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# Fixed points of bilateral multivalued contractions

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**Abstract.** In this paper, the concepts of Jaggi, Dass-Gupta and Caristi-Ćirić-type bilateral multivalued contractions are introduced in the framework of metric spaces. New conditions for the existence of fixed points for such contractions are analyzed. A few consequences in single-valued mappings which include the conclusion of the main result of Chen et al. [On bilateral contractions. Mathematics, 2019, 7, 538] are obtained. In addition, nontrivial example are provided to support the validity of the results obtained herein.

## 1. Introduction

Banach [1] presented one of the outstanding results concerning contraction mapping which appeared in Banach thesis in 1922. This famous result is known as Banach Contraction mapping principle. It states that every contraction mapping on a complete metric space has a unique fixed point. Fixed point theory becomes a subject of great interest due to its application in mathematics and other areas of research. Fixed point theorem in metric spaces plays a significant role to construct methods to solve the problem in mathematics and sciences. Many researchers worked in this area and extended the result either by considering a more general space or imposing some conditions on the domain of the contraction mapping or by considering a more general contractive conditions. Meanwhile, the main result in [1] has been modified and applied in different directions. In some generalizations of the contraction mapping principle, the inequality is weakened, see, for instance [7], and in others, the topology of the underlying space is weakened, see [13] and the references therein. Along the line, one prominent improvement of the Banach fixed point theorem was presented by Hardy-Rogers [6].

Recently, Roldán et al. [23] established some new fixed point theorems for a family of contractions depending on two functions and some parameters under the name multiparametric contractions and pointed out significant number of Hardy-Roger's type contractions in the setting of both metric and $b$-metric spaces. Other important versions of the Banach contraction mapping principle were independently presented by Ćirić [7], Reich [4] and Rus [10]. Banach contraction mapping principle has been generalized by many researchers in various ways (see, for example, ([2], [5], [14], [15], [22], [27]) and the references therein). Following the Banach Contraction mapping principle, the concept of multivalued contraction was

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introduced by Nadler [3] and the corresponding fixed point result was proposed therein. Moreover, [24] extended the concept of weak KKM set-valued mapping from topological vector spaces to hyperconvex metric spaces. Shagari et. al [21] introduced the idea of $\alpha_L$-compatible mappings of an $L$-fuzzy map. Using this idea together with the technique of Meir-Keeler $(M - K)$ contraction, some common fixed point theorems for $L$-fuzzy compatible maps were obtained.

One of the initial nonlinear forms of the contraction mapping principle was given by Jaggi [9] and Dass-Gupta [8], who used rational inequalities in their functional equations. Moreover, Shagari et. al [26] introduced the concepts of Jaggi and Dass-Gupta type bilateral multi-valued contractions and under some suitable conditions, the existence of fixed points for such mappings were established. Also, [16] introduced bilateral contractions which merges two significant approaches in fixed point theory: Caristi-type and Jaggi-type contractions. An inherent property of the existing fixed point results via the bilateral contraction is that the fixed point of the concerned mapping is not necessarily unique; for example, see [16, Example 2]. This restriction is an indication that fixed point theorems using bilateral notions are more suitable for fixed point theory of point-to-set-valued maps.

In 2018, Karapinar [12] studied Jaggi's inequalities that imply the existence and uniqueness of fixed points in metric spaces from the view point of partial metric spaces. In addition, Karapinar and Fulga [20] provided a new hybrid-type contraction that is a combination of a Jaggi-type contraction and interpolative-type contraction in the framework of complete metric spaces and established the existence and uniqueness of fixed point. Moreover, Karapinar et. al [19] applied new fixed point theorems of Jaggi and Geraghty-type on fractional and ordinary differential equations. A hybrid of Jaggi-Meir-Keeler-type contraction which combined some existing results was introduced by Karapinar and Fulga [28]. Also, Alqahtani and Karapınar [18] introduced the notion of a bilateral contraction that unified the ideas of Ćirić and Caristi-type contractions via simulation functions.

Following the above chain of developments, this paper aims at proposing the concept of bilateral contraction from the case of single-valued mappings to multivalued mappings. For this purpose, we introduce the notion of Jaggi-type, Dass-Gupta-type and Caristi-Ciric-type multivalued contractions and then establish the corresponding fixed point theorems.

## 2. Preliminaries

We recall some basic definitions and preliminaries that will be needed in this paper.
Let $(X, d)$ be a metric space, $CB(X)$ be a collection of non-empty closed and bounded subset of $X$ and $A, B \in CB(X)$. The Hausdorff metric $H$ on $CB(X)$ induced by the metric $d$ is given by

$$H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}$$

$$D(a, B) = \{\inf_{b\in B} d(a, b)\}.$$

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

It is known that $H$ is a metric on $CB(X)$ and $H$ is called the Hausdorff metric or Pompeiu-Hausdorff metric induced by $d$. It is also known that $(CB(X), H)$ is a complete metric space whenever $(X, d)$ is a complete metric space.

**Definition 2.1.** *Let $(X, d)$ be a complete metric space and $T : X \longrightarrow X$.*

*(i) (Jaggi [9]) There exists $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$ such that* $d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)}$

*(ii) (Dass and Gupta [8]) There exists $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$ such that* $d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 \frac{[1+d(x,Tx)]d(y,Ty)}{[1+d(x,y)]}$

(iii) *(Ćirić [7]) There exists a constant $\lambda$, $0 \le \lambda < 1$, such that, for each $x, y \in X$,*
$d(Tx, Ty) \le \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Not long ago, Chen et al. [16] introduced the notion of Bilateral contraction in the following manner:

**Definition 2.2.** *[16] Let $(X, d)$ be a complete metric space. A self mapping $T : X \longrightarrow X$ is called a Jaggi type bilateral contraction if there is $\vartheta : X \longrightarrow [0, \infty)$ such that, $d(x, Tx) > 0$ implies*

$$d(Tx, Ty) \le [\vartheta(x) - \vartheta(Tx)]R_T(x, y),$$

*for all distinct $x, y \in X$, where*

$$R_T(x, y) = Max\{d(x, y), \frac{d(x, Tx)d(y, Ty)}{d(x, y)}\}.$$

**Definition 2.3.** *[16] Let $(X, d)$ be a complete metric space. A self mapping $T : X \longrightarrow X$ is called a Dass-Gupta type bilateral contraction if there is $\vartheta : X \longrightarrow [0, \infty)$ such that, $d(x, Tx) > 0$ implies*

$$d(Tx, Ty) \le [\vartheta(x) - \vartheta(Tx)]Q_T(x, y),$$

*for all $x, y \in X$, where*

$$Q_T(x, y) = Max\{d(x, y), \frac{(1 + d(x, Tx))d(y, Ty)}{1 + d(x, y)}\}.$$

**Definition 2.4.** *[18, Definition 2.1] Let $T$ be a self mapping on a complete metric space $(X, d)$. If there exists $\xi \in Z$ and $\vartheta : X \longrightarrow [0, \infty)$ such that, $d(x, Tx) > 0$ implies*

$$\xi(d(Tx, Ty), (\vartheta(x) - \vartheta(Tx))C_T(x, y)) \ge 0,$$

*in which*

$$C_T(x, y) = Max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}\},$$

*for all $x, y \in X$, then $T$ is called a bilateral contraction of Ćirić-Caristi.*

**Definition 2.5.** *[17] Let $(X, d)$ be a complete metric space. A self mapping $T : X \longrightarrow X$ is called a Ćirić - Caristi type contraction if there is a mapping $\vartheta : X \longrightarrow \mathbb{R}_+$ such that, $d(x, Tx) > 0$ implies*

$$d(Tx, Ty) \le [\vartheta(x) - \vartheta(Tx)]N(x, y),$$

*for all $x, y \in X$, where*

$$N(x, y) = Max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

**Lemma 1:**[11] Let $(X, d)$ be a metric space. Let $A, B \in X$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that

$$d(a, b) \le qH(A, B).$$

## 3. Main Results

In 2021, Noorwali and Yesilkaya [25] introduced the concept of new hybrid contractions that combine Jaggi hybrid type contractions and Suzuki-type contractions with $\omega$-orbital admissible and established fixed point result in single-valued mapping. Motivated by the results of [25] and [16], we introduce the notion of Jaggi, bilateral, Dass-Gupta and Caristi-Ćirić-type multivalued contractions. In each case, we establish the corresponding fixed point theorem in the setting of metric spaces.

**Definition 1:** Let $(X, d)$ be a metric space. A multivalued mapping $T : X \longrightarrow CB(X)$ is called a Jaggi-type bilateral multivalued contraction if there is $\vartheta : X \longrightarrow \mathbb{R}_+$ such that $D(x, Tx) > 0$ implies

$$H(Tx, Ty) \leq [\vartheta(x) - \vartheta(Tx)]R_T(x, y), \tag{1}$$

for all distinct $x, y \in X$, where

$$R_T(x, y) = \max\left\{d(x, y), \frac{D(x, Tx)D(y, Ty)}{1 + d(x, y)}\right\}.$$

**Theorem 1:** Let $(X, d)$ be a complete metric space and $T : X \longrightarrow CB(X)$ be a multivalued mapping. Moreover, if the following conditions are satisfied:

C1: $T$ is a Jaggy-type bilateral multivalued contraction;

C2: there exists

$$K := \sup\{\vartheta(x) - \vartheta(Tx) : d(x, y) > 0\}$$

where $\vartheta : X \longrightarrow \mathbb{R}_+$, then there exists $u \in X$ such that $u \in Tu$.

**Proof:** Let $x_0 \in X$ be arbitrary. Since $T : X \longrightarrow CB(X)$, then $Tx_o \in CB(X)$. This implies that $Tx_o \neq \emptyset$. Therefore, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. For this $x_1 \in X$, $Tx_1 \in CB(X)$ implies that $Tx_1 \neq \emptyset$. Hence, there exists $x_2 \in X$ such that $x_2 \in Tx_1$. Similarly, $Tx_2 \in CB(X)$ implies that $Tx_2 \neq \emptyset$. It follows that, there exists $x_3 \in X$ such that $x_3 \in Tx_2$. Continuing in this manner, we construct a sequence $\{x_n\}$ in $X$ such that $x_{n+1} = Tx_n$, for $n = 0, 1, 2, ....$ Note that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} \in Tx_{n_0+1}$ and the proof is finished. Hence, we presume that $x_n \neq x_{n+1}$ for all $n$. By Lemma 1, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned}
d(x_1, x_2) &\leq qH(Tx_0, Tx_1)\\
&\leq q[\vartheta(x_0) - \vartheta(x_1)]R_T(x_0, x_1)\\
&= q[\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0, x_1), \frac{D(x_0, Tx_0).D(x_1, Tx_1)}{1 + d(x_0, x_1)}\}\\
&\leq q[\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0, x_1), \frac{d(x_0, x_1).d(x_1, x_2)}{1 + d(x_0, x_1)}\}\\
&\leq q[\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0, x_1), \frac{d(x_0, x_1).d(x_1, x_2)}{d(x_0, x_1)}\}\\
&\leq q[\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0, x_1), d(x_1, x_2)\}\\
&\leq qk\max\{d(x_0, x_1), d(x_1, x_2)\}\\
&\leq \lambda\max\{d(x_0, x_1), d(x_1, x_2)\},
\end{aligned}$$

where $\lambda \leq qk < 1$ . Hence, we see that

$$d(x_1, x_2) \leq \lambda\max\{d(x_0, x_1), d(x_1, x_2)\}$$

Thus, we have two cases to consider as follows:
Case 1: Suppose that the $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1)$. Then, we see that

$$d(x_1, x_2) \leq \lambda d(x_0, x_1).$$

Case 2: Suppose that the $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$. We see that

$$\begin{aligned} d(x_1, x_2) &\leq \lambda d(x_1, x_2) \\ (1-\lambda)d(x_1, x_2) &\leq 0 \\ d(x_1, x_2) &\leq 0, \end{aligned}$$

and this is a contradiction. Similarly, by Lemma 1, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq qH(Tx_1, Tx_2) \\ &\leq q[\vartheta(x_1) - \vartheta(x_2)]R_T(x_1, x_2) \\ &= q[\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), \frac{D(x_1, Tx_1).D(x_2, Tx_2)}{1 + d(x_1, x_2)}\} \\ &\leq q[\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), \frac{d(x_1, x_2).d(x_2, x_3)}{1 + d(x_1, x_2)}\} \\ &\leq q[\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), \frac{d(x_1, x_2).d(x_2, x_3)}{d(x_1, x_2)}\} \\ &\leq q[\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), d(x_2, x_3)\} \\ &\leq qk \max\{d(x_1, x_2), d(x_2, x_3)\} \\ &\leq \lambda \max\{d(x_1, x_2), d(x_2, x_3)\}, \end{aligned}$$

where $\lambda \leq qk < 1$. Hence, we see that

$$d(x_2, x_3) \leq \lambda \max\{d(x_1, x_2), d(x_2, x_3)\}$$

Hence we have two cases to consider.
Case 1: Suppose that $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_1, x_2)$.
We see that

$$d(x_2, x_3) \leq \lambda d(x_1, x_2).$$

Case 2: Suppose that the $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_2, x_3)$. We see that

$$\begin{aligned} d(x_2, x_3) &\leq \lambda d(x_2, x_3) \\ (1-\lambda)d(x_2, x_3) &\leq 0 \\ d(x_2, x_3) &\leq 0, \end{aligned}$$

and this is a contradiction. Continuing in this manner inductively, we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).$$

Now, we show that, the sequence $(x_n)$ in $X$ is Cauchy. Let $m, n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + ... + \lambda^{m-1} d(x_0, x_1) \\ &= (\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}) d(x_0, x_1) \\ &= \sum_{i=n}^{n+i-1} \lambda^i d(x_0, x_1) \\ &\leq \sum_{i=n}^{\infty} \lambda^i d(x_0, x_1) \end{aligned}$$

$\longrightarrow 0$ as n, m $\longrightarrow \infty$. Hence, $(x_n)$ in $X$ is a Cauchy sequence. The completeness of $(X, d)$ guarantees that we can find $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now,

$$\begin{aligned}
D(u, Tu) &\leq d(u, x_{n+1}) + D(x_{n+1}, Tu) \\
&\leq d(u, x_{n+1}) + H(Tx_n, Tu) \\
&\leq d(u, x_{n+1}) + [\vartheta(x_n) - \vartheta(u)]R_T(x_n, u) \\
&\leq d(u, x_{n+1}) + [\vartheta(x_n) - \vartheta(u)] \max\{d(x_n, u), \frac{D(x_n, Tx_n).D(u, Tu)}{1 + d(x_n, u)}\} \\
&\leq d(u, x_{n+1}) + [\vartheta(x_n) - \vartheta(u)] \max\{d(x_n, u), \frac{d(x_n, x_{n+1}).D(u, Tu)}{1 + d(x_n, u)}\} \\
&\leq d(u, x_{n+1}) + [\vartheta(x_n) - \vartheta(u)] \max\{d(x_n, u), \frac{d(x_n, x_{n+1}).D(u, Tu)}{d(x_n, u)}\} \\
&\leq d(u, x_{n+1}) + k \max\{d(x_n, u), \frac{d(x_n, x_{n+1}).D(u, Tu)}{d(x_n, u)}\}.
\end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality gives $D(u, Tu) \leq 0 + k \max\{0, 0\} = 0$. This implies that $u \in Tu$.

**Definition 2:** Let $(X, d)$ be a metric space. A mapping $T : X \longrightarrow CB(X)$ is called a Dass - Gupta type bilateral multivalued contraction if there is a mapping $\vartheta : X \longrightarrow \mathbb{R}_+$ such that, $D(x, Tx) > 0$ implies

$$H(Tx, Ty) \leq [\vartheta(x) - \vartheta(Tx)]Q_T(x, y), \tag{2}$$

for all $x, y \in X$, where

$$Q_T(x, y) = \max\{d(x, y), \frac{(1 + D(x, Tx))D(y, Ty)}{1 + d(x, y)}\}.$$

**Theorem 2:** Let $(X, d)$ be a complete metric space and $T : X \longrightarrow CB(X)$ be a multivalued mapping. Moreover, if the following conditions are satisfied:

C1: $T$ is a Dass-Gupta type bilateral multivalued contraction;

C2: there exists $k := \sup\{\vartheta(x) - \vartheta(Tx) : d(x, y) > 0\}$,

where $\vartheta : X \longrightarrow \mathbb{R}_+$, then there exists $u \in X$ such that $u \in Tu$.

**Proof:** Let $x_0 \in X$ be arbitrary and defined a sequence $(x_n)$ in $X$ by $x_{n+1} \in Tx_n$ $n = 0, 1, 2, \ldots.$ Note that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} \in Tx_{n_0+1}$ and the proof is finished. For this, assume that $x_n \neq x_{n+1}$ for all $n$. By Lemma 1, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned}
d(x_1, x_2) &\leq qH(Tx_0, Tx_1) \\
&\leq q[\vartheta(x_0) - \vartheta(x_1)]Q_T(x_0, x_1) \\
&= q[\vartheta(x_0) - \vartheta(x_1)] \max\{d(x_0, x_1), \frac{(1 + D(x_0, Tx_0)).D(x_1, Tx_1)}{1 + d(x_0, x_1)}\} \\
&\leq q[\vartheta(x_0) - \vartheta(x_1)] \max\{d(x_0, x_1), \frac{(1 + d(x_0, x_1)).d(x_1, x_2)}{1 + d(x_0, x_1)}\} \\
&\leq q[\vartheta(x_0) - \vartheta(x_1)] \max\{d(x_0, x_1), d(x_1, x_2)\} \\
&\leq qk \max\{d(x_0, x_1), d(x_1, x_2)\} \\
&\leq \lambda \max\{d(x_0, x_1), d(x_1, x_2)\},
\end{aligned}$$

where $\lambda \leq qk < 1$. Therefore, we see that

$$d(x_1, x_2) \leq \lambda \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

We investigate two cases as follows:
Case 1: Suppose that $\max\{d(x_0,x_1), d(x_1,x_2)\} = d(x_0,x_1)$. We see that $d(x_1,x_2) \leq \lambda d(x_0,x_1)$.
Case 2: Suppose that the $\max\{d(x_0,x_1), d(x_1,x_2)\} = d(x_1,x_2)$. We see that

$$\begin{aligned} d(x_1,x_2) &\leq \lambda d(x_1,x_2) \\ (1-\lambda)d(x_1,x_2) &\leq 0 \\ d(x_1,x_2) &\leq 0 \end{aligned}$$

and this is a contradiction. Similarly, by Lemma 1, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\leq qH(Tx_1,Tx_2) \\ &\leq q[\vartheta(x_1) - \vartheta(x_2)]Q_T(x_1,x_2) \\ &= q[\vartheta(x_1) - \vartheta(x_2)]\max\{d(x_1,x_2), \frac{(1+D(x_1,Tx_1)).D(x_2,Tx_2)}{1+d(x_1,x_2)}\} \\ &\leq q[\vartheta(x_1) - \vartheta(x_2)]\max\{d(x_1,x_2), \frac{(1+d(x_1,x_2)).d(x_2,x_3)}{1+d(x_1,x_2)}\} \\ &\leq q[\vartheta(x_1) - \vartheta(x_2)]\max\{d(x_1,x_2), d(x_2,x_3)\} \\ &\leq qk\max\{d(x_1,x_2), d(x_2,x_3)\} \\ &\leq \lambda\max\{d(x_1,x_2), d(x_2,x_3)\}, \end{aligned}$$

where $\lambda \leq qk < 1$, and hence

$$d(x_2,x_3) \leq \lambda\max\{d(x_1,x_2), d(x_2,x_3)\}$$

Again, we consider two cases as follows:
Case 1: Suppose that the $\max\{d(x_1,x_2), d(x_2,x_3)\} = d(x_1,x_2)$
We see that $d(x_2,x_3) \leq \lambda d(x_1,x_2)$.
Case 2: Suppose that the $\max\{d(x_1,x_2), d(x_2,x_3)\} = d(x_2,x_3)$. We see that

$$\begin{aligned} d(x_2,x_3) &\leq \lambda d(x_2,x_3) \\ (1-\lambda)d(x_2,x_3) &\leq 0 \\ d(x_2,x_3) &\leq 0, \end{aligned}$$

and this is a contradiction. Continuing in this manner inductively, we have

$$d(x_n,x_{n+1}) \leq \lambda^n d(x_0,x_1).$$

Now, we show that the sequence $(x_n)$ in $X$ is Cauchy. Let $m,n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{m-1},x_m) \\ &\leq \lambda^n d(x_0,x_1) + \lambda^{n+1}d(x_0,x_1) + ... + \lambda^{m-1}d(x_0,x_1) \\ &= (\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1})d(x_0,x_1) \\ &= \sum_{i=n}^{n+i-1}\lambda^i d(x_0,x_1) \\ &\leq \sum_{i=n}^{\infty}\lambda^i d(x_0,x_1) \end{aligned}$$

$\longrightarrow 0$ as n, m $\longrightarrow \infty$. Hence, $(x_n)$ in $X$ is a Cauchy sequence. The completeness of $(X,d)$ guarantees $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$.

Now,

$$\begin{aligned}
D(u,Tu) &\leq d(u,x_{n+1})+D(x_{n+1},Tu)\\
&\leq d(u,x_{n+1})+H(Tx_n,Tu)\\
&\leq d(u,x_{n+1})+[\vartheta(x_n)-\vartheta(u)]Q_T(x_n,u)\\
&\leq d(u,x_{n+1})+[\vartheta(x_n)-\vartheta(u)]\max\{d(x_n,u),\frac{(1+D(x_n,Tx_n)).D(u,Tu)}{1+d(x_n,u)}\}\\
&\leq d(u,x_{n+1})+[\vartheta(x_n)-\vartheta(u)]\max\{d(x_n,u),\frac{(1+d(x_n,x_{n+1})).D(u,Tu)}{1+d(x_n,u)}\}\\
&\leq d(u,x_{n+1})+k\max\{d(x_n,u),\frac{(1+d(x_n,x_{n+1})).D(u,Tu)}{1+d(x_n,u)}\}.
\end{aligned}$$

Letting $n\longrightarrow\infty$ in the above inequality gives

$$\begin{aligned}
D(u,Tu) &\leq 0+k\max\{0,D(u,Tu)\}\\
&\leq kD(u,Tu)\\
(1-k)D(u,Tu) &\leq 0\\
D(u,Tu) &\leq 0.
\end{aligned}$$

This implies that $u\in Tu$.

**Definition 3:** Let $(X,d)$ be a metric space. A mapping $T:X\longrightarrow CB(X)$ is called a Ciric - Caristi type bilateral multivalued contraction if there is a mapping $\vartheta:X\longrightarrow\mathbb{R}_+$ such that $D(x,Tx)>0$ implies

$$H(Tx,Ty)\leq[\vartheta(x)-\vartheta(Tx)]N(x,y), \tag{3}$$

for all $x,y\in X$, where

$$N(x,y)=\max\{d(x,y),D(x,Tx),D(y,Ty),D(x,Ty),D(y,Tx)\}.$$

**Theorem 3:** Let $(X,d)$ be a complete metric space and $T:X\longrightarrow CB(X)$ be a multivalued mapping. Moreover, if the following conditions are satisfied;

CR1: $T$ is a Ćirić - Caristi type bilateral multivalued contraction;

CR2: there exists

$$k:=Sup\{\vartheta(x)-\vartheta(Tx):d(x,y)>0\},$$

where $\vartheta:X\longrightarrow\mathbb{R}_+$, then there exists $u\in X$ such that $u\in Tu$.

**Proof:** Let $x_0\in X$ be arbitrary and define a sequence $(x_n)$ in $X$ by $x_{n+1}\in Tx_n$ $n=0,1,2,....$ Note that if there exists $n_0\in\mathbb{N}$ such that $x_{n_0}=x_{n_0+1}$, then $x_{n_0}\in Tx_{n_0+1}$ and the proof is finished. For this, assume that $x_n\neq x_{n+1}$ for all $n$. By Lemma 1, for $x_1\in Tx_0$, we can find $x_2\in Tx_1$ such that

$$\begin{aligned}
d(x_1,x_2) &\leq qH(Tx_0,Tx_1)\\
&\leq q[\vartheta(x_0)-\vartheta(x_1)]N(x_0,x_1)\\
&= q[\vartheta(x_0)-\vartheta(x_1)]\max\{d(x_0,x_1),D(x_0,Tx_0),D(x_1,Tx_1),D(x_1,Tx_0),D(x_0,Tx_1)\}\\
&\leq q[\vartheta(x_0)-\vartheta(x_1)]\max\{d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_1,x_1),d(x_0,x_2)\}\\
&\leq q[\vartheta(x_0)-\vartheta(x_1)]\max\{d(x_0,x_1),d(x_1,x_2),d(x_0,x_2)\}\\
&\leq qk\max\{d(x_0,x_1),d(x_1,x_2),d(x_0,x_2)\}\\
&\leq \lambda\max\{d(x_0,x_1),d(x_1,x_2),d(x_0,x_2)\}
\end{aligned}$$

where $\lambda \leq qk < 1$, and hence

$$\begin{aligned}
d(x_1, x_2) &\leq \lambda \max\{d(x_0, x_1), d(x_1, x_2), d(x_0, x_2)\} \\
&\leq \lambda \max\{d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2)\} \\
&\leq \lambda[d(x_0, x_1) + d(x_1, x_2)] \\
&= \lambda d(x_0, x_1) + \lambda d(x_1, x_2) \\
(1-\lambda)d(x_1, x_2) &= \lambda d(x_0, x_1) \\
d(x_1, x_2) &\leq (\frac{\lambda}{1-\lambda})d(x_0, x_1).
\end{aligned}$$

Similarly, by Lemma 1, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned}
d(x_2, x_3) &\leq qH(Tx_1, Tx_2) \\
&\leq q[\vartheta(x_1) - \vartheta(x_2)]N(x_1, x_2) \\
&= q[\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), D(x_2, Tx_1), D(x_1, Tx_2)\} \\
&\leq q[\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_2, x_2), d(x_1, x_3)\} \\
&\leq q[\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_3)\} \\
&\leq qk \max\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_3)\} \\
&\leq \lambda \max\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_3)\},
\end{aligned}$$

where $\lambda \leq qk < 1$, and hence

$$\begin{aligned}
d(x_2, x_3) &\leq \lambda \max\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_3)\} \\
&\leq \lambda \max\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_2) + d(x_2, x_3)\} \\
&\leq \lambda[d(x_1, x_2) + d(x_2, x_3)] \\
&= \lambda d(x_1, x_2) + \lambda d(x_2, x_3)] \\
(1-\lambda)d(x_2, x_3) &= \lambda d(x_1, x_2) \\
d(x_2, x_3) &\leq \frac{\lambda}{1-\lambda} d(x_1, x_2) \\
&\leq \frac{\lambda}{1-\lambda}(\frac{\lambda}{1-\lambda} d(x_0, x_1)) \\
&= (\frac{\lambda}{1-\lambda})^2 d(x_0, x_1).
\end{aligned}$$

Continuing in this manner inductively, we have

$$d(x_n, x_{n+1}) \leq (\frac{\lambda}{1-\lambda})^n d(x_0, x_1).$$

Now, we show that the sequence $(x_n)$ in $X$ is Cauchy. Let $m, n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\
&\leq (\frac{\lambda}{1-\lambda})^n d(x_0, x_1) + (\frac{\lambda}{1-\lambda})^{n+1} d(x_0, x_1) + ... + (\frac{\lambda}{1-\lambda})^{m-1} d(x_0, x_1) \\
&= [(\frac{\lambda}{1-\lambda})^n + (\frac{\lambda}{1-\lambda})^{n+1} + ... + (\frac{\lambda}{1-\lambda})^{m-1}] d(x_0, x_1) \\
&= \sum_{i=n}^{n+i-1} [\frac{\lambda}{1-\lambda})^i d(x_0, x_1] \\
&\leq \sum_{i=n}^{\infty} [\frac{\lambda}{1-\lambda}]^i d(x_0, x_1)
\end{aligned}$$

$\longrightarrow 0$ as n, m $\longrightarrow \infty$. Hence, $(x_n)$ in $X$ is a Cauchy sequence. The completeness of $(X, d)$ guarantees $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now,

$$\begin{aligned}
D(u, Tu) &\leq d(u, x_{n+1}) + D(x_{n+1}, Tu) \\
&\leq d(u, x_{n+1}) + H(Tx_n, Tu) \\
&\leq d(u, x_{n+1}) + [\vartheta(x_n) - \vartheta(u)]N(x_n, u) \\
&\leq d(u, x_{n+1}) + [\vartheta(x_n) - \vartheta(u)] \max\{d(x_n, u), D(x_n, Tx_n), D(u, Tu), \\
&\quad D(u, Tx_n), D(x_n, Tu)\} \\
&\leq d(u, x_{n+1}) + k \max\{d(x_n, u), D(x_n, Tx_n), D(u, Tu), D(u, Tx_n), D(x_n, Tu)\}.
\end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality gives,

$$\begin{aligned}
D(u, Tu) &\leq 0 + k \max\{d(u, u), D(u, Tu), D(u, Tu), D(u, Tu), D(u, Tu)\} \\
&\leq k \max\{0, D(u, Tu)\} \\
&\leq kD(u, Tu) \\
(1 - k)D(u, Tu) &\leq 0 \\
D(u, Tu) &\leq 0.
\end{aligned}$$

This implies that $u \in Tu$.

**Example:** Let $X = \{(3, 3), (3, 4), (5, 5)\}$ be equipped with the taxicab metric $d : X \times X \longrightarrow \mathbb{R}$, given by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Obviously, $(X, d)$ is a complete metric space. Consider a multivalued mapping $T : X \longrightarrow CB(X)$ defined as follows:

$$Tx = \begin{cases} (3, 4) & \text{if } x = (3, 3), \\ (5, 5) & \text{if } x \neq (3, 3). \end{cases}$$

Also, defined $\vartheta : X \longrightarrow \mathbb{R}$ as follows: $\vartheta(3, 3) = 16$, and $\vartheta(3, 4) = 10$. Now, we examine the following cases:
Case I: for $x = (3, 3)$, we have

$$\begin{aligned}
D((3, 3), T(3, 3)) &= \inf\{d((3, 3), y) : y \in T(3, 3)\} \\
&= d((3, 3), (3, 4)) \\
&= 1.
\end{aligned}$$

Case II: for $x = (3, 4)$, we have

$$\begin{aligned}
D((3, 4), T(3, 4)) &= \inf\{d((3, 4), y) : y \in T(3, 4)\} \\
&= d((3, 4), (5, 5)) \\
&= 3.
\end{aligned}$$

Case III: for $x = (5, 5)$, we have

$$\begin{aligned}
D((5, 5), T(5, 5)) &= \inf\{d((5, 5), y) : y \in T(5, 5)\} \\
&= d((5, 5), (5, 5)) \\
&= 0.
\end{aligned}$$

Now, for $x \in X$ with $D(x, Tx) > 0$ i.e
$x \in \{(3, 3), (3, 4)\}$, we have

$H(T(3,3),T(3,4)) = H((3,3),(3,4)) = 3$, $\vartheta(3,3) - \vartheta(3,4) = 6$,

$$\begin{aligned} R_T((3,3),(3,4)) &= \max\{d((3,3),(3,4)), \frac{D((3,3),T(3,3))D((3,4),T(3,4))}{1+d((3,3),(3,4))}\} \\ &= \max\{1, \frac{1.3}{1+1}\} \\ &= \max\{1, \frac{3}{2}\} \\ &= \frac{3}{2}. \end{aligned}$$

Hence,
$H(T(3,3),T(3,4)) = 3 \le 6(\frac{3}{2}) = [\vartheta(3,3) - \vartheta(3,4)]R_T((3,3),(3,4))$.
Thus, for all $x, y \in X$ with $x \neq y$, $D(x,Tx) > 0$ and $D(y,Ty) > 0$ imply $H(Tx,Ty) \le [\vartheta(x) - \vartheta(Tx)]R_T(x,y)$, where $R_T(x,y) = \max\{d(x,y), \frac{D(x,Tx)D(y,Ty)}{1+d(x,y)}\}$. It follows that all the hypotheses of Theorem 1 are satisfied. We see that $T$ has a fixed point.

By defining the multivalued mapping $T : X \longrightarrow CB(X)$ as $Tx = \{gx\}$, for all $x \in X$, where $g : X \longrightarrow X$ is a single valued mapping, we have the following result.

**Corollary 3.1.** *[16] Suppose that $g$ is continuous and forms a Jaggi-type bilateral contraction on a complete metric $(X,d)$. Then, $g$ possesses at least a fixed point.*

**Corollary 3.2.** *[16] Suppose that $g$ forms a Dass-Gupta-type bilateral contraction on a complete metric $(X,d)$. Then, $g$ possesses at least a fixed point*

**Corollary 3.3.** *[17] Suppose that $g$ is a self-mapping on a complete metric space $(X,d)$. If there is a mapping $\vartheta : X \longrightarrow \mathbb{R}_+$ such that $d(x,gx) > 0$ implies $d(gx,gy) \le [\vartheta(x) - \vartheta(gx)]N(x,y)$, in which $N(x,y) = \max\{d(x,y), d(x,gx), d(y,gy), d(x,gy), d(y,gx)\}$. for all $x, y \in M$. Then, $g$ has a fixed point in $X$.*

## 4. Conclusion

This paper broadened the scope of fixed point theory of multivalued mappings by incorporating the bilateral approaches. To this end, Jaggi-type bilateral multivalued contraction, Dass-Gupta type bilateral multivalued contraction and Ciric-Caristi type multivalued contraction are initiated and the corresponding fixed point theorems are proved, with example illustrating the hypotheses of the main results. The ideas in this work, being discussed in the setting of metric spaces, are completely fundamental. Hence, they can be improved upon when presented in the framework of generalized metric spaces such as $b$-metric spaces, $F$-metric spaces and some other pseudo-metric or quasi metric spaces.

## Competing Interests

The authors declare that they have no competing interests.

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