, ..., \beta_n, p\Big) : \beta_i \in \{x, z\}, i = 0, 1, ..., n\Big\}$$
$$= \mathtt{N}(x, z, p),$$

and

$$\mathfrak{N}\Big(\underbrace{z, z, ..., z}_{m\text{–times}}, y, y, ..., y, q\Big) \leq \max\Big\{\mathfrak{N}\Big(\gamma_0, \gamma_1, ..., \gamma_n, q\Big) : \gamma_i \in \{z, y\}, i = 0, 1, ..., n\Big\}$$
$$= \mathfrak{N}(z, y, q).$$

Using the monotonicity of $\diamond$, we obtain

$$\mathfrak{N}\Big(\underbrace{x, x, ..., x}_{m\text{–times}}, z, z, ..., z, p\Big) \diamond \mathfrak{N}\Big(\underbrace{z, z, ..., z}_{m\text{–times}}, y, y, ..., y, q\Big) \leq \mathtt{N}(x, z, p) \diamond \mathtt{N}(z, y, q).$$
$$\implies \mathtt{N}(x, y, p+q) \leq \mathtt{N}(x, z, p) \diamond \mathtt{N}(z, y, q).$$

This completes the proof of the theorem. □

Remark 2.9. If $(\mathfrak{M}, \mathfrak{N})$ is multiplicity independent on an *IF𝒢M*–space $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$, then $(\mathfrak{X}, \mathtt{M}, \mathtt{N}, *, \diamond)$ ia an *IFM*–space, where

$$\mathtt{M}(x, y, p) = \mathfrak{M}\Big(\underbrace{x, x, ..., x}_{l\text{–times}}, y, y, ..., y, p\Big)$$

and

$$\mathtt{N}(x, y, p) = \mathfrak{N}\Big(\underbrace{x, x, ..., x}_{m\text{–times}}, y, y, ..., y, p\Big)$$

for any $1 \leq m, n \leq n$.

Proof. The result is the direct conclusion of the Remark 2.8. □

Proposition 2.10. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF𝒢M–space, where $\eta * \eta = \eta$ and $\eta \diamond \eta = \eta$ for all $\eta \in [0, 1]$. Then the following hold:*

$$(a)\ \mathfrak{M}\Big(\underbrace{x, x, ..., x}_{l\text{–times}}, z, ..., z, p\Big) \geq \mathfrak{M}\Big(x, z, ..., z, \frac{p}{2^{l-1}}\Big) \text{ and } \mathfrak{N}\Big(\underbrace{x, x, ..., x}_{l\text{–times}}, z, ..., z, p\Big) \leq \mathfrak{N}\Big(x, z, ..., z, \frac{p}{2^{l-1}}\Big).$$

$$(b)\ \mathfrak{M}\Big(\underbrace{x, x, ..., x}_{l\text{–times}}, z, ..., z, p\Big) \geq \mathfrak{M}\Big(z, x, ..., x, \frac{p}{2^{n-l}}\Big) \text{ and } \mathfrak{N}\Big(\underbrace{x, x, ..., x}_{l\text{–times}}, z, ..., z, p\Big) \leq \mathfrak{N}\Big(z, x, ..., x, \frac{p}{2^{n-l}}\Big).$$

Proof. (a): By using the condition (**IFG**–6), we get

$$\begin{aligned}\mathfrak{M}\Big(\underbrace{x,x,...,x,}_{l\text{–times}}z,...,z,p\Big) &\geq \mathfrak{M}\Big(\underbrace{x,x,...,x,}_{(l-1)\text{–times}}z,...,z,\frac{p}{2}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2}\Big)\\ &\geq \mathfrak{M}\Big(\underbrace{x,x,...,x,}_{(l-2)\text{–times}}z,...,z,\frac{p}{2^2}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2^2}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2}\Big).\end{aligned}$$

Since $\eta*\eta=\eta$ for all $\eta\in[0,1]$, we have

$$\begin{aligned}\mathfrak{M}\Big(\underbrace{x,x,...,x,}_{l\text{–times}}z,...,z,p\Big) &\geq \mathfrak{M}\Big(\underbrace{x,x,...,x,}_{(l-2)\text{–times}}z,...,z,\frac{p}{2^2}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2^2}\Big)\\ &\geq \mathfrak{M}\Big(\underbrace{x,x,...,x,}_{(l-3)\text{–times}}z,...,z,\frac{p}{2^3}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2^3}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2^2}\Big)\\ &\geq \mathfrak{M}\Big(\underbrace{x,x,...,x,}_{(l-3)\text{–times}}z,...,z,\frac{p}{2^3}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2^3}\Big)\\ &\vdots\\ &\geq \mathfrak{M}\Big(x,z,...,z,\frac{p}{2^{l-1}}\Big)*\mathfrak{M}\Big(x,z,...,z,\frac{p}{2^{l-1}}\Big)\\ &= \mathfrak{M}\Big(x,z,...,z,\frac{p}{2^{l-1}}\Big).\end{aligned}$$

By using the condition (**IFG**–13), we get

$$\begin{aligned}\mathfrak{N}\Big(\underbrace{x,x,...,x,}_{l\text{–times}}z,...,z,p\Big) &\leq \mathfrak{N}\Big(\underbrace{x,x,...,x,}_{(l-1)\text{–times}}z,...,z,\frac{p}{2}\Big)\diamond\mathfrak{N}\Big(x,z,...,z,\frac{p}{2}\Big)\\ &\leq \mathfrak{N}\Big(\underbrace{x,x,...,x,}_{(l-2)\text{–times}}z,...,z,\frac{p}{2^2}\Big)\diamond\mathfrak{N}\Big(x,z,...,z,\frac{p}{2^2}\Big)\diamond\mathfrak{M}\Big(x,z,...,z,\frac{p}{2}\Big).\end{aligned}$$

Since $\eta \diamond \eta = \eta$ for all $\eta \in [0,1]$, we have

$$\begin{aligned}
\mathfrak{N}\Big(\underbrace{x,x,...,x}_{l\text{–times}},z,...,z,p\Big) &\leq \mathfrak{N}\Big(\underbrace{x,x,...,x,}_{(l-2)\text{–times}}\ z,...,z,\frac{p}{2^2}\Big) \diamond \mathfrak{N}\Big(x,z,...,z,\frac{p}{2^2}\Big)\\
&\leq \mathfrak{N}\Big(\underbrace{x,x,...,x,}_{(l-3)\text{–times}}\ z,...,z,\frac{p}{2^3}\Big) \diamond \mathfrak{N}\Big(x,z,...,z,\frac{p}{2^3}\Big) \diamond \mathfrak{N}\Big(x,z,...,z,\frac{p}{2^2}\Big)\\
&\geq \mathfrak{N}\Big(\underbrace{x,x,...,x,}_{(l-3)\text{–times}}\ z,...,z,\frac{p}{2^3}\Big) * \mathfrak{N}\Big(x,z,...,z,\frac{p}{2^3}\Big)\\
&\ \vdots\\
&\leq \mathfrak{N}\Big(x,z,...,z,\frac{p}{2^{l-1}}\Big) \diamond \mathfrak{N}\Big(x,z,...,z,\frac{p}{2^{l-1}}\Big)\\
&= \mathfrak{N}\Big(x,z,...,z,\frac{p}{2^{l-1}}\Big).
\end{aligned}$$

(b): By using part (a), we have

$$\begin{aligned}
\mathfrak{M}\Big(\underbrace{x,x,...,x}_{l\text{–times}},z,...,z,p\Big) &= \mathfrak{M}\Big(\underbrace{z,z,...,z,}_{(n+1-l)\text{–times}}\ x,...,x,p\Big)\\
&\geq \mathfrak{M}\Big(z,x,...,x,\frac{p}{2^{n-l}}\Big).
\end{aligned}$$

and

$$\mathfrak{N}\Big(\underbrace{x,x,...,x}_{l\text{–times}},z,...,z,p\Big) \leq \mathfrak{N}\Big(z,x,...,x,\frac{p}{2^{n-l}}\Big).$$

□

The following is a description of the implementation of a topology induced by the $(\mathfrak{M},\mathfrak{N})$ in the *IF𝒢M*–space $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$.

Definition 2.11. Let $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ be an *IF𝒢M*–space and $x_0 \in \mathfrak{X}$. The open ball with center x_0 and radius $r \in (0,1)$ with respect to $p > 0$, is the set

$$\mathbf{B}_{x_0}^{(\mathfrak{M},\mathfrak{N})}\big(p,r\big) = \Big\{y \in \mathfrak{X} : \mathfrak{M}\big(x_0,y,y,...,y,p\big) > 1-r \text{ and } \mathfrak{N}\big(x_0,y,y,...,y,p\big) < r\Big\}.$$

Remark 2.12. Let $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ be an *IF𝒢M*–space. Define

$$\mathcal{T}^{(\mathfrak{M},\mathfrak{N})} = \Big\{\mathbf{T} \subset \mathfrak{X} : \text{ for each } x \in \mathbf{T}, \exists\, r \in (0,1) \text{ and } p > 0 \text{ such that } \mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\big(p,r\big) \subset \mathbf{T}\Big\}.$$

Then $\mathcal{T}^{(\mathfrak{M},\mathfrak{N})}$ is a topology on $\mathfrak{X}$ induced by $(\mathfrak{M},\mathfrak{N})$. Clearly, the set $\Big\{\mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\Big(\frac{1}{n},\frac{1}{n}\Big)\Big\}$ is a local base at $x \in \mathfrak{X}$ and so the topology $\mathcal{T}^{(\mathfrak{M},\mathfrak{N})}$ is first countable. Also, we notice that every open ball is an open set in the topology $\mathcal{T}^{(\mathfrak{M},\mathfrak{N})}$.

Proposition 2.13. *Let $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ be an IF𝒢M–space. Suppose $\mathfrak{M}\big(x_0,x_1,...,x_n,p\big) > 1-\eta$ and $\mathfrak{N}\big(x_0,x_1,...,x_n,p\big) < \eta$ for some $\eta \in (0,1)$ and $p > 0$. Then*

(a) *If* $n(\{x_0, x_1, ..., x_n\}) \geq 3$*, then* $x_i \in \mathbf{B}_{x_0}^{(\mathfrak{M},\mathfrak{N})}\big(p, \eta\big)$ *for each* $i \in \{0, 1, ..., n\}$.

(b) *If* $(\mathfrak{M}, \mathfrak{N})$ *is multiplicity independent and* $n(\{x_0, x_1, ..., x_n\}) \geq 2$*, then* $x_i \in \mathbf{B}_{x_0}^{(\mathfrak{M},\mathfrak{N})}\big(p, \eta\big)$ *for each* $i \in \{0, 1, ..., n\}$.

Proof. (a) Let $n(\{x_0, x_1, ..., x_n\}) \geq 3$. Clearly, for each $i \in \{0, 1, ..., n\}$ we have $\{x_0, x_i, ..., x_i\} \subsetneq \{x_0, x_1, ..., x_n\}$. Therefore, by using (**IFG**–3) and (**IFG**–10), we find

$$\mathfrak{M}\Big(x_0, x_i, ..., x_i, p\Big) \geq \mathfrak{M}\Big(x_0, x_1, ..., x_n, p\Big) > 1 - \eta$$

and

$$\mathfrak{N}\Big(x_0, x_i, ..., x_i, p\Big) \leq \mathfrak{N}\Big(x_0, x_1, ..., x_n, p\Big) < \eta.$$

Thus $x_i \in \mathbf{B}_{x_0}^{(\mathfrak{M},\mathfrak{N})}\big(p, \eta\big)$ for each $i \in \{0, 1, ..., n\}$.

(b) Let $(\mathfrak{M}, \mathfrak{N})$ is multiplicity independent and $n(\{x_0, x_1, ..., x_n\}) = 2$. Then $\mathfrak{M}\big(x_0, x_i, ..., x_i, p\big) = \mathfrak{M}\big(x_0, x_1, ..., x_n, p\big)$ and $\mathfrak{N}\big(x_0, x_i, ..., x_i, p\big) = \mathfrak{N}\big(x_0, x_1, ..., x_n, p\big)$ for each $i \in \{0, 1, ..., n\}$. Hence the result follows. □

Theorem 2.14. *Consider* $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *to be an IF*$\mathcal{G}$*M–space, where* $\eta * \eta = \eta$ *and* $\eta \diamond \eta = \eta$ *for all* $\eta \in [0, 1]$*. Then* $\Big(\mathfrak{X}, \mathcal{T}^{(\mathfrak{M},\mathfrak{N})}\Big)$ *is Hausdorff.*

Proof. Let $x, y \in \mathfrak{X}$ so that $x \neq y$. Then $0 < \mathfrak{M}\big(x, y, ..., y, p\big) < 1$ and $0 < \mathfrak{N}\big(x, y, ..., y, p\big) < 1$, $\forall\ p > 0$. Let $\eta_1 = \mathfrak{M}\big(x, y, ..., y, p\big)$ and $\eta_2 = \mathfrak{N}\big(x, y, ..., y, p\big)$. Choose $\eta \in (0, 1)$ such that $\eta = \max\{\eta_1, 1 - \eta_2\}$. For given $\eta < \eta_0 < 1$, there are $\eta_3, \eta_4 \in (0, 1)$ such that $\eta_3 > \eta_0$ and $(1 - \eta_4) < 1 - \eta_0$. Put $\eta_5 = \max\{\eta_3, \eta_4\}$. Now, consider the open balls $\mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\big(\frac{p}{2^n}, 1 - \eta_5\big)$ and $\mathbf{B}_y^{(\mathfrak{M},\mathfrak{N})}\big(\frac{p}{2^n}, 1 - \eta_5\big)$. We show that $\mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\big(\frac{p}{2^n}, 1 - \eta_5\big) \bigcap \mathbf{B}_y^{(\mathfrak{M},\mathfrak{N})}\big(\frac{p}{2^n}, 1 - \eta_5\big) = \phi$. Let on contrary,

$$z \in \mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\Big(\frac{p}{2^n}, 1 - \eta_5\Big) \bigcap \mathbf{B}_y^{(\mathfrak{M},\mathfrak{N})}\Big(\frac{p}{2^n}, 1 - \eta_5\Big).$$

Then $\eta_1 = \mathfrak{M}\big(x, y, ..., y, p\big) \geq \mathfrak{M}\big(x, z, ..., z, \frac{p}{2}\big) * \mathfrak{M}\big(z, y, ..., y, \frac{p}{2}\big)$. By using Proposition 2.10, we get $\mathfrak{M}\big(z, y, ..., y, \frac{p}{2}\big) \geq \mathfrak{M}\big(y, z, ..., z, \frac{p}{2^n}\big)$. Consequently,

$$\begin{aligned}
\eta_1 = \mathfrak{M}\Big(x, y, ..., y, p\Big) &\geq \mathfrak{M}\Big(x, z, ..., z, \frac{p}{2}\Big) * \mathfrak{M}\Big(y, z, ..., z, \frac{p}{2^n}\Big) \\
&\geq \mathfrak{M}\Big(x, z, ..., z, \frac{p}{2^n}\Big) * \mathfrak{M}\Big(y, z, ..., z, \frac{p}{2^n}\Big) \\
&\geq \eta_5 * \eta_5 \geq \eta_3 * \eta_3 > \eta_0 > \eta_1.
\end{aligned}$$

Also $\eta_2 = \mathfrak{N}\big(x, y, ..., y, p\big) \leq \mathfrak{N}\big(x, z, ..., z, \frac{p}{2}\big) \diamond \mathfrak{N}\big(z, y, ..., y, \frac{p}{2}\big)$. Again, by using proposition 2.10, we get $\mathfrak{N}\big(z, y, ..., y, \frac{p}{2}\big) \leq \mathfrak{N}\big(y, z, ..., z, \frac{p}{2^n}\big)$. Hence,

$$\begin{aligned}
\eta_2 = \mathfrak{N}\Big(x, y, ..., y, p\Big) &\leq \mathfrak{N}\Big(x, z, ..., z, \frac{p}{2}\Big) \diamond \mathfrak{N}\Big(y, z, ..., z, \frac{p}{2^n}\Big) \\
&\leq \mathfrak{N}\Big(x, z, ..., z, \frac{p}{2^n}\Big) \diamond \mathfrak{N}\Big(y, z, ..., z, \frac{p}{2^n}\Big) \\
&\leq (1 - \eta_5) \diamond (1 - \eta_5) \\
&\leq (1 - \eta_4) \diamond (1 - \eta_4) \\
&< (1 - \eta_0) < \eta_2.
\end{aligned}$$

Thus, we have a contradiction and so $\Big(\mathfrak{X}, \mathcal{T}^{(\mathfrak{M},\mathfrak{N})}\Big)$ is Hausdorff. □

Remark 2.15. Theorem 2.14 is also true even when the conditions $\eta * \eta = \eta$ and $\eta \diamond \eta = \eta$, $\forall\ \eta \in [0,1]$ are substituted with the condition that the pair $(\mathfrak{M},\mathfrak{N})$ exhibits multiplicity independence.

Proof. In the proof of Theorem 2.14, take $\eta_3,\eta_4 \in (0,1)$ such that $\eta_3 * \eta_3 \geq \eta_0$ and $(1-\eta_4)\diamond(1-\eta_4) \leq (1-\eta_0)$. Now, consider the open balls $\mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\left(\frac{p}{2}, 1-\eta_5\right)$ and $\mathbf{B}_y^{(\mathfrak{M},\mathfrak{N})}\left(\frac{p}{2}, 1-\eta_5\right)$. Then we can easily see that $\mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\left(\frac{p}{2}, 1-\eta_5\right) \bigcap \mathbf{B}_y^{(\mathfrak{M},\mathfrak{N})}\left(\frac{p}{2}, 1-\eta_5\right) = \phi$. □

Definition 2.16. Let $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ be an $IF\mathcal{G}M$–space. A sequence (x_k) in $\mathfrak{X}$ is said to be convergent to some $x \in \mathfrak{X}$ with respect to the $(\mathfrak{M},\mathfrak{N})$ if, for every $\eta \in (0,1)$ and $p > 0$, $\exists\ m_0 \in \mathbb{N}$ such that $\mathfrak{M}\left(x, x_{i_1}, ..., x_{i_n}, p\right) > 1-\eta$ and $\mathfrak{N}\left(x, x_{i_1}, ..., x_{i_n}, p\right) < \eta$ for all $i_1, i_2, ..., i_n \geq m_0$.

Theorem 2.17. *Let $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ be an $IF\mathcal{G}M$–space and $\mathcal{T}^{(\mathfrak{M},\mathfrak{N})}$ be the topology on $\mathfrak{X}$. Then a sequence (x_k) in $\mathfrak{X}$ is convergent to x iff $\mathfrak{M}\left(x_k, x_k, ..., x_k, x, p\right) \to 1$ and $\mathfrak{N}\left(x_k, x_k, ..., x_k, x, p\right) \to 0$ as $k \to \infty$ for every $p > 0$.*

Proof. Assume that (x_k) converges to x. Then, for every $p > 0$ and $\eta \in (0,1)$, $\exists\ m_0 \in \mathbb{N}$ such that $x_k \in \mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\left(p, \eta\right)$, $\forall\ k \geq m_0$. Consequently, for all $k \geq m_0$, we obtain

$$\mathfrak{M}\left(x, x_k, x_k, ..., x_k, p\right) > 1-\eta$$

and

$$\mathfrak{N}\left(x, x_k, x_k, ..., x_k, p\right) < \eta.$$

This implies that $1-\mathfrak{M}\left(x, x_k, x_k, ..., x_k, p\right) < \eta$ and $\mathfrak{N}\left(x, x_k, x_k, ..., x_k, p\right) < \eta$. As a result, we find $\mathfrak{M}\left(x_k, x_k, ..., x_k, x, p\right) \to 1$ and $\mathfrak{N}\left(x_k, x_k, ..., x_k, x, p\right) \to 0$, as $k \to \infty$.

Conversely, suppose that $\mathfrak{M}\left(x_k, x_k, ..., x_k, x, p\right) \to 1$ and $\mathfrak{N}\left(x_k, x_k, ..., x_k, x, p\right) \to 0$ as $k \to \infty$ for every $p > 0$. Then, for given $\eta \in (0,1)$, $\exists\ m_0 \in \mathbb{N}$ so that $1-\mathfrak{M}\left(x, x_k, x_k, ..., x_k, p\right) < \eta$ and $\mathfrak{N}\left(x, x_k, x_k, ..., x_k, p\right) < \eta$, $\forall\ k \geq m_0$. Hence $x_k \in \mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\left(p, \eta\right)$, $\forall\ k \geq m_0$. Thus (x_k) is convergent to x. □

Definition 2.18. Let $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ be an $IF\mathcal{G}M$–space. A sequence (x_k) in $\mathfrak{X}$ is said to be Cauchy with respect to the $(\mathfrak{M},\mathfrak{N})$ if, for every $\eta \in (0,1)$ and $p > 0$, $\exists\ m_0 \in \mathbb{N}$ such that $\mathfrak{M}\left(x_{i_0}, x_{i_1} ..., x_{i_n}, p\right) > 1-\eta$ and $\mathfrak{N}\left(x_{i_0}, x_{i_1} ..., x_{i_n}, p\right) < \eta$ for all $i_0, i_1, ..., i_n \geq m_0$.

An $IF\mathcal{G}M$–space $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ is said to be complete if every Cauchy sequence (x_k) in $\mathfrak{X}$ is convergent.

Theorem 2.19. *Consider $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ to be an $IF\mathcal{G}M$–space. Then every convergent sequence (x_k) is Cauchy in $\mathfrak{X}$.*

3. Statistical convergence in $IF\mathcal{G}M$–spaces

Our aim in this section is to explore the concept of statistical convergence of sequences in the $IF\mathcal{G}M$–space $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$. In order to accomplish this, let us recall some notions as follows:

Let $A \subseteq \mathbb{N}$. The asymptotic (or natural) density of the set A, denoted by $d(A)$, is defined as:

$$d(A) = \lim_{k\to\infty} \frac{1}{k}\left|\{n \leq k : n \in A\}\right|,$$

provided the limit exists. Here, $|B|$ denotes the cardinality of the set B. A real sequence (x_k) is called statistically convergent to $l \in \mathbb{R}$ if

$$d\left(\{k \in \mathbb{N} : |x_k - l| > \xi\}\right) = 0$$

holds for every $\xi > 0$ and the limit is denoted by $x_k \xrightarrow{st} l$ (see [15], [32]).

Definition 3.1. ([1]) Consider the n–product of $\mathbb{N}$, i.e., $\mathbb{N}^n = \prod_{i=1}^{n} \mathbb{N}^i$. Let $Y \subseteq \mathbb{N}^n$ and $Y(k) = \Big\{(i_1, i_2, ..., i_n) \in Y : i_1, i_2, ..., i_n \leq k\Big\}$. Then, the n–dimensional asymptotic density of the set Y is defined as

$$\mathrm{d}_n(Y) = \lim_{k\to\infty} \frac{n!}{k^n} |Y(k)|.$$

For a subset $A \subseteq \mathbb{N}$, the n–dimensional asymptotic density of the set A is defined as $d_n(A) = \lim_{k\to\infty} \frac{n!}{k^n}|A(k)|$, where $A(k) = \Big\{(k_1, k_2, ..., k_n) \in A^n : k_1, k_2, ..., k_n \leq k\Big\}$.

Definition 3.2. ([1]) Consider a subset $A = \{k_m : m \in \mathbb{N}\}$ of $\mathbb{N}$. Then A is called statistically dense in $\mathbb{N}$ if $d_n(A) = \lim_{k\to\infty} \frac{n!}{k^n}|A(k)| = 1$.

Now we are ready to introduce our main result as follows:

Definition 3.3. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space and $x \in \mathfrak{X}$. For any $\eta \in (0, 1)$ and $p > 0$, the (p, η)–vicinity of $x \in \mathfrak{X}$ with respect to $(\mathfrak{M}, \mathfrak{N})$ is defined by

$$\mathbf{V}_x^{(\mathfrak{M},\mathfrak{N})}(p, \eta) = \Big\{\Big(x_1, x_2, ..., x_n\Big) \in \mathfrak{X}^n : \mathfrak{M}\Big(x, x_1, x_2, ..., x_n, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x, x_1, x_2, ..., x_n, p\Big) < \eta\Big\}.$$

Definition 3.4. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space. A sequence (x_k) in $\mathfrak{X}$ is statistically convergent to a $x \in \mathfrak{X}$ with respect to $(\mathfrak{M}, \mathfrak{N})$ if, for every $\eta \in (0, 1)$ and $p > 0$,

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) \leq 1 - \eta \text{ or } \mathfrak{N}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) \geq \eta\Big\}\Big) = 0.$$

In such a case, we write $x_k \xrightarrow{st-(\mathfrak{M},\mathfrak{N})} x$ or $st\text{–}\lim_k x_k = x$.

Lemma 3.5. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space. Suppose (x_k) is a sequence in $\mathfrak{X}$. Then, for given $p > 0$ and $\eta \in (0, 1)$, the following are equivalent:*

(a) $st\text{–}\lim_k x_k = x$.

(b) $d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) > 1 - \eta \textit{ and } \mathfrak{N}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) < \eta\Big\}\Big) = 1.$

(c) $d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) \leq 1 - \eta\Big\}\Big) = 0$ *and*
$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{N}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) \geq \eta\Big\}\Big) = 0.$

(d) $d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) > 1 - \eta\Big\}\Big) = 1$ *and*
$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{N}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) < \eta\Big\}\Big) = 1.$

Theorem 3.6. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space and let (x_k) be a sequence in $\mathfrak{X}$. Then, for given $\eta \in (0, 1)$ and $p > 0$, the following are equivalent:*

(a) $st\text{–}\lim_k x_k = x$.

(a) $d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \Big(x_{i_1}, x_{i_2}, ..., x_{i_n}\Big) \notin \mathbf{V}_x^{(\mathfrak{M},\mathfrak{N})}(p, \eta)\Big\}\Big) = 0.$

Proof. The proof is a direct consequence of the definition of statistical convergence. □

Theorem 3.7. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space. Suppose (x_k) is a sequence in $\mathfrak{X}$ such that $st\text{–}\lim_k x_k = x$. Then, for any $\eta \in (0, 1)$ and $p > 0$, we have*

$$d_n\Big(\Big\{(i_1, i_2, ...i_n) \in \mathbb{N}^n : x_{i_j} \notin \mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\Big(p, \eta\Big) \textit{ for every } j \in \{1, 2, ..., n\}\Big\}\Big) = 0.$$

Proof. For given $\eta \in (0,1)$ and $p > 0$, set

$$\mathbf{A}(p,\eta) = \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) < \eta\Big\}$$

and

$$\mathbf{D}(p,\eta) = \Big\{(i_1, i_2, ...i_n) \in \mathbb{N}^n : x_{i_j} \in \mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\Big(p,\eta\Big) \text{ for every } j \in \{1, 2, ..., n\}\Big\}.$$

Since $st\text{–}\lim_k x_k = x$, so $d_n\Big(\mathbf{A}(p,\eta)\Big) = 1$. Let $(i_1, i_2, ...i_n) \in \mathbf{A}(p,\eta)$. Then $\mathfrak{M}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) > 1 - \eta$ and $\mathfrak{N}\Big(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, p\Big) < \eta$. By using Proposition 2.13, we can conclude that $x_{i_j} \in \mathbf{B}_x^{(\mathfrak{M},\mathfrak{N})}\Big(p,\eta\Big)$ for every $j \in \{1, 2, ..., n\}$. Therefore $(i_1, i_2, ...i_n) \in \mathbf{D}(p,\eta)$ and so $\mathbf{A}(p,\eta) \subseteq \mathbf{D}(p,\eta)$. This implies that $d_n(\mathbf{D}(p,\eta)) = 1$. Hence the result follows. □

Theorem 3.8. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space. If a sequence (x_k) in $\mathfrak{X}$ is convergent to $x \in \mathfrak{X}$ then $st\text{–}\lim_k x_k = x$.*

Proof. Suppose that (x_k) is convergent to $x \in \mathfrak{X}$. Then, for every $\eta \in (0,1)$ and $p > 0$, $\exists\ m_0 \in \mathbb{N}$ so that $\mathfrak{M}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) > 1 - \eta$ and $\mathfrak{N}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) < \eta$, $\forall\ i_1, i_2, ..., i_n \geq m_0$. Define

$$\mathbf{A}(m) = \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1, i_2, ..., i_n \leq m,\ \mathfrak{M}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) < \eta\Big\}.$$

Clearly,

$$|\mathbf{A}(m)| \geq \binom{m - m_0}{n}. \implies \lim_{m\to\infty} \frac{n!}{m^n}|\mathbf{A}(m)| \geq \lim_{m\to\infty} \frac{n!}{m^n}\binom{m - m_0}{n} = 1.$$

Consequently, $\lim_{m\to\infty} \frac{n!}{m^n}|(\mathbf{A}(m))^c| = 0$. This implies that $st\text{–}\lim_k x_k = x$. □

We provide the following example to support the fact that the converse of the above Theorem 3.3 does not hold.

Example 3.9. Consider the *IF$\mathcal{G}$M*–space $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ defined in Example 2.2, where $\mathfrak{X} = \mathbb{R}$ and $g : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^+$ such that $g(x_0, x_1, ..., x_n) = \max_{0\leq i,j\leq n} \Big\{|x_i - x_j|\Big\}$. Now, define the sequence (x_k) in $\mathbb{R}$ by

$$x_k = \left\{\begin{array}{ll} k, & \text{if } k = i^3, \\ 1, & \text{otherwise} \end{array}\right\} i \in \mathbb{N}.$$

Then, (x_k) is statistically convergent to 1, but not convergent.

Theorem 3.10. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space. Suppose (x_k) is a sequence in $\mathfrak{X}$ such that (x_k) is statistically convergent. Then $st\text{–}\lim_k x_k$ is unique.*

Proof. Assume that $st\text{–}\lim_k x_k = x$ and $st\text{–}\lim_k x_k = z$. We need to show that $x = z$. For given $\eta \in (0,1)$, choose $\eta_1 \in (0,1)$ such that $(1 - \eta_1) * (1 - \eta_1) > 1 - \eta$ and $\eta_1 \diamond \eta_1 < \eta$. For given $p > 0$, consider the following sets:

$$\mathbf{A}(p,\eta_1) = \left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\left(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\right) \leq 1 - \eta_1 \text{ or } \mathfrak{N}\left(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\right) \geq \eta_1\right\},$$

$$\mathbf{B}(p,\eta_1) = \left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\left(z, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\right) \leq 1 - \eta_1 \text{ or } \mathfrak{N}\left(z, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\right) \geq \eta_1\right\},$$

$$\mathbf{A}^c(p,\eta_1) = \left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\left(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\right) > 1 - \eta_1 \text{ and } \mathfrak{N}\left(x, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\right) < \eta_1\right\},$$

$$\mathbf{B}^c(p,\eta_1) = \Big\{(i_1,i_2,...,i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(z, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\Big) > 1-\eta_1 \text{ and } \mathfrak{N}\Big(z, x_{i_1}, x_{i_2}, ..., x_{i_n}, \frac{p}{2}\Big) < \eta_1\Big\}.$$

Since $st-\lim_k x_k = x$ and $st-\lim_k x_k = z$, we have $d_n(\mathbf{A}(p,\eta_1)) = 0$ and $d_n(\mathbf{B}(p,\eta_1)) = 0$. Also, by Lemma 3.5, we have $d_n(\mathbf{A}^c(p,\eta_1) = d_n(\mathbf{B}^c(p,\eta_1) = 1$. Thus $d_n(\mathbf{A}(p,\eta_1) \cup \mathbf{B}(p,\eta_1)) = 0$, implies $d_n((\mathbf{A}(p,\eta_1) \cup \mathbf{B}(p,\eta_1))^c) = d_n(\mathbf{A}(p,\eta_1)^c \cap \mathbf{B}(p,\eta_1)^c) = 1$. Let $(i_1,i_2,...,i_n) \in \mathbf{A}(p,\eta_1)^c \cap \mathbf{B}(p,\eta_1)^c$. Then, by using (**IFG**–6), (**IFG**–3) and the part (3) of Definition 1.4, we get

$$\begin{aligned}\mathfrak{M}\Big(x, z, ..., z, p\Big) &\geq \mathfrak{M}\Big(x, x_{i_n}, x_{i_n} ..., x_{i_n}, \frac{p}{2}\Big) * \mathfrak{M}\Big(x_{i_n}, z, z, ..., z, \frac{p}{2}\Big)\\ &\geq \mathfrak{M}\Big(x, x_{i_1}, x_{i_2} ..., x_{i_n}, \frac{p}{2}\Big) * \mathfrak{M}\Big(z, x_{i_i}, x_{i_2} ..., x_{i_n}, \frac{p}{2}\Big)\\ &\geq (1-\eta_1) * (1-\eta_1)\\ &> 1-\eta.\end{aligned}$$

Also, by using (**IFG**–13), (**IFG**–10) and part (3′) of Definition 1.4, we have

$$\begin{aligned}\mathfrak{N}\Big(x, z, ..., z, p\Big) &\leq \mathfrak{N}\Big(x, x_{i_n}, x_{i_n} ..., x_{i_n}, \frac{p}{2}\Big) \diamond \mathfrak{N}\Big(x_{i_n}, z, z, ..., z, \frac{p}{2}\Big)\\ &\leq \mathfrak{N}\Big(x, x_{i_1}, x_{i_2} ..., x_{i_n}, \frac{p}{2}\Big) \diamond \mathfrak{N}\Big(z, x_{i_i}, x_{i_2} ..., x_{i_n}, \frac{p}{2}\Big)\\ &\leq \eta_1 \diamond \eta_1\\ &< \eta.\end{aligned}$$

Since $\eta \in (0,1)$ is arbitrary, we conclude that $\mathfrak{M}\Big(x, z, ..., z, p\Big) = 1$ and $\mathfrak{N}\Big(x, z, ..., z, p\Big) = 0$, $\forall\ p > 0$. Hence $x = z$. □

Definition 3.11. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space and (x_{k_m}) be a subsequence of (x_k) in $\mathfrak{X}$. Then (x_{k_m}) is called a statistically dense subsequence of (x_k) if the index set $\{k_m : m \in \mathbb{N}\}$ is statistically dense in $\mathbb{N}$, i.e., $d_n(\{k_m : m \in \mathbb{N}\}) = 1$.

Theorem 3.12. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space and let (x_k) be a sequence in $\mathfrak{X}$. Then, the following are equivalent:*

(1) *$st-\lim_k x_k = x$, for some $x \in \mathfrak{X}$.*

(2) *There exists a convergent sequence (z_k) in $\mathfrak{X}$ with $x_k = z_k$ for almost all k.*

(3) *There exists a subsequence (x_{k_m}) of (x_k) such that (x_{k_m}) is statistically dense and (x_{k_m}) is convergent.*

(4) *There exists a subsequence (x_{k_m}) of (x_k) such that (x_{k_m}) is statistically dense and (x_{k_m}) is convergent.*

Proof. ((1) $\implies$ (2)): Let $x \in \mathfrak{X}$ and $st-\lim_k x_k = x$. By Lemma 3.5, for any $\eta \in (0,1)$ and $p > 0$, it follows that

$$d_n\Big(\Big\{(i_1,i_2,...,i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) > 1-\eta \text{ and } \mathfrak{N}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) < \eta\Big\}\Big) = 1.$$

For $j \in \mathbb{N}$, consider the set

$$K(p,j) = \Big\{(i_1,i_2,....,i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) > 1-\frac{1}{j} \text{ and } \mathfrak{N}\Big(x, x_{i_1}, ..., x_{i_n}, p\Big) < \frac{1}{j}\Big\}.$$

Clearly, $K(p, j+1) \subseteq K(p, j)$ for every $j \in \mathbb{N}$. Since $st-\lim_k x_k = x$, we have

$$d_n(K(p, j)) = 1, \ (j \in \mathbb{N}). \tag{2}$$

Choose an arbitrary $\left(i_1^1, i_2^1, ..., i_n^1\right) \in K(p, 1)$. Put $q_1 = \max\left\{i_1^1, i_2^1, ..., i_n^1\right\}$. Since Equation 2 holds, there are $\left(i_1^2, i_2^2, ..., i_n^2\right) \in K(p, 2)$ and $q_2 = \max\left\{i_1^2, i_2^2, ..., i_n^2\right\}$ such that $q_2 > q_1$ and for each $k \geq q_2$ implies

$$\lim_{k\to\infty} \frac{n!}{k^n}\left|\left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1, i_2, ..., i_n \leq k, \mathfrak{M}\left(x, x_{i_1}, ..., x_{i_n}, p\right) > 1 - \frac{1}{2} \text{ and } \mathfrak{N}\left(x, x_{i_1}, ..., x_{i_n}, p\right) < \frac{1}{2}\right\}\right| > \frac{1}{2}.$$

Further again by Equation 2, $\exists \left(i_1^3, i_2^3, ..., i_n^3\right) \in K(p, 3)$ with $q_3 = \max\left\{i_1^3, i_2^3, ..., i_n^3\right\} > q_2$ such that for each $k \geq q_3$,

$$\lim_{k\to\infty} \frac{n!}{k^n}\left|\left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1, i_2, ..., i_n \leq k, \mathfrak{M}\left(x, x_{i_1}, ..., x_{i_n}, p\right) > 1 - \frac{1}{3} \text{ and } \mathfrak{N}\left(x, x_{i_1}, ..., x_{i_n}, p\right) < \frac{1}{3}\right\}\right| > \frac{2}{3}.$$

As a result of continuing this process, by induction, we can build up an increasing sequence (q_j) of natural numbers, $q_j = \max\left\{i_1^j i_2^j, ..., i_n^j\right\}$ such that $\left(i_1^j, i_2^j, ..., i_n^j\right) \in K(p, j)$ and for each $k \geq q_j$,

$$\lim_{k\to\infty} \frac{n!}{k^n}\left|\left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1, i_2, ..., i_n \leq k, \mathfrak{M}\left(x, x_{i_1}, ..., x_{i_n}, p\right) > 1 - \frac{1}{j} \text{ and } \mathfrak{N}\left(x, x_{i_1}, ..., x_{i_n}, p\right) < \frac{1}{j}\right\}\right| > \frac{j-1}{j}.$$

Now consider the sets

$$\mathcal{K}_1 = \left\{l \in \mathbb{N} : 1 < l < q_1\right\} \text{ and}$$

$$\mathcal{K}_2 = \bigcup_{j\in\mathbb{N}} \left\{l = \max\{k_1, k_2, ..., k_n\} : (k_1, k_2, ..., k_n) \in K(p, j), q_j \leq l < q_{j+1}\right\}.$$

Define $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ and the sequence

$$z_l = \left\{\begin{array}{ll} x_l, & \text{if } l \in \mathcal{K}, \\ x, & \text{otherwise.} \end{array}\right\}$$

For $\eta \in (0, 1)$, choose $j \in \mathbb{N}$ such that $\frac{1}{j} < \eta$ and hence $1 - \frac{1}{j} > 1 - \eta$. It follows that the sequence (z_l) converges to x with respect to $(\mathfrak{M}, \mathfrak{N})$.

Now , for fixed $k \in \mathbb{N}$ and $q_j \leq k < q_{j+1}$, we have

$$\begin{aligned} &\left\{ (i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1, i_2, ..., i_n \leq k; x_{i_m} \neq z_{i_m}, \ m \in \{1, 2, ..., n\}\right\} \\ &\subseteq \left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1, i_2, ..., i_n \leq k\right\} \Big\backslash \left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1, i_2, ..., i_n \leq k;\right. \\ &\quad \left.\mathfrak{M}\left(x, x_{i_1}, ..., x_{i_n}, p\right) > 1 - \frac{1}{j} \text{ and } \mathfrak{N}\left(x, x_{i_1}, ..., x_{i_n}, p\right) < \frac{1}{j}\right\}. \end{aligned}$$

This implies that

$$\begin{aligned}
&\lim_{k\to\infty}\frac{n!}{k^n}\left|\left\{ (i_1,i_2,...,i_n)\in\mathbb{N}^n : i_1,i_2,...,i_n\le k; x_{i_m}\neq z_{i_m},\ m\in\{1,2,...,n\}\right\}\right|\\
&\quad\le \lim_{k\to\infty}\frac{n!}{k^n}\left|\left\{(i_1,i_2,...,i_n)\in\mathbb{N}^n : i_1,i_2,...,i_n\le k\right\}\right| - \lim_{k\to\infty}\frac{n!}{k^n}\left|\left\{(i_1,i_2,...,i_n)\in\mathbb{N}^n : i_1,i_2,...,i_n\le k;\right.\right.\\
&\qquad \left.\left.\mathfrak{M}\left(x,x_{i_1},...,x_{i_n},p\right) > 1-\frac{1}{j} \text{ and } \mathfrak{N}\left(x,x_{i_1},...,x_{i_n},p\right) < \frac{1}{j}\right\}\right|\\
&\quad\le 1-\lim_{k\to\infty}\frac{n!}{k^n}\left|\left\{(i_1,i_2,...,i_n)\in\mathbb{N}^n : i_1,i_2,...,i_n\le k;\right.\right.\\
&\qquad \left.\left.\mathfrak{M}\left(x,x_{i_1},...,x_{i_n},p\right) > 1-\frac{1}{j} \text{ and } \mathfrak{N}\left(x,x_{i_1},...,x_{i_n},p\right) < \frac{1}{j}\right\}\right|\\
&\quad\le \frac{1}{j} < \eta.
\end{aligned}$$

Since η is arbitrary, we conclude that

$$d_n\left(\left\{(i_1,i_2,...,i_n)\in\mathbb{N}^n : i_1,i_2,...,i_n\le k; x_{i_m}\neq z_{i_m},\ m\in\{1,2,...,n\}\right\}\right) = 0.$$

Hence $x_k = z_k$ for almost all k.

((2) $\implies$ (3)): Let (z_k) be a convergent sequence in $\mathfrak{X}$ such that $x_k = z_k$ for almost all $k\in\mathbb{N}$. Then, the set $A = \{k\in\mathbb{N} : x_k = z_k\}$ has $d_n(A) = 1$. Hence, $(z_k)_{k\in A}$ is a statistically dense subsequence of (x_k) which is convergent.

((3) $\implies$ (4)): The proof directly follows from Theorem 3.8.

((4) $\implies$ (1)): Let (x_{k_m}) be a subsequence of (x_k) such that (x_{k_m}) is statistically dense and (x_{k_m}) is statistically convergent to $x\in\mathfrak{X}$. Consider the index set $B = \{k_m : m\in\mathbb{N}\}$. Then $d_n(B) = 1$. Now, for any $\eta\in(0,1)$ and $p > 0$, we obtain

$$\begin{aligned}
&\left\{(i_1,i_2,...,i_n)\in\mathbb{N}^n : i_1,i_2,...,i_n\le k; \mathfrak{M}\left(x,x_{i_1},...,x_{i_n},p\right) > 1-\eta \text{ and } \mathfrak{N}\left(x,x_{i_1},...,x_{i_n},p\right) < \eta\right\}\\
&\quad\supseteq \left\{(i_1,i_2,...,i_n)\in\mathbb{B}^n : i_1,i_2,...,i_n\le k; \mathfrak{M}\left(x,x_{i_1},...,x_{i_n},p\right) > 1-\eta \text{ and } \mathfrak{N}\left(x,x_{i_1},...,x_{i_n},p\right) < \eta\right\}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\lim_{k\to\infty}\frac{n!}{k^n}\left|\left\{(i_1,i_2,...,i_n)\in\mathbb{N}^n : i_1,i_2,...,i_n\le k;\ \mathfrak{M}\left(x,x_{i_1},...,x_{i_n},p\right) > 1-\eta \text{ and } \mathfrak{N}\left(x,x_{i_1},...,x_{i_n},p\right) < \eta\right\}\right|\\
&\quad\ge \lim_{k\to\infty}\frac{n!}{k^n}\left|\left\{(i_1,i_2,...,i_n)\in\mathbb{B}^n : i_1,i_2,...,i_n\le k;\ \mathfrak{M}\left(x,x_{i_1},...,x_{i_n},p\right) > 1-\eta \text{ and } \mathfrak{N}\left(x,x_{i_1},...,x_{i_n},p\right) < \eta\right\}\right|.\\
&\quad= 1.
\end{aligned}$$

Consequently, (x_k) is statistically convergent to x. □

As a direct consequence of the above Theorem 3.12, we have the corollary as follows:

Corollary 3.13. *Consider $(\mathfrak{X},\mathfrak{M},\mathfrak{N},*,\diamond)$ to be an IF$\mathcal{G}$M–space and (x_k) to be a sequence in $\mathfrak{X}$ such that (x_k) is statistically convergent. Then (x_k) has a convergent subsequence.*

The converse of the above statement is not generally true, i.e., there can exist a non–statistically convergent sequence that has a convergent subsequence. We have the following example in support of our statement.

Example 3.14. Let $(\mathbb{R}, g)$ be a g–metric space with order n, where

$$g(x_0, x_1, ..., x_n) = \max_{0\leq i,j\leq n} \Big\{|x_i - x_j|\Big\}, \forall\, x_0, x_1, ..., x_n \in \mathbb{R}.$$

Consider the tuple $(\mathfrak{M}, \mathfrak{N})$ defined in Example 2.2. Let $\eta_1 * \eta_2 = \min\{\eta_1, \eta_2\}$ and $\eta_1 \diamond \eta_2 = \max\{\eta_1, \eta_2\}$, $\forall$ $\eta_1, \eta_2 \in [0, 1]$. Then $(\mathbb{R}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is an $IF\mathcal{G}M$–space. Define (x_k) by

$$x_k = \left\{ \begin{array}{ll} \frac{1}{k}, & \text{if } k = i^2, \\ k^2, & \text{otherwise} \end{array} \right\} i \in \mathbb{N}.$$

Then (x_{i^2}) is a subsequence of (x_k) and it is convergent to 0. Notice that (x_k) is not statistically convergent.

4. Statistically Cauchy sequence in $IF\mathcal{G}M$–spaces

In this section, we introduce the concept of statistically Cauchy sequences in the $IF\mathcal{G}M$–space $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ and investigate some properties.

Definition 4.1. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an $IF\mathcal{G}M$–space. A sequence (x_k) in $\mathfrak{X}$ is statistically Cauchy with respect to $(\mathfrak{M}, \mathfrak{N})$ if, for every $\eta \in (0, 1)$ and $p > 0$, $\exists\, N = N(\eta) \in \mathbb{N}$ such that

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_N, p\Big) \leq 1 - \eta \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_N, p\Big) \geq \eta\Big\}\Big) = 0.$$

Now, we investigate the relationship between the statistically convergent and the statistically Cauchy sequences in an $IF\mathcal{G}M$–space as follows:

Theorem 4.2. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an $IF\mathcal{G}M$–space and (x_k) to be a sequence in $\mathfrak{X}$ such that (x_k) is statistically convergent. Then (x_k) is statistically Cauchy.*

Proof. Let $st\text{–}\lim_k x_k = x$. For given $\eta \in (0, 1)$, select $\eta_1 \in (0, 1)$ so that $(1 - \eta_1) * (1 - \eta_1) > 1 - \eta$ and $\eta_1 \diamond \eta_1 < \eta$. For $p > 0$, consider the following sets:

$$P(\eta_1) = \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, \frac{p}{2}\Big) \leq 1 - \eta_1 \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, \frac{p}{2}\Big) \geq \eta_1\Big\}$$

and

$$P(\eta_1)^c = \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, \frac{p}{2}\Big) > 1 - \eta_1 \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, \frac{p}{2}\Big) < \eta_1\Big\}.$$

Since $st\text{–}\lim_k x_k = x$, so $d_n(P(\eta_1)) = 0$ and $d_n(P(\eta_1)^c) = 1$. Let $(j_1, j_2, ..., j_n) \in P(\eta_1)^c$. Then, we have $\mathfrak{M}\Big(x_{j_1}, x_{j_2}, ..., x_{j_n}, x, \frac{p}{2}\Big) > 1 - \eta_1$ and $\mathfrak{N}\Big(x_{j_1}, x_{j_2}, ..., x_{j_n}, x, \frac{p}{2}\Big) < \eta_1$. Fix $j_k \in \mathbb{N}$, for some $k \in \{1, 2, ..., n\}$. Then

$$\mathfrak{M}\Big(x_{j_k}, x, ..., x, \frac{p}{2}\Big) \geq \mathfrak{M}\Big(x_{j_1}, x_{j_2}, ..., x_{j_n}, x, \frac{p}{2}\Big) > 1 - \eta_1 \text{ and}$$

$$\mathfrak{N}\Big(x_{j_k}, x, ..., x, \frac{p}{2}\Big) \leq \mathfrak{N}\Big(x_{j_1}, x_{j_2}, ..., x_{j_n}, x, \frac{p}{2}\Big) < \eta_1.$$

For $(i_1, i_2, ..., i_n) \in P(\eta_1)^c$, we have

$$\begin{aligned} \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) &\geq \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, \frac{p}{2}\Big) * \mathfrak{M}\Big(x_{j_k}, x, ..., x, x, \frac{p}{2}\Big) \\ &> (1 - \eta_1) * (1 - \eta_1) \\ &> 1 - \eta \end{aligned}$$

and

$$\mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) \leq \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, \frac{p}{2}\Big) \diamond \mathfrak{N}\Big(x_{j_k}, x, ..., x, x, \frac{p}{2}\Big)$$
$$< \eta_1 \diamond \eta_1$$
$$< \eta.$$

This implies that

$$P(\eta_1)^c \subseteq \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) < \eta\Big\}.$$

Consequently,

$$d_n\Big(P(\eta_1)^c\Big) \leq d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) < \eta\Big\}\Big).$$

Therefore,

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) < \eta\Big\}\Big) = 1$$

and thus

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) \leq 1 - \eta \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_{j_k}, p\Big) \geq \eta\Big\}\Big) = 0.$$

This completes the proof of the theorem. □

The converse statement of theorem 4.2 is not true. Let us look at the following example to demonstrate this:

Example 4.3. Let $\mathfrak{X} = (0, 1]$. Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$, the *IF$\mathcal{G}$M*–space which is defined in Example 3.9. Consider the sequence (x_k) defined by

$$x_k = \left\{ \begin{array}{ll} 1, & \text{if } k = i^3, \\ \frac{1}{k}, & \text{otherwise} \end{array} \right\} i \in \mathbb{N}.$$

Then (x_k) is statistically Cauchy sequence, but not statistically convergent.

Definition 4.4. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space. Then $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is called statistically complete iff every statistically Cauchy sequence in $\mathfrak{X}$ is statistically convergent.

Theorem 4.5. *If an IF$\mathcal{G}$M–space $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is statistically complete then it is complete.*

Proof. Suppose $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is statistically complete. Let (x_k) be a Cauchy sequence in $\mathfrak{X}$. Then it is statistically Cauchy in $\mathfrak{X}$. Since $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is statistically complete, (x_k) is statistically convergent. By Corollary 3.13, we have a convergent subsequence (x_{k_m}) of (x_k). Let (x_{k_m}) converges to x. For given $\eta \in (0, 1)$, there are $\eta_1, \eta_2, \eta_3, \eta_4 \in (0, 1)$ such that $(1 - \eta_2) * (1 - \eta_2) > 1 - \eta_1$, $\eta_3 \diamond \eta_3 < \eta_1$, $(1 - \eta_1) * (1 - \eta_4) > 1 - \eta$ and $\eta_1 \diamond \eta_4 < \eta$. Put $\eta_5 = \min\{\eta_2, \eta_3\}$.

Since (x_k) is Cauchy, for given $p > 0$, $\exists\, m_0 \in \mathbb{N}$ such that for all $i_0, i_1, ..., i_n \geq m_0$, we have

$$\mathfrak{M}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{4}\Big) > 1 - \eta_5$$

and

$$\mathfrak{N}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{4}\Big) < \eta_5.$$

Also, (x_{k_m}) is convergent to x. Therefore, there is $m_1 \in \mathbb{N}$ such that $\mathfrak{M}\Big(x_{i_{k_1}}, x_{i_{k_2}}, ..., x_{i_{k_n}}, x, \frac{p}{4}\Big) > 1 - \eta_4$ and $\mathfrak{N}\Big(x_{i_{k_1}}, x_{i_{k_2}}, ..., x_{i_{k_n}}, x, \frac{p}{4}\Big) < \eta_4$ for all $i_{k_1}, i_{k_2}, ..., i_{k_n} \geq m_1$. Let $M = \max\{m_0, m_1\}$. For $i_0, i_1, ..., i_n, i_{k_1}, i_{k_2}, ..., i_{k_n} \geq M$, by using (**IFG**–3) and (**IFG**–6), we have

$$\begin{aligned}
\mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) &\geq \mathfrak{M}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{2}\Big) * \mathfrak{M}\Big(x_{i_0}, x_{i_0}, ..., x_{i_0}, x, \frac{p}{2}\Big) \\
&\geq \mathfrak{M}\Big(x_{i_{k_n}}, x_{i_{k_n}}, ..., x_{i_{k_n}}, x, \frac{p}{4}\Big) * \mathfrak{M}\Big(x_{i_0}, x_{i_0}, ..., x_{i_0}, x_{i_{k_n}} \frac{p}{4}\Big) * \mathfrak{M}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{2}\Big) \\
&\geq \mathfrak{M}\Big(x_{i_{k_1}}, x_{i_{k_2}}, ..., x_{i_{k_n}}, x, \frac{p}{4}\Big) * \mathfrak{M}\Big(x_{i_0}, x_{i_{k_1}}, ..., x_{i_{k_n}}, \frac{p}{4}\Big) * \mathfrak{M}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{4}\Big) \\
&> (1 - \eta_4) * (1 - \eta_5) * (1 - \eta_5) \\
&> (1 - \eta_4) * (1 - \eta_2) * (1 - \eta_2) \\
&> (1 - \eta_4) * (1 - \eta_1) \\
&> (1 - \eta).
\end{aligned}$$

Again, by using (**IFG**–10) and (**IFG**–13), we have

$$\begin{aligned}
\mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) &\leq \mathfrak{N}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{2}\Big) * \mathfrak{N}\Big(x_{i_0}, x_{i_0}, ..., x_{i_0}, x, \frac{p}{2}\Big) \\
&\leq \mathfrak{N}\Big(x_{i_{k_n}}, x_{i_{k_n}}, ..., x_{i_{k_n}}, x, \frac{p}{4}\Big) * \mathfrak{N}\Big(x_{i_0}, x_{i_0}, ..., x_{i_0}, x_{i_{k_n}} \frac{p}{4}\Big) * \mathfrak{N}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{2}\Big) \\
&\leq \mathfrak{N}\Big(x_{i_{k_1}}, x_{i_{k_2}}, ..., x_{i_{k_n}}, x, \frac{p}{4}\Big) * \mathfrak{N}\Big(x_{i_0}, x_{i_{k_1}}, ..., x_{i_{k_n}}, \frac{p}{4}\Big) * \mathfrak{N}\Big(x_{i_0}, x_{i_1}, ..., x_{i_n}, \frac{p}{4}\Big) \\
&< \eta_4 \diamond \eta_5 * \eta_5 \\
&< \eta_4 \diamond \eta_3 \diamond \eta_3 \\
&< \eta_4 \diamond \eta_1 \\
&< \eta.
\end{aligned}$$

This implies that (x_k) converges to x and hence $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is complete. □

Definition 4.6. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space. A sequence (x_k) in $\mathfrak{X}$ is called statistically bounded with respect to $(\mathfrak{M}, \mathfrak{N})$ if, for an arbitrary $x_0 \in \mathfrak{X}$, there exist $\eta_0 \in (0, 1)$ and $p_0 > 0$ such that

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p_0\Big) \leq 1 - \eta_0 \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p_0\Big) \geq \eta_0\Big\}\Big) = 0.$$

Theorem 4.7. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space. If a sequence (x_k) in $\mathfrak{X}$ is statistically Cauchy then it is statistically bounded.*

Proof. Let the sequence (x_k) is statistically Cauchy in $\mathfrak{X}$. Then, for every $p > 0$ and $\eta \in (0, 1)$, there is $N = N(\eta) \in \mathbb{N}$ such that the set

$$\mathbf{E}(p, \eta) = \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_N, \frac{p}{2}\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_N, \frac{p}{2}\Big) < \eta\Big\}$$

has asymptotic density 1, i.e., $d_n(\mathbf{E}(p, \eta)) = 1$. Fix $x_0 \in \mathfrak{X}$ and let $\mathfrak{M}\Big(x_N, x_N, ..., x_N, x_0, \frac{p}{2}\Big) = \alpha$ and $\mathfrak{N}\Big(x_N, x_N, ..., x_N, x_0, \frac{p}{2}\Big) = \gamma$. Since $\alpha, \gamma \in (0, 1)$, there are $\beta, \delta \in (0, 1)$ such that $(1 - \eta) * \alpha > 1 - \beta$ and $\eta \diamond \gamma < \delta$. Let $(i_1, i_2, ..., i_n) \in \mathbf{E}(p, \eta)$. Then

$$\begin{aligned}
\mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) &\geq \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_N, \frac{p}{2}\Big) * \mathfrak{M}\Big(x_N, x_N, ..., x_N, x_0, \frac{p}{2}\Big) \\
&> (1 - \eta) * \alpha \\
&> 1 - \beta
\end{aligned}$$

and

$$\mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) \leq \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_N, \frac{p}{2}\Big) \diamond \mathfrak{N}\Big(x_N, x_N, ..., x_N, x_0, \frac{p}{2}\Big)$$
$$< \eta \diamond \gamma$$
$$< \delta.$$

Take $\eta_0 = \max\{\beta, \delta\}$. Then

$$\mathbf{E}(p, \eta) \subseteq \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) > 1 - \eta_0 \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) < \eta_0\Big\}.$$

Consequently,

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) > 1 - \eta_0 \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) < \eta_0\Big\}\Big) = 1,$$

which implies that

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) \leq 1 - \eta_0 \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x_0, p\Big) \geq \eta_0\Big\}\Big) = 0.$$

Thus (x_k) is statistically bounded. □

Corollary 4.8. *In an IF$\mathcal{G}$M–space $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$, every statistically convergent sequence is statistically bounded.*

Proof. The result follows from Theorem 4.2 and Theorem 4.7. □

Aside from Theorem 3.12, the following theorem can also be asserted:

Theorem 4.9. *Consider $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an IF$\mathcal{G}$M–space. Then, for a sequence (x_k) in $\mathfrak{X}$, the following are equivalent:*

(a) *The sequence (x_k) is statistically Cauchy.*

(b) *There exists a statistically dense subsequence (x_{k_m}) of (x_k) and (x_{k_m}) is Cauchy in $\mathfrak{X}$.*

5. Statistical limit points and statistical cluster points

In this section, we extend the ideas of thin subsequence, nonthin subsequence, statistical limit points and statistical cluster points introduced in [17] in the framework of *IF$\mathcal{G}$M*–spaces.

Definition 5.1. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space. Then $x \in \mathfrak{X}$ is said to be a limit point of a sequence (x_k) in $\mathfrak{X}$ with respect to $(\mathfrak{M}, \mathfrak{N})$, if there is a subsequence (x_{k_m}) of (x_k) that converges to x. We denote $Ł^{(\mathfrak{M},\mathfrak{N})}(x_k)$ to refer to the set of all limit points of the sequence (x_k).

Definition 5.2. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space and (x_{k_m}) be a subsequence of a sequence (x_k) in $\mathfrak{X}$. Denote $\mathbf{K} = \{k_m : m \in \mathbb{N}\} \subset \mathbb{N}$. We say that (x_{k_m}) is a thin subsequence of (x_k) if $d_n(\mathbf{K}) = 0$. If $d_n(\mathbf{K}) \neq 0$ then we say that (x_{k_m}) is a nonthin subsequence of (x_k).

Definition 5.3. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space. Then $x \in \mathfrak{X}$ is called a statistical limit point of the sequence (x_k) in $\mathfrak{X}$ with respect to $(\mathfrak{M}, \mathfrak{N})$, if there exists a nonthin subsequence (x_{k_m}) of (x_k) that converges to x. We denote $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k)$ to refer to the set of all statistical limit points of (x_k).

Definition 5.4. Let $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an *IF$\mathcal{G}$M*–space. Then $x \in \mathfrak{X}$ is said to be a statistical cluster point of the sequence (x_k) in $\mathfrak{X}$ with respect to $(\mathfrak{M}, \mathfrak{N})$ if, for every $\eta \in (0, 1)$ and $p > 0$,

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) < \eta\Big\}\Big) \neq 0.$$

By $\Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$, we denote the set of all statistical clusters points of (x_k).

Theorem 5.5. *Consider* $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *to be an IF$\mathcal{G}$M–space and* (x_k) *to be a sequence in* $\mathfrak{X}$*. Then*

$$\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) \subseteq \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k) \subseteq Ł^{(\mathfrak{M},\mathfrak{N})}(x_k).$$

Proof. Let (x_k) be a sequence in $\mathfrak{X}$ and $x \in \Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k)$. Then, there exists a subsequence (x_{k_m}) of (x_k) such that the index set $A = \{k_m : m \in \mathbb{N}\} \subset \mathbb{N}$ has non zero n–dimensional asymptotic density, i.e.,

$$d_n(A) = \lim_{k\to\infty} \frac{n!}{k^n} \left|\left\{(k_1, k_2, ..., k_n) \in A^n : k_1, k_2, ..., k_n \leq k\right\}\right| = t > 0$$

and (x_{k_m}) converges to x. Since

$$\left\{(k_1, k_2, ..., k_n) \in A^n : \mathfrak{M}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) > 1 - \eta \text{ and } \mathfrak{N}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) < \eta\right\}$$
$$\subseteq \left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\right) > 1 - \eta \text{ and } \mathfrak{N}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\right) < \eta\right\}$$

true for every $\eta \in (0, 1)$ and $p > 0$, it follows that

$$\left\{(k_1, k_2, ..., k_n) \in A^n : k_j \in \mathbb{N},\ j = 1, 2, ..., n\right\} \setminus \left\{(k_1, k_2, ..., k_n) \in A^n : \mathfrak{M}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) \leq 1 - \eta \right.$$
$$\left. \text{or } \mathfrak{N}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) \geq \eta\right\} \subseteq \left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\right) > 1 - \eta \right.$$
$$\left. \text{and } \mathfrak{N}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\right) < \eta\right\}.$$

Also, the subsequence (x_{k_m}) is convergent to x. Therefore,

$$\left\{(k_1, k_2, ..., k_n) \in A^n : \mathfrak{M}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) \leq 1 - \eta \text{ or } \mathfrak{N}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) \geq \eta\right\}$$

is a finite subset of $\mathbb{N}^n$. As a result,

$$\begin{aligned} &d_n\left(\left\{ (i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\right) > 1 - \eta \text{ and } \mathfrak{N}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\right) < \eta\right\}\right) \\ &\geq d_n\left(\left\{(k_1, k_2, ..., k_n) \in A^n : k_j \in \mathbb{N},\ j = 1, 2, ..., n\right\}\right) - \\ &\quad d_n\left(\left\{(k_1, k_2, ..., k_n) \in A^n : \mathfrak{M}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) \leq 1 - \eta \text{ or } \mathfrak{N}\left(x_{k_1}, x_{k_2}, ..., x_{k_n}, x, p\right) \geq \eta\right\}\right) \\ &\geq d_n\left(\left\{(k_1, k_2, ..., k_n) \in A^n : k_j \in \mathbb{N},\ j = 1, 2, ..., n\right\}\right) - 0 \\ &\geq t > 0. \end{aligned}$$

Consequently, $x \in \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$ and so $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) \subseteq \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$.

Now let $y \in \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$. Then, for every $p > 0$ and $\eta \in (0, 1)$, we have

$$d_n(\mathbf{H}(p, \eta)) = d_n\left(\left\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\right) > 1 - \eta \text{ and } \mathfrak{N}\left(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\right) < \eta\right\}\right) > 0.$$

Let $(k_1, k_2, ..., k_n) \in \mathbf{H}(p, \eta)$. Then, by Proposition 2.13, we have $x_{k_j} \in \mathbf{B}_y^{(\mathfrak{M},\mathfrak{N})}\left(p, \eta\right)$ for each $j \in \{0, 1, ..., n\}$. Set

$$B = \left\{k_j \in \mathbb{N} : x_{k_j} \in \mathbf{B}_y^{(\mathfrak{M},\mathfrak{N})}\left(p, \eta\right)\right\}.$$

It is clear that $d_n(B) = d_n(\mathbf{H}(p, \eta)) > 0$ and thus (x_{k_j}) is a nonthin subsequence of (x_k) along B. Since B contains an infinite number of positive integers, $y \in Ł^{(\mathfrak{M},\mathfrak{N})}(x_k)$. Therefore, $\Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k) \subseteq Ł^{(\mathfrak{M},\mathfrak{N})}(x_k)$. □

Theorem 5.6. *Consider* $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *to be an IF$\mathcal{G}$M–space. Suppose* (x_k) *is a sequence in* $\mathfrak{X}$ *such that* $st\text{–}\lim_k x_k = x$*. Then* $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) = \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k) = \{x\}$.

Proof. Let $st-\lim_k x_k = x$. From Definition 3.4 and Definition 5.4, it follows that $x \in \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$. Suppose there is $y \in \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$ such that $x \neq y$. Then, for every $\eta \in (0,1)$ and $p > 0$, we find

$$d_n(\mathbf{F}(p,\eta)) = \Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) < \eta\Big\}\Big) \neq 0$$

and

$$d_n(\mathbf{G}(p,\eta)) = \Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) < \eta\Big\}\Big) \neq 0.$$

Since $x \neq y$, we have $\mathbf{F}(p,\eta) \cap \mathbf{G}(p,\eta) = \phi$. Consequently, $\mathbf{F}(p,\eta)^c \supseteq \mathbf{G}(p,\eta)$, i.e.,

$$\begin{aligned}&\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) \leq 1 - \eta \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) \geq \eta\Big\}\\ &\qquad \supseteq \Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) < \eta\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}&d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) \leq 1 - \eta \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) \geq \eta\Big\}\Big)\\ &\qquad \geq d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) < \eta\Big\}\Big).\end{aligned} \tag{3}$$

As $st-\lim_k x_k = x$, we have

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) \leq 1 - \eta \text{ or } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, x, p\Big) \geq \eta\Big\}\Big) = 0.$$

Therefore, from Equation 3, it follows that

$$d_n\Big(\Big\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : \mathfrak{M}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) > 1 - \eta \text{ and } \mathfrak{N}\Big(x_{i_1}, x_{i_2}, ..., x_{i_n}, y, p\Big) < \eta\Big\}\Big) = 0.$$

This contradict the fact that $y \in \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$. Thus $\Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k) = \{x\}$.

Now, suppose $st-\lim_k x_k = x$. Then $x \in \Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k)$, as implied by Definition 5.3 and Theorem 3.12. Hence, by Theorem 5.5, we have $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) = \{x\}$. □

The converse of Theorem 5.6 is not true, i.e., it is possible to have a sequence (x_k) in $\mathfrak{X}$ such that $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) = \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k) = \{x\}$, but (x_k) is not statistically convergent to x.

Example 5.7. Consider $(\mathbb{R}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ to be an *IF𝒢M*–space defined in Example 2.2. Define

$$x_k = \left\{\begin{array}{ll} 1, & \text{if } k = 2i \\ 2, & \text{otherwise} \end{array}\right\} i \in \mathbb{N}.$$

Then $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) = \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k) = \{1, 2\}$ but (x_k) is not statistically convergent.

Theorem 5.8. *Consider* $(\mathfrak{X}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *to be an IF𝒢M–space. Suppose* (x_k) *and* (y_k) *are two sequences in* $\mathfrak{X}$ *such that* $x_k = y_k$ *for almost all k. Then* $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) = \Lambda^{(\mathfrak{M},\mathfrak{N})}(y_k)$ *and* $\Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k) = \Gamma^{(\mathfrak{M},\mathfrak{N})}(y_k)$.

Proof. Let $x_k = y_k$ for almost all k. Thus, the set $K_1 = \{k \in \mathbb{N} : x_k \neq y_k\}$ has zero n–dimensional asymptotic density, i.e., $d_n(K_1) = 0$. Let $x \in \Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k)$. Then, there is a subsequence (x_{k_m}) of (x_k) so that (x_{k_m}) is convergent to x and the index set $K_2 = \{k_m : m \in \mathbb{N}\}$ has non zero n–dimensional asymptotic density, i.e., $d_n(K_2) \neq 0$. Consider the set

$$K_3 = \Big\{k_m \in K_2 : m \in \mathbb{N}, x_{k_j} \neq y_{k_j}\Big\}.$$

It is clear that $d_n(k_3) = 0$, i.e.,

$$\lim_{k\to\infty} \frac{n!}{k^n} \left| \left\{ (k_1, k_2, ..., k_n) \in K_2^n : k_1, k_2, ..., k_n \leq k, x_{k_j} \neq y_{k_j} \right\} \right| = 0.$$

This implies that

$$\lim_{k\to\infty} \frac{n!}{k^n} \left| \left\{ (k_1, k_2, ..., k_n) \in K_2^n : k_1, k_2, ..., k_n \leq k, x_{k_j} = y_{k_j} \right\} \right| > 0,$$

which is equivalent to

$$d_n(I) = d_n\left(\left\{ k_m \in K_2 : x_{k_m} = y_{k_m} \right\} \right) > 0.$$

Now consider the sequence (y_{k_m}) along I. Then (y_{k_m}) is a nonthin subsequence of (y_k) that converges to x, and so $x \in \Lambda^{(\mathfrak{M},\mathfrak{N})}(y_k)$. This implies that $\Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k) \subseteq \Lambda^{(\mathfrak{M},\mathfrak{N})}(y_k)$. By symmetry, we also have $\Lambda^{(\mathfrak{M},\mathfrak{N})}(y_k) \subseteq \Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k)$. Hence $\Lambda^{(\mathfrak{M},\mathfrak{N})}(y_k) = \Lambda^{(\mathfrak{M},\mathfrak{N})}(x_k)$.

Similarly, we can show that $\Gamma^{(\mathfrak{M},\mathfrak{N})}(y_k) = \Gamma^{(\mathfrak{M},\mathfrak{N})}(x_k)$. □

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