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Generalization of simulation functions for finding best proximity pair, best proximity point and best proximity coincidence point

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Abstract. In the setup of metric spaces, many recent studies established a significant variety of control type mappings and illustrated some fixed point results. To represent various contractivity conditions, Khojasteh et al. have established the idea of a simulation function and came up with certain conclusions about the fixed point. For two nonlinear operators utilizing this type of control function, in this study we explore the existence of the best proximity coincidence point in this study by generalizing the concept of simulation functions. Finally, we applied our results for a new type of nonlinear integral equations.

1. Introduction and preliminaries

Let Ω be a metric space and let $\Gamma, S : \Omega \to \Omega$ be two mappings. A point $x \in X$ is called a fixed point of Γ if $\Gamma x = x$ and a point $x \in \Omega$ is called a coincidence point of Γ and S if $\Gamma x = Sx$. Also, a point $x \in \Omega$ is called a common fixed point of Γ and S if $\Gamma x = Sx = x$. Suppose that A and B be nonempty subsets of Ω. Let:

$$\begin{aligned} A_\circ &= \{u \in A : d(u, v) = d(A, B) \ for\ some\ v \in B\}, \\ B_\circ &= \{v \in B : d(u, v) = d(A, B) \ for\ some\ u \in A\}. \end{aligned}$$

If there is a pair $(\vartheta_\circ, \varsigma_\circ) \in A \times B$ for which $d(\vartheta_\circ, \varsigma_\circ) = d(A, B)$, where $d(A, B)$ is the distance between A and B, then the pair $(\vartheta_\circ, \varsigma_\circ)$ is said to be a best proximity pair for (A, B). Best proximity pair evolves as an expansion of the concept of best approximation.

The best proximity points of (A, B) can be find by considering a map $\Gamma : A \cup B \to A \cup B$. A point $u \in A \cup B$ is called a best proximity point of the pair (A, B), if $d(u, \Gamma u) = d(A, B)$ and the family of all best proximity points of (A, B), $P_\Gamma(A, B)$, is

$$P_\Gamma(A, B) = \{u \in A \cup B : d(u, \Gamma u) = d(A, B)\}.$$

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The best proximity point is also an expansion of the concept of fixed-point because, if $A \cap B \neq \emptyset$, then every best proximity point is a fixed point of Γ. There are a number of contributing publications on the subject of fixed points and common fixed-point theorems (see [7–9, 20, 25, 28]). Common best proximity points have become one of the most studied topics in the field of fixed point theory. These notions generalize the concept of a common fixed point and allow us to deal with non-self-mappings. There are many more results on common best proximity points in the literature; for instance, we can see [10, 24, 30].

Eldred et al. [11] and Sankar Raj et al. [32] presented a best proximity point theorem for relatively nonexpansive mappings. A best proximity point theorem for contraction mappings has been obtained by Basha [31]. The best proximity point theorems for various variants of contractions have been explored in [12, 14, 15, 17–19, 26].

Recently, Khojasteh et al. have explained in [23] the idea of the simulation function in order to express different contractivity conditions. Hence, it could be possible to treat several fixed-point problems from a unique common point of view. Furthermore, Roldan et al. in [29] slightly modified their notion of simulation functions, and they have investigated the existence and uniqueness of coincidence points for two nonlinear operators by using this kind of control functions.

Karapınar has introduced new fixed point results via simulation functions [22]. Alsubaie et al. extended the simulation function via rational expressions [4]. Heidary et al. gave a common fixed-point theorem for Suzuki-type contractions via generalized Ψ-simulation functions [16]. Agarwal et al. introduced the notion of interpolative Rus-Reich-Ćirić type $\mathcal{Z}$-contractions in the setting of complete metric spaces, and they considered some immediate consequences of their results [1]. Alqahtani et al. investigated the existence of a fixed point for some contractions with the help of simulation functions [3, 4]. In continuation, Alghamdi et al. gave a note on extended $\mathcal{Z}$-contraction.

Gabeleh et al. proposed a new type of simulation functions and, using Meir-Keeler condensing operators and the notion of measure of non-compactness, established a new generalisation of Darbo's fixed point theorem [13]. Monfared et al. [27] introduced α_μ-admissible mappings, Z_μ-contractions and N_μ-contractions via simulation functions. They proved some new fixed-point theorems for some classes of contractions via α-admissible simulation mappings as well. The existence of fixed points for certain operators via simulation functions has been investigated in the context of complete M-metric spaces by Asadi et al. [5]. Also, Asadi et al. gave a new approach to generalising Darbo's fixed point problem by using simulation functions with an application to integral equations [6].

At the end, Karapınar and Khojasteh have investigated the existence of the best proximity points of certain mappings via simulation functions in the framework of complete metric spaces.

Definition 1.1. *[29] A simulation function is a mapping $\xi : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:*
(1) $\xi(0,0) = 0$;
(2) $\xi(t,s) < s - t$ for all $t, s > 0$;
(3) if $\{t_n\}$ and $\{s_n\}$ be sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$.

Suppose $\mathcal{Z}$ be the set of all simulation functions. Now, in the following we give a new generalization of the simulation functions.

Definition 1.2. *Let α be an arbitrary positive real number. An α-simulation function is a mapping $\xi_\alpha : [\alpha,\infty) \times [\alpha,\infty) \to \mathbb{R}$ which is satisfying the following conditions:*
(1) $\xi_\alpha(\alpha,\alpha) = 0$;
(2) $\xi_\alpha(t,s) < s - t$ for all $t, s > \alpha$;
(3) if $\{t_n\}$ and $\{s_n\}$ be sequences in (α,∞) such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > \alpha$, then $\limsup_{n\to\infty} \xi_\alpha(t_n, s_n) < 0$.

Let $\mathcal{Z}_\alpha$ be the set of all α-simulation functions. Before presenting our main results using simulation functions, we give some examples.

Example 1.3. *Let $\alpha > 0$, $k \in [0,1)$ and let $\xi_\alpha : [\alpha,\infty) \times [\alpha,\infty) \to \mathbb{R}$ be the function defined by:*

$$\xi_\alpha(t,s) = \begin{cases} 2(s-t), & if \quad s < t, \\ k(s-\alpha) - (t-\alpha), & otherwise. \end{cases}$$

Then ξ_α is an α-simulation function. Verification of (1) and (2) follows from

$$\begin{cases} \alpha < s < t & \Rightarrow \xi_\alpha(t,s) = 2(s-t) < s-t, \\ \\ \alpha < t \le s & \Rightarrow \xi_\alpha(t,s) = k(s-\alpha) - (t-\alpha) < (s-\alpha) - (t-\alpha). \end{cases}$$

Example 1.4. *Let $\alpha > 0$, $k \in [0,1)$ and let $\xi_\alpha : [\alpha,\infty) \times [\alpha,\infty) \to \mathbb{R}$ be the function defined by:*

$$\xi_\alpha(t,s) = \begin{cases} 2(s-t), & if \quad s < t, \\ \sqrt{s\alpha} - t, & otherwise. \end{cases}$$

Then ξ_α is an α-simulation function.

Example 1.5. *Let ϕ and φ be two altering distance functions such that $\varphi(t) < t \le \phi(t)$ for all $t > 0$. Then the mapping*

$$\xi_\alpha(t,s) = \varphi(s) - \phi(t), \ \forall\, t,s \in [\alpha,\infty)$$

is an α-simulation function. If, in the previous example, $\phi(t) = t$ and $\varphi(t) = kt + (1-k)\alpha$ for all $t > 0$, where $k \in [0,1)$, then we obtain the following particular case of simulation functions:

$$\xi_\alpha(t,s) = ks + (1-k)\alpha - t, \ \forall\, t,s \in [\alpha,\infty).$$

Example 1.6. *If $\phi : [\alpha,\infty) \to [0,\infty)$ be a continuous function such that $\phi(t) = 0$ iff $t = \alpha$, and we put*

$$\xi_\alpha(t,s) = s - \phi(s) - t, \quad \forall\, s,t \in [\alpha,\infty),$$

then ξ_α is an α-simulation function.

2. Main Results

In this section, we use the notion of α-simulation function for finding the best proximity pairs, best proximity points and best proximity coincidence points for a non-self-mapping defined on a metric space.

Definition 2.1. *Let (Ω, d) be a metric space, A, B be nonempty subsets of Ω and let $\Gamma, S : A \cup B \to A \cup B$ be two mappings. Let $\alpha = d(A,B)$. We say that the pair of operators (Γ, S) is a pair of cyclic $(\mathcal{Z})$-contractions if*

$$\Gamma(A) \subseteq S(B) \subseteq B, \ \Gamma(B) \subseteq S(A) \subseteq A$$

and there exists a pair $(\xi, \xi_\alpha) \in \mathcal{Z} \times \mathcal{Z}_\alpha$ such that

$$\xi(d(\Gamma x, \Gamma y), d(Sx, Sy)) \ge 0, \ \forall\, x,y \in A, \ or\ x,y \in B \ \ such\ that\ d(Sx,Sy) > 0,$$

and

$$\xi_\alpha(d(\Gamma x, \Gamma y), d(Sx, Sy)) \ge 0, \ \forall\, x \in A \ and\ \forall\, y \in B \ such\ that\ d(Sx,Sy) > \alpha.$$

Definition 2.2. *Let (Ω, d) be a metric space. Given two mappings $\Gamma, S : A \cup B \to A \cup B$, we say that $\{\kappa_n\} \subseteq \Omega$ is a Picard sequence of the pair of cyclic $(\mathcal{Z})$-contractions (Γ, S) if*

$$S\kappa_{n+1} = \Gamma\kappa_n.$$

Since $\Gamma(A) \subseteq S(B) \subseteq B$ and $\Gamma(B) \subseteq S(A) \subseteq A$, it is well known that there is a Picard sequence of (Γ, S) for every point $\kappa_0 \in A$. Also, if $\kappa_0 \in A$, then we have $\{\kappa_{2n}\} \subseteq A$ and $\{\kappa_{2n+1}\} \subseteq B$.

Definition 2.3. *Let A and B be nonempty subsets of a metric space Ω and let $\Gamma, S : A \cup B \to A \cup B$. A point $x \in A$ is called a best proximity coincidence point (BPC point) of the pair (Γ, S) if*

$$d(\Gamma x, Sx) = d(A, B).$$

Of course, on the other hand, $(\Gamma x, Sx)$ is a best proximity pair. It is notable that if we have $A \cap B \neq \emptyset$, the point $x \in A$ is a coincidence point of (Γ, S), i.e., $\Gamma x = Sx$. Also, if we have $S = I$ (i.e., the identity mapping), then the point $x \in A$ is a best proximity point of Γ. Finally, if we have $A \cap B \neq \emptyset$ and S be the identity mapping, then the point $x \in A$ is a fixed point of Γ.

In the following, we give the main theorem of this section.

Theorem 2.4. *Let (Ω, d) be a metric space, A, B nonempty closed subsets of Ω and let $\Gamma, S : A \cup B \to A \cup B$ be mappings such that (Γ, S) be a pair of cyclic $(\mathcal{Z})$-contractions in (Ω, d) and suppose that there exists a Picard sequence $\{\kappa_n\}_{n\geq 0}$ of (Γ, S). Also assume that $(\Gamma(A), d)$ or $(S(B), d)$ is complete. Then (Γ, S) have, at least, a BPC point.*

Proof. If $\{\kappa_n\}_{n\geq 0}$ contains a *BPC* point of (Γ, S), then the proof is finished. Suppose that $\{\kappa_n\}_{n\geq 0}$ does not contain any *BPC* point of (Γ, S), that is,

$$d(S\kappa_n, S\kappa_{n+1}) > d(A, B), \quad \forall\, n \geq 0.$$

We divide the proof into four steps.
Step 1. We claim that $\lim\limits_{n\to\infty} d(S\kappa_n, S\kappa_{n+1}) = d(A, B)$. Using the definition of the α-simulation function, we have

$$\begin{aligned} 0 \leq \xi_\alpha(d(\Gamma\kappa_n, \Gamma\kappa_{n+1}), d(S\kappa_n, S\kappa_{n+1})) &= \xi_\alpha(d(S\kappa_{n+1}, S\kappa_{n+2}), d(S\kappa_n, S\kappa_{n+1})) \\ &< d(S\kappa_n, S\kappa_{n+1}) - d(S\kappa_{n+1}, S\kappa_{n+2}), \end{aligned}$$

which means that $d(A, B) \leq d(S\kappa_{n+1}, S\kappa_{n+2}) < d(S\kappa_n, S\kappa_{n+1})$ for all $n \in \mathbb{N}$. Let $d_n = d(S\kappa_n, S\kappa_{n+1})$, then $\{d_n\}$ is a non-increasing sequence of nonnegative real numbers, hence it is convergent. Let $r = \lim_{n\to\infty} d_n$. We prove $r = d(A, B)$. If $t_n := d_{n+1}$ and $s_n := d_n$, then $t_n < s_n$ and so $\limsup_{n\to\infty} \xi_\alpha(t_n, s_n) < 0$, which is a contradiction. Therefore, $d(A, B) = \lim_{n\to\infty} d_n$.

Step 2. In this step, we prove that $\lim_{n\to\infty} d(S\kappa_{n-1}, S\kappa_{n+1}) = 0$. Using the definition of the simulation function, we have

$$\begin{aligned} 0 \leq \xi(d(\Gamma\kappa_{n-1}, \Gamma\kappa_{n+1}), d(S\kappa_{n-1}, S\kappa_{n+1})) &= \xi(d(S\kappa_n, S\kappa_{n+2}), d(S\kappa_{n-1}, S\kappa_{n+1})) \\ &< d(S\kappa_{n-1}, S\kappa_{n+1}) - d(S\kappa_n, S\kappa_{n+2}), \end{aligned}$$

which means that $0 \leq d(S\kappa_n, S\kappa_{n+2}) < d(S\kappa_{n-1}, S\kappa_{n+1})$ for all $n \in \mathbb{N}$. Let $b_n = d(S\kappa_{n-1}, S\kappa_{n+1})$, then $\{b_n\}$ is a non-increasing sequence of nonnegative real numbers, so it is convergent. Let $\delta = \lim_{n\to\infty} b_n$. We prove that $\delta = 0$. If $t_n := b_{n+1}$ and $s_n := b_n$, then $t_n < s_n$ and so $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$, which is a contradiction. Therefore, $\lim_{n\to\infty} b_n$=0.

Step 3. We claim that $\{S\kappa_{2n}\}$ is Cauchy in (Ω, d). We reason by contradiction. Suppose that $\{S\kappa_{2n}\}$ is not Cauchy in (Ω, d). Hence, there is $\epsilon_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $m_k, n_k \in \mathbb{N}$ so that $m_k > n_k \geq k$ and $d(S\kappa_{2m_k}, S\kappa_{2n_k}) \geq \epsilon_0$.

We can assume that m_k is the smallest index for which the above relation is valid, that is,

$$d(S\kappa_{2m(k)-2)}, S\kappa_{2n(k)}) < \epsilon_0 \ \forall\, k \in \mathbb{N}. \tag{1}$$

By using (1), we deduce that

$$\begin{aligned} \epsilon_0 &\leq d(S\kappa_{2m(k)}, S\kappa_{2n(k)}) \\ &\leq d(S\kappa_{2m(k)}, S\kappa_{2m(k)-2}) + d(S\kappa_{2m(k)-2}, S\kappa_{2n(k)}) \\ &< d(S\kappa_{2m(k)}, S\kappa_{2m(k)-2}) + \epsilon_0, \end{aligned}$$

for all $k \in \mathbb{N}$ and also using step 1, it follows that

$$\epsilon_0 = \lim_{k\to\infty} d(S\kappa_{2m(k)}, S\kappa_{2n(k)}).$$

Moreover, by

$$d(S\kappa_{2m(k)}, S\kappa_{2n(k)}) \le d(S\kappa_{2m(k)}, S\kappa_{2m(k)+2}) + d(S\kappa_{2m(k)+2}, S\kappa_{2n(k)+2}) + d(S\kappa_{2n(k)+2}, S\kappa_{2n(k)})$$

and

$$d(S\kappa_{2m(k)+2}, S\kappa_{2n(k)+2}) \le d(S\kappa_{2m(k)+2}, S\kappa_{2m(k)}) + d(S\kappa_{2m(k)}, S\kappa_{2n(k)}) + d(S\kappa_{2n(k)}, S\kappa_{2n(k)+2})$$

for all $k \in \mathbb{N}$ and also using step 1, it follows that

$$\epsilon_0 = \lim_{k\to\infty} d(S\kappa_{2m(k)+2}, S\kappa_{2n(k)+2}).$$

In particular, there is $n_1 \in \mathbb{N}$ such that

$$d(S\kappa_{2m(k)}, S\kappa_{2n(k)}) > \epsilon_0/2 > 0,\ d(S\kappa_{2m(k)+2}, S\kappa_{2n(k)+2}) > \epsilon_0/2 > 0,\ \ \forall\, k \ge n_1.$$

Using the fact that the pair (Γ, S) is a pair of cyclic $\mathcal{Z}$-contractions with respect to ξ, we deduce that

$$\begin{aligned} 0 &\le \xi(d(\Gamma\kappa_{2m(k)+2}, \Gamma\kappa_{2n(k)+2}), d(S\kappa_{2m(k)}, S\kappa_{2n(k)})) \\ &= \xi(d(S\kappa_{2m(k)+3}, S\kappa_{2n(k)+3}), d(S\kappa_{2m(k)}, S\kappa_{2n(k)})) \\ &< d(S\kappa_{2m(k)}, S\kappa_{2n(k)}) - d(S\kappa_{2m(k)+3}, S\kappa_{2n(k)+3}) \end{aligned}$$

$\forall\, k \ge n_1$.

In particular,

$$0 < d(S\kappa_{2m(k)+1}, S\kappa_{2n(k)+1}) < d(S\kappa_{2m(k)}, S\kappa_{2n(k)}),\ \ \forall\, k \ge n_1.$$

Employ the sequences $t_k = d(S\kappa_{2m(k)+1}, S\kappa_{2n(k)+1})$ and $s_k = d(S\kappa_{2m(k)}, S\kappa_{2n(k)})$. Then $t_k < s_k$ and $\limsup_{k\to\infty} \xi(t_k, s_k) < 0$, which is a contradiction. Therefore, we must admit that the sequence $\{S\kappa_{2n}\}$ is Cauchy in (Ω, d).

Step 4. Now, we prove that (Γ, S) have a *BPC* point. Note that $S\kappa_{2n+1} = \Gamma\kappa_{2n} \in \Gamma(A) \subseteq S(B)$. Since $S(B)$ is complete, there is $u \in S(B)$ such that $S\kappa_{2n+1} \to u$, that is,

$$\lim_{n\to\infty} d(S\kappa_{2n+1}, u) = 0 \textit{ and so } \lim_{n\to\infty} d(\Gamma\kappa_{2n}, u) = 0. \tag{2}$$

Let $v \in B$ be a point such that $Sv = u$. We are going to show that v is a *BPC* point of (Γ, S). Now,

$$d(A, B) \le d(Sv, \Gamma\kappa_{2n-1}) \le d(Sv, \Gamma\kappa_{2n}) + d(\Gamma\kappa_{2n}, \Gamma\kappa_{2n-1}).$$

Thus, by step 1 and (2) we infer that $d(Sv, \Gamma\kappa_{2n-1})$ converges to $d(A, B)$. On the other hand,

$$0 \le \xi(d(\Gamma\kappa_{2n-1}, \Gamma v), d(S\kappa_{2n-1}, Sv)) < d(S\kappa_{2n-1}, Sv) - d(\Gamma\kappa_{2n-1}, \Gamma v),$$

and so

$$d(\Gamma\kappa_{2n-1}, \Gamma v) < d(S\kappa_{2n-1}, Sv),\ \forall\, n \in \mathbb{N}.$$

Therefore

$$\lim_{n\to\infty} d(\Gamma\kappa_{2n-1}, \Gamma v) = 0. \tag{3}$$

Hence,

$$d(A, B) \le d(Sv, \Gamma v) \le d(Sv, \Gamma\kappa_{2n-1}) + d(\Gamma\kappa_{2n-1}, \Gamma v).$$

Hence, if $n \to \infty$ by (2) and (3) we have $d(Sv, \Gamma v) = d(A, B)$. □

In the following we give some consequences of Theorem 2.4.

Corollary 2.5. *Let (Ω, d) be a metric space, A, B nonempty closed subsets of Ω and let $\Gamma, S : A \cup B \to A \cup B$ be mappings such that $\Gamma(A) \subseteq S(B) \subseteq B$ and $\Gamma(B) \subseteq S(A) \subseteq A$ and suppose that there exists a Picard sequence $\{\kappa_n\}_{n\geq 0}$ of (Γ, S). If ϕ and φ be two altering distance functions such that $\varphi(t) < t \leq \phi(t)$ for every $t > 0$ and*

$$\phi(d(\Gamma x, \Gamma y)) \leq \varphi(d(Sx, Sy)), \ \forall\, x, y \in A \text{ or } x, y \in B$$

and

$$d(\Gamma x, \Gamma y) \leq kd(Sx, Sy) + (1-k)d(A, B), \ \forall\, x \in A, \ y \in B,$$

also assume that $(\Gamma(A), d)$ or $(S(B), d)$ is complete, then (Γ, S) have a BPC point.

Proof. It follows from Theorem 2.4 using the simulation function $\xi(t, s) = \varphi(s) - \phi(t)$ for all $s, t \in [0, \infty)$, and α-simulation function $\xi_\alpha(t, s) = ks + (1-k)\alpha - t$ for all $s, t \in [\alpha, \infty)$. □

Corollary 2.6. *Let (Ω, d) be a metric space, A, B nonempty closed subsets of Ω and let $\Gamma : A \cup B \to A \cup B$ be a mapping such that $\Gamma(A) \subseteq B$ and $\Gamma(B) \subseteq A$. Let ϕ and φ be two altering distance functions such that $\varphi(t) < t \leq \phi(t)$ for every $t > 0$ and*

$$\phi(d(\Gamma x, \Gamma y)) \leq \varphi(d(x, y)), \ \forall\, x, y \in A \text{ or } x, y \in B$$

and

$$d(\Gamma x, \Gamma y) \leq kd(x, y) + (1-k)d(A, B), \ \forall\, x \in A, \ y \in B.$$

Also, assume that $(\Gamma(A), d)$ is complete. Then Γ have a best proximity point.

Proof. It follows from Corollary 2.5, if S be the identity mapping on Ω. □

Theorem 2.7. *Let A and B be two nonempty closed and convex subsets of a strictly convex Banach space Ω. Let $\Gamma, S : A \cup B \to A \cup B$ be mappings such that (Γ, S) be a cyclic $\mathcal{Z}$-contraction. Also, assume that $\Gamma(A)$ or $S(B)$ is complete. Then (Γ, S) have a unique BPC point.*

Proof. By Theorem 2.4 (Γ, S) have at least a *BPC* point. Suppose that $x, y \in A$ such that $x \neq y$. Since A and B are convex, $\frac{\Gamma x + \Gamma y}{2} \in B$ and $\frac{Sx + Sy}{2} \in A$. Also, since $\|\Gamma x - Sx\| = d(A, B)$ and $\|\Gamma y - Sy\| = d(A, B)$, then from strictly convexity assumption of Banach space Ω, $\|\frac{\Gamma x + \Gamma y}{2} - \frac{Sx + Sy}{2}\| < d(A, B)$, which is a contradiction. Therefore, $x = y$. Hence, the proof is completed. □

In the following, we give some examples for finding the *BPC* point.

Example 2.8. *Let A and B be subsets of $\mathbb{R}^2$ defined by*

$$A = \{(x, 0) : x \geq 1\}, \ B = \{(0, y) : y \geq 1\}.$$

Suppose that

$$\Gamma(x, y) = (\sqrt[4]{y}, \sqrt[4]{x}), \ S(x, y) = (\sqrt{x}, \sqrt{y})$$

and

$$\xi(s, t) = ks - t, \ \ \xi_\alpha(s, t) = \sqrt{\sqrt{2}s} - t$$

where $k \in [0, 1)$.

Then (Γ, S) is a cyclic $\mathcal{Z}$-contraction on $\mathbb{R}^2$ and $\|\Gamma(1, 0) - S(1, 0)\| = d(A, B)$. Here, $d(A, B) = \sqrt{2}$. For $(x, 0) \in A$ and $(0, y) \in B$ we put

$$t = \|\Gamma(x, 0) - \Gamma(0, y)\|, \ s = \|S(x, 0) - S(0, y)\|.$$

Hence

$$t = \sqrt{\sqrt{x} + \sqrt{y}}, \ s = \sqrt{x + y}.$$

Since

$$\sqrt{\sqrt{x} + \sqrt{y}} \leq \sqrt{\sqrt{2}\sqrt{x + y}},$$

therefore

$$\xi_\alpha(s,t) = \sqrt{\sqrt{2}s} - t < s - t, \quad \textit{for all } s,t > \sqrt{2} = d(A,B)$$

and so ξ_α *is a* α*-simulation function.*

Example 2.9. *Let* A *and* B *be subsets of* $\mathbb{R}^2$ *defined by,*

$$A = [1,2] \times [1,2],\ B = [-2,-1] \times [-2,-1]$$

and $x_\circ \in A$. *Define mappings* $\Gamma, S : A \cup B \to A \cup B$ *by*

$$\Gamma(x,y) = \begin{cases} (-\frac{x}{3} - \frac{2}{3}, -\frac{y}{3} - \frac{2}{3}), & (x,y) \in A, \\ \\ (-\frac{x}{3} + \frac{2}{3}, -\frac{y}{3} + \frac{2}{3}), & (x,y) \in B, \end{cases} \tag{4}$$

$$S(x,y) = \begin{cases} (\frac{x}{2} + \frac{1}{2}, \frac{y}{2} + \frac{1}{2}), & (x,y) \in A, \\ \\ (\frac{x}{2} - \frac{1}{2}, \frac{y}{2} - \frac{1}{2}), & (x,y) \in B. \end{cases} \tag{5}$$

Obviously, $\Gamma(A) \subseteq S(B) \subseteq B$ *and* $\Gamma(B) \subseteq S(A) \subseteq A$. *Initially, we show that*

$$\|\Gamma(x,y) - \Gamma(u,w)\| \leqslant \|S(x,y) - S(u,w)\|,$$

for all $(x,y) \in A$ *and* $(u,w) \in B$. *We have*

$$\begin{aligned} \|\Gamma(x,y) - \Gamma(u,w)\| &= \|(-\frac{x}{3} - \frac{2}{3} - [-\frac{u}{3} + \frac{2}{3}], -\frac{y}{3} - \frac{2}{3} - [-\frac{w}{3} + \frac{2}{3}])\| \\ &\leqslant \sqrt{|\frac{u-x}{3} - \frac{4}{3}|^2 + |\frac{w-y}{3} - \frac{4}{3}|^2} \\ &\leqslant \sqrt{|\frac{u-x}{2} - 1|^2 + |\frac{w-y}{2} - 1|^2} \\ &\leqslant \|S(x,y) - S(u,w)\|, \end{aligned}$$

and $S(x^*, y^*) = (x^*, y^*) = (1,1)$ *such that*

$$\|\Gamma(x^*, y^*) - S(x^*, y^*)\| = \|(1,1) - (-1,-1)\| = 2\sqrt{2} = dist(A,B).$$

3. An application to integral equations

In this section, we apply Corollary 2.6 to study the existence and uniqueness of a solution to a new kind of nonlinear integral equations. Let $f : [-b,b] \times \mathbb{R} \to \mathbb{R}$ and $G : [-b,b] \times [-b,b] \to [0,\infty)$ are continuous functions. Let $X = C([-b,b], \mathbb{R})$ be the set of real continuous functions on $[-b,b]$. We endow X with the standard norm

$$\|u\|_\infty = \max_{t \in [-b,b]} |u(t)|.$$

It is well known that $(X, \|.\|_\infty)$ is a Banach space. For $0 < a < b$, we consider the nonlinear integral equation

$$\|u(t) - \int_{-b}^{b} G(t,s) f(s, u(s)) ds\| = 2a, \forall\ t \in [a,b]. \tag{6}$$

We suppose that

$$\sup_{t \in [-b,b]} \int_a^b |G(t,s)| ds \leq 1. \tag{7}$$

Also, we suppose

$$-b \leq \int_a^b G(-t,s)f(s,u(s))ds \leq -a, \forall\ t \in [a,b], \tag{8}$$

and

$$a \leq \int_a^b G(t,-s)f(-s,u(-s))ds \leq b, \forall\ t \in [a,b]. \tag{9}$$

Finally, we suppose that for all $s \in [-b,b]$, if $x(s) \in [a,b]$ and $y(s) \in [-b,-a]$, then

$$|f(s,x(s)) - f(s,y(s))| \leq k|x(s) - y(s)| + 2a(1-k) \tag{10}$$

and if $x(s), y(s) \in [a,b]$, then

$$|f(s,x(s)) - f(s,y(s))| \leq k|x(s) - y(s)|, \tag{11}$$

where $k \in (0,1)$. Now, define the set

$$\Theta = \{u \in X : -b \leq u(t) \leq b,\ for\ t \in [-b,b]\}.$$

Theorem 3.1. *Under the assumptions (7)–(12), problem (6) has a solution $u^a \in \Theta$.*

Proof. Let A and B be closed subsets of X such that

$$A = \{u \in \Theta : u([a,b]) \subseteq [a,b], u([-b,a]) = \{a\}\}$$

and

$$B = \{u \in \Theta : u([-b,-a]) \subseteq [-b,-a], u([-a,b]) = \{-a\}\}.$$

It is clear that $d(A,B) = 2a$. Define the mapping $\Gamma : X \to X$ by

$$\Gamma u(t) = \int_{-b}^b G(t,s)f(s,u(s))ds,\ t \in [-b,b].$$

We shall prove that

$$\Gamma(A) \subseteq B \textit{ and } \Gamma(B) \subseteq A. \tag{12}$$

Let $u \in A$, that is, $u(s) \in [a,b]$ for all $s \in [a,b]$. (8) implies that $-b \leq \Gamma u(t) \leq -a$ for all $t \in [-b,-a]$. Hence, $\Gamma u \in B$, i.e., $\Gamma(A) \subseteq B$. Also, let $u \in B$, that is, $u(s) \in [-b,-a]$ for all $s \in [-a,-b]$. (9) implies that $a \leq \Gamma u(t) \leq b$. Hence, $\Gamma u \in A$, i.e., $\Gamma(B) \subseteq A$. Now, using conditions (7) and (12), for all $t \in [a,b]$, $u \in A$ and $v \in B$, we have

$$\begin{aligned}
|\Gamma u(t) - \Gamma v(t)| &\leq \int_a^b G(t,s)|f(s,u(s)) - f(s,-a)|ds \\
&\leq k\int_a^b G(t,s)|u(s) - (-a)|ds + (1-k)2a\int_a^b G(t,s)ds \\
&\leq k\|u - v\|_\infty + (1-k)2a\int_a^b G(t,s)ds \\
&\leq k\|u - v\|_\infty + (1-k)2a.
\end{aligned}$$

Similarly, for all $t \in [-b,-a]$, $u \in A$ and $v \in B$, we have

$$\begin{aligned}
|\Gamma u(t) - \Gamma v(t)| &\leq \int_{-b}^{-a} G(t,s)|f(s,a) - f(s,v(s))|ds \\
&\leq k\|u - v\|_\infty + (1-k)2a.
\end{aligned}$$

For all $t \in [-a, a]$, $u \in A$ and $v \in B$, we have

$$\begin{aligned} |\Gamma u(t) - \Gamma v(t)| &\leq \int_{-a}^{a} G(t,s)|f(s,a) - f(s,-a)|ds \\ &\leq k\|u - v\|_\infty + (1-k)2a \end{aligned}$$

and so for all $t \in [-b, b]$, $u \in A$ and $v \in B$, we have

$$\begin{aligned} |\Gamma u(t) - \Gamma v(t)| &\leq \int_{-b}^{b} G(t,s)|f(s,u(s)) + f(s,v(s))|ds \\ &\leq k\|u - v\|_\infty + (1-k)2a. \end{aligned}$$

Hence

$$\|\Gamma u - \Gamma v\|_\infty \leq k\|u - v\|_\infty + (1-k)d(A,B).$$

Also, for all $t \in [-b, b]$ and $u, v \in A$, we have

$$\begin{aligned} |\Gamma u(t) - \Gamma v(t)| &\leq \int_{-b}^{b} G(t,s)|f(s,u(s)) - f(s,v(s))|ds \\ &\leq k\int_{a}^{b} G(t,s)|u(s) - v(s)|ds \\ &\leq k\|u - v\|_\infty. \end{aligned}$$

Hence,

$$\|\Gamma u - \Gamma v\|_\infty \leq k\|u - v\|_\infty.$$

Now, all the conditions of Corollary 2.6 are satisfied and we deduce that Γ has a best proximity point $u^* \in A$, that is, $u^* \in \Theta$ is a solution for (6). □

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