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On the weak solution for the nonlocal parabolic problem with p-Kirchhoff term via topological degree

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Abstract. In this work, we study the existence of weak solution for the following nonlinear parabolic initial boundary value problem associated to the p-Kirchhoff-type equation,

$$\frac{\partial u}{\partial t} - \mathcal{M}\Big(\int_\Omega \Big(A(x,t,\nabla u) + \frac{1}{p}|u|^p\Big)dx\Big)\mathrm{div}\Big(a(x,t,\nabla u) - |\nabla u|^{p-2}\nabla u\Big) = f$$

in $Q. = \Omega \times (0,T)$ where $\Omega \subset \mathbb{R}^n$ $(N \geq 2)$ is a bounded domain with Lipschitz boundar $\partial\Omega$, $\mathcal{M} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the p-Kirchhoff-type function and $a : Q \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function. Under some appropriate assumptions, we obtain the existence of a weak solution for the problem above by using Berkovits and Mustonen topological degree theory, in the space $L^p(0,T,W_0^{1,p}(\Omega))$.

1. Introduction

In this article, we are interested in the following class of p-Kirchhoff parabolic problem:

$$(\mathcal{P}) \begin{cases} \dfrac{\partial u}{\partial t} - \mathcal{M}\big(\mathcal{K}(u)\big)\mathrm{div}\,\Big(a(x,t,\nabla u) - |\nabla u|^{p-2}\nabla u\Big) = f(x,t) & \text{in} \quad Q := \Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \\ u(x,t) = 0 & \text{on} \quad \Gamma = \partial\Omega \times (0,T). \end{cases}$$

Where

$$\mathcal{K}(u) = \int_\Omega (A(x,t,\nabla u) + \frac{1}{p}|\nabla u|^p)\,dx.$$

In the problem $(\mathcal{P})$ and in the sequel, Ω designantes a bounded and open domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$, and we denote by Q the cylinder $\Omega \times (0,T)$ that $\Gamma = \partial\Omega \times (0,T)$ is its lateral surface, where $T > 0$ is a fixing time, the terme $-\mathrm{div}\,(a(x,t,\nabla u)$ is a Leray-Lions operator acting from $\mathcal{H}$ to its dual $\mathcal{H}^*$, such that

$$\mathcal{H} = L^p(0,T,W_0^{1,p}(\Omega)) \text{ and } \mathcal{H}^* = L^{p'}(0,T,W^{-1,p'}(\Omega)) \quad (p \geq 2),$$

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where $\frac{1}{p}+\frac{1}{p'}=1$. The right-hand side f is assumed to belong to $\mathcal{H}^*$ and the Kirchhoff type function, $\mathcal{M}:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is continuous and satisfying certain assumptions.
The problem $(\mathcal{P})$ is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0 \tag{1}$$

introduced in 1883 by Kirchhoff see [16] for mor details. This equation is an extension of the classical d'Alembert's wave equation. The parameters in (1) have the following meanings: h is the cross-section area, E is the Young modulus, ρ is the mass density, L is the length of the string, and ρ_0 is the initial tension. In the last few years there has been a lot of interest in problems of the Kirchhoff type, see for example [3, 5, 9–11, 13, 15, 19, 23–25, 30] in which the authors have used variational, Galerkin approximation method, topological methods and sub- and super-solutions concept, to get the existence of solutions. This interest is due to their contributions to the modelling of many physical and biological phenomena. We refer the reader to [12, 18] for some interesting results and further references.
Motivated by the above, we consider $(\mathcal{P})$ to study the existence of at least one weak solutions solution, using a different approach. This approach is based on the topological degree theory for operators of the type type $L+S$, where L is a linear densely defined maximal monotone map and S is a bounded demicontinuous map of type (S_+) with respect to a domain of L. For more information on the history of this theory, the reader is referred to ([1, 4, 6–8, 14, 17, 21, 22, 26–28]).
The rest of the paper is organized as follows. In section 2, we give some mathematical preliminaries about the functional framework in which we will treat our problem. In Section 3, we introduce some classes of operators and then the associated topological degree. The last Section, is devoted to giving the proof of the main result of our paper.

2. Mathematical preliminaries

In this section, we recall some necessary definitions and basic properties of the functional framework required to investigate the problem $(\mathcal{P})$.

Let $\Omega\subseteq\mathbb{R}^N$ be a bounded open set with smooth boundary, $p\geq 2$ and $p'=\dfrac{p}{p-1}$, we will denote by $L^p(\Omega)$ the space of all measurable functions φ defined in Ω such that

$$\|\varphi\|_{L^p(\Omega)}=\Big(\int_\Omega|\varphi(x)|^pdx\Big)^{1/p}<\infty.$$

We define the Sobolev Space

$$W^{1,p}(\Omega)=\Big\{\varphi\in L^p(\Omega)\ :\ \nabla\varphi\in L^p(\Omega)\Big\},$$

with respect to the norm

$$\|\varphi\|_{W^{1,p}(\Omega)}=\Big(\|\varphi\|^p_{L^p(\Omega)}+\|\nabla\varphi\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

We define the functional space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$.
Note that, according to the Poincaré inequality, the norm $\|\cdot\|_{W^{1,p}(\Omega)}$ is equivalent to the norm $\|\cdot\|_{1,p}$ setting by

$$\|\varphi\|_{1,p}=\|\nabla\varphi\|_{L^p(\Omega)}\ \text{ for }\ \varphi\in W_0^{1,p}(\Omega).$$

Remember that the Sobolev space $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space and the embedding $W_0^{1,p}(\Omega)\hookrightarrow\hookrightarrow L^p(\Omega)$ is compact (see [29]).
In this work, we consider the following space

$\mathcal{H} := L^p(0,T;W_0^{1,p}(\Omega)) \quad (T > 0),$

that is a separable and reflexive Banach space with the norm

$$|\varphi|_{\mathcal{H}} = \Big(\int_0^T \|\varphi\|^p_{W^{1,p}(\Omega)} dt\Big)^{1/p}.$$

Thanks to Poincaré inequality, the expression

$$\|\varphi\|_{\mathcal{H}} = \Big(\int_0^T \|\nabla\varphi\|^p_{L^p(\Omega)} dt\Big)^{1/p},$$

is a norm defined on $\mathcal{H}$ and is equivalent to the norm $|\varphi|_{\mathcal{H}}$.
Note that the $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a separable and reflexive Banach space.

3. Classes of mappings and Topological degree

Now, we give some results and properties from the Berkovits and Mustonen degree theory for demicontinuous operators of generalized (S_+) type in real reflexive Banach. In what follows, let $\mathcal{X}$ be a real separable reflexive Banach space with dual $\mathcal{X}^*$ and with continuous dual pairing $\langle\cdot,\cdot\rangle$ and given a nonempty subset Ω of X, and $\rightharpoonup$ represents the weak convergence.
Let $\mathcal{T} : \mathcal{X} \longrightarrow 2^{\mathcal{X}^*}$ be a multi-values mapping. We denote by $G(T)$ the graph of $\mathcal{T}$, given by

$$G(\mathcal{T}) = \Big\{(u,v) \in \mathcal{X}\times\mathcal{X}^* \ : \ v \in \mathcal{T}(u)\Big\}.$$

Definition 3.1. *The multi-values mapping $\mathcal{T}$ is called*

1. *monotone, if for each pair of elements $(u_1,u_1), (v_2,v_2)$ in $G(\mathcal{T})$, we have the inequality*

$$\langle u_1 - u_2, v_1 - v_2\rangle \geq 0.$$

2. *maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from $\mathcal{X}$ to $2^{\mathcal{X}^*}$. An equivalent version of the last clause is that for any $(u_0,v_0) \in \mathcal{X}\times\mathcal{X}^*$ for which $\langle u_0 - u, v_0 - v\rangle \geq 0$, for all $(u,v) \in G(T)$, we have $(u_0,v_0) \in G(T)$.*

Definition 3.2. *let $\mathcal{Y}$ be another real Banach space. A mapping F: $D(F) \subset \mathcal{X} \to \mathcal{Y}$ is said to be*

1. *bounded, if it takes any bounded set into a bounded set.*
2. *demicontinuous, if for each sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.*
3. *of type (S_+), if for any sequence $(u_n) \subset D(F)$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.*

In the sequel, let $L : D(L) \subset \mathcal{X} \to \mathcal{X}^*$ be a linear maximal monotone map such that $D(L)$ is dense in $\mathcal{X}$.
For each open and bounded subset G on $\mathcal{X}$, we consider the following classes of operators:

$$\mathcal{F}_G(\Omega) := \Big\{L + S : \overline{G}\cap D(L) \to \mathcal{X}^* \mid S \ \text{ is bounded, demicontinuous and of type } (S_+) \text{ with respect to } D(L) \text{ from } G \text{ to } \mathcal{X}^*\Big\},$$

$$\mathcal{H}_G := \Big\{L + S(t) \ : \ \overline{G}\cap D(L) \to \mathcal{X}^* \mid S(t) \ \text{ is a bounded homotopy of type } (S_+) \text{ with respect to } D(L) \text{ from } \overline{G} \text{ to } \mathcal{X}^*\Big\}.$$

Remark 3.3. *([6]). Remark that the class $\mathcal{H}_G$ contains all affine homotopy*

$$L + (1-t)S_1 + tS_2 \quad \textit{with} \quad (L + S_i) \in \mathcal{F}_G \quad \textit{and } i = 1, 2.$$

We give the Berkovits and Mustonen topological degree for a class of demicontinuous operator satisfying condition $(S_+)_T$ for more details see [6].

Theorem 3.4. *Let L be a linear maximal monotone densely defined map from $D(L) \subset \mathcal{X}$ to $\mathcal{X}^*$. There exists a unique degree function*

$$d : \Big\{(F, G, h) \ : \ F \in \mathcal{F}_G, \ G \textit{ an open bounded subset in } \mathcal{X}, \ h \notin F\Big(\partial G \cap D(L)\Big)\Big\} \longrightarrow \mathbb{Z},$$

which satisfies the following properties :

1. *(Normalization)* $L+J$ *is a normalising map, where J is the duality mapping of $\mathcal{X}$ into $\mathcal{X}^*$, that is, $d(L+J, G, h) = 1$, when $h \in (L+J)(G \cap D(L))$.*
2. *(Additivity) Let $F \in \mathcal{F}_G$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F\Big((\overline{G} \backslash (G_1 \cup G_2)) \cap D(L)\Big)$ then we have*
$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h) \ .$$
3. *(Homotopy invariance) If $F(t) \in \mathcal{H}_G$ and $h(t) \notin F(t)(\partial G \cap D(L))$ for every $t \in [0, 1]$, where $h(t)$ is a continuous curve in $\mathcal{X}^*$, then*
$$d(F(t), \ G, \ h(t)) = \ \textit{constant}, \qquad \forall t \in [0, 1].$$
4. *(Existence) if $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in $G \cap D(L)$.*

Lemma 3.5. *Let $L + S \in \mathcal{F}_{\mathcal{X}}$ and $h \in \mathcal{X}^*$. Suppose that there exists $R > 0$ such that*

$$\langle Lu + Su - h, u\rangle > 0, \tag{2}$$

for any $u \in \partial B_R(0) \cap D(L)$. Then

$$(L + S)(D(L)) = \mathcal{X}^*. \tag{3}$$

Proof. Let $\varepsilon > 0, \ t \in [0, 1]$ and

$$F_\varepsilon(t, u) = Lu + (1-t)Ju + t(Su + \varepsilon Ju - h).$$

As $0 \in L(0)$ and applying the boundary condition (2), we have

$$\begin{aligned}
\langle F_\varepsilon(t, u), u\rangle &= \langle t(Lu + Su - h, u\rangle + \langle(1-t)Lu + (1-t+\varepsilon)Ju, u\rangle \\
&\geq \langle(1-t)Lu + (1-t+\varepsilon)Ju, u\rangle \\
&= (1-t)\langle Lu, u\rangle + (1-t+\varepsilon)\langle Ju, u\rangle \\
&\geq (1-t+\varepsilon)\|u\|^2 = (1-t+\varepsilon)R^2 > 0.
\end{aligned}$$

Which means that $0 \notin F_\varepsilon(t, u)$. Since J and $S + \varepsilon J$ are bounded, continuous and of type (S_+), $\{F_\varepsilon(t, \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Hence, by using the normalisation and invariance under homotopy, we get

$$d(F_\varepsilon(t, \cdot), B_R(0), 0) = d(L + J, B_R(0), 0) = 1.$$

As a result, there exists $u_\varepsilon \in D(L)$ such that $0 \in F_\varepsilon(t, \cdot)$.
If we take $t = 1$ and when $\varepsilon \to 0^+$, then we have $h \in Lu + Su$ for some $u \in D(L)$. Since $h \in \mathcal{X}^*$ is arbitrary, we deduce that $(L + S)(D(L)) = \mathcal{X}^*$. $\square$

4. Basic assumptions and main result

We use the following assumptions: $a(x,t,\xi) : Q \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory, and a continuous derivative with respect to ξ of the continuous mapping $A(x,t,\xi) : Q \times \mathbb{R}^N \longrightarrow \mathbb{R}$. for all $(x,t) \in Q$ and all $\xi, \xi' \in \mathbb{R}^N$, with $(\xi \neq \xi')$.

(h_1) $\quad A(x,t,0) = 0$ and $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$,

(h_2) $\quad \alpha|\xi|^p \leq a(x,t,\xi)\cdot\xi \leq pA(x,t,\xi)$,

(h_3) $\quad |a(x,t,\xi)| \leq k(x,t) + \beta\,|\xi|^{p-1}$,

(h_4) $\quad [a(x,t,\xi) - a(x,t,\xi')]\cdot(\xi - \xi') > 0$,

where α, β are some real positive number and $k(x,t)$ is a positive function in $L^q(Q)$.

In order to obtain the existence of weak solutions, the authors always assume that the Kirchhoff function $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non-decreasing function, and satisfies the following conditions:

(M_0) there exist two positive constant m_0 and m_1, such that, $m_0 \leq \mathcal{M}(t) \leq m_1$, for all $t \in [0, +\infty[$.

Lemma 4.1. *([2]). Assume that* (h_2)-(h_4) *hold, let* $(u_n)_n$ *be a sequence in* $L^p\Big(0, T, W_0^{1,p}(\Omega)\Big)$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^p\Big(0, T, W_0^{1,p}(\Omega)\Big)$ *and*

$$\int_Q \Big[a(x,t,\nabla u_n) - a(x,t,\nabla u)\Big]\nabla(u_n - u)dx \longrightarrow 0. \tag{4}$$

Then $u_n \longrightarrow u$ *strongly in* $L^p\Big(0, T, W_0^{1,p}(\Omega)\Big)$.

Let us consider the following functional

$$\mathcal{G}(u) = \int_0^T \widehat{\mathcal{M}}\Big(\int_\Omega \Big(A(x,\nabla u) + \frac{1}{p}|\nabla u|^p\Big)dx\Big)dt \qquad \forall u \in \mathcal{H},$$

where $\widehat{\mathcal{M}} : [0,+\infty[\longrightarrow [0,+\infty[$ be the primitive of the function $\mathcal{M}$, defned by

$$\widehat{\mathcal{M}}(n) = \int_0^n \mathcal{M}(\xi)d\xi.$$

It is well known that $\mathcal{G}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u \in \mathcal{H}$ is the functional $\mathcal{G}'(u) \in \mathcal{H}^*$ setting by

$$\langle \mathcal{G}'(u), \varphi\rangle =$$
$$\int_0^T \Big\{\mathcal{M}\Big(\int_\Omega (A(x,t,\nabla u) + \frac{1}{p}|\nabla u|^p)\,dx\Big)\Big[\int_\Omega a(x,t,\nabla u)\nabla\varphi\,dx + \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\Big]\Big\}dt$$

for all $\varphi \in \mathcal{H}$.

Lemma 4.2. *Suppose that the assumption* (h_2)-(h_4) *and* (M_0) *hold, then*

- $\mathcal{G}'$ *is continuous and bounded mapping.*
- *the mapping* $\mathcal{G}'$ *is of class* (S_+).

Proof. • Given that $\mathcal{G}$ is continuously Gâteaux differentiable and whose Gâteaux derivatives at point $u \in \mathcal{H}$, is $\mathcal{G}'u \in \mathcal{H}^*$ with

$$\langle \mathcal{G}'u, \varphi \rangle = \int_0^T \Big\{ \mathcal{M}\Big(\int_\Omega (A(x,t,\nabla u) + \frac{1}{p}|\nabla u|^p)\, dx \Big) \Big[\int_\Omega a(x,t,\nabla u)\nabla\varphi \, dx + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \, dx \Big] \Big\} dt,$$

$\forall \varphi \in \mathcal{H}$.
Therefore $\mathcal{G}'$ is the Fréchet derivative of $\mathcal{G}$. So we can conclude that the operator $\mathcal{G}'$ is continuous.
Now, we prove that the operator $\mathcal{G}'$ is bounded.

$$\begin{aligned}
|\, \langle \mathcal{G}'u, \varphi \rangle \,| &= \Big| \int_0^T \Big\{ \mathcal{M}\Big(\int_\Omega (Ax,t,\nabla u) + \frac{1}{p}|\nabla u|^p) dx \Big) \\
&\qquad \times \Big(\int_\Omega a(x,t,\nabla u)\nabla\varphi \, dx + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx \Big\} dt \Big| \\
&\le \int_0^T m_1 \times \Big(\int_\Omega |a(x,t,\nabla u)| \cdot |\nabla\varphi| \, dx + \int_\Omega |\nabla u|^{p-1} \cdot |\nabla\varphi| dx \Big) dt \\
&\le m_1 \Big(\int_0^T \int_\Omega |a(x,t,\nabla u)| \cdot |\nabla\varphi| \, dxdt + \int_0^T \Big(\int_\Omega |\nabla u|^{p-1} \cdot |\nabla\varphi| dx \Big) dt \Big) \\
&\le 2m_1 \int_0^T \Big(\|a(x,t,\nabla u)\|_{L^{p'}(\Omega)} \cdot \|\nabla\varphi\|_{L^p(\Omega)} \Big) dt + \int_0^T \|\nabla u\|_{L^p(\Omega)}^{\frac{p}{p'}} \cdot \|\nabla\varphi\|_{L^p(\Omega)} dt \\
&\le C \int_0^T \Big(\|a(x,t,\nabla u)\|_{L^{p'}(\Omega)} \|\varphi\|_{1,p} dt + C \int_0^T \|u\|_{L^p(\Omega)}^{\frac{p}{p'}} \|\varphi\|_{1,p} \, dt.
\end{aligned}$$

From the growth condition (h_2), we can easily show that $\|a(x,t,\nabla u)\|_{L^{p'}(\Omega)}$ is bounded for all $u \in W_0^{1,p}(\Omega)$.
Then

$$|\langle \mathcal{G}', \varphi \rangle| \le C_1 \int_0^T \|\varphi\|_{1,p} + C_2 \int_0^T \|\varphi\|_{1,p}.$$

By the continuous embedding $\mathcal{X} \hookrightarrow L^1\Big(0,T,W^{1,p}(\Omega)\Big)$, we concludes that

$$|\langle \mathcal{G}', \varphi \rangle| \le Const \, \|\varphi\|_{\mathcal{H}}.$$

Which means that the operator $\mathcal{G}'$ is bounded.

• Next, we verify that the operator $\mathcal{G}'$ is of type (S_+).
Assume that $(u_n)_n \subset \mathcal{H}$ and

$$\begin{cases} u_n \rightharpoonup u \qquad \text{in} \quad \mathcal{H} \\ \limsup\limits_{n\to\infty} \langle \mathcal{G}'u_n, u_n - u \rangle \le 0. \end{cases} \tag{5}$$

We will show that $u_n \to u$ in $\mathcal{H}$.

On the one hand, in fact $u_n \rightharpoonup u$ in $\mathcal{H}$, so $(u_n)_n$ is a bounded sequence in $\mathcal{H}$ and since $\mathcal{H}$ embeds compactly in $L^p(Q)$, then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \to u$ in $L^p(Q)$.
On the other hand, we have

$$\limsup_{n\to\infty} \langle \mathcal{G}'u_n, u_n - u \rangle = \limsup_{n\to\infty} \langle \mathcal{G}'u_n - \mathcal{G}'u, \; u_n - u \rangle = \lim_{n\to\infty} \langle \mathcal{G}'u_n - \mathcal{G}'u, \; u_n - u \rangle \le 0. \tag{6}$$

Then

$$\lim_{n\to\infty}\Big(\int_0^T \mathcal{M}\big(\mathcal{K}(u_n)\big)\Big[\int_\Omega a(x,t,\nabla u_n)\nabla(u_n-u)dx+\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)dx\Big]dt$$
$$-\int_0^T \mathcal{M}\big(\mathcal{K}(u)\big)\Big[\int_\Omega a(x,t,\nabla u)\nabla(u_n-u)dx+\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(u_n-u)dx\Big]dt\Big)\leq 0.$$

Or by (A_1) we have for any $x\in\Omega$ and $\xi\in\mathbb{R}^n$

$$A(x,\xi)=\int_0^1 \frac{d}{ds}A(x,s\xi)ds=\int_0^1 a(x,t,s\xi)\xi ds,$$

by combining (h_3), Fubini's theorem and Young's inequality we have

$$\begin{aligned}\int_\Omega A(x,t,\nabla u)dx &= \int_\Omega\int_0^1 a(x,t,s\nabla u_n)\nabla u ds\,dx\\ &= \int_0^1 \Big[\int_\Omega a(x,t,s\nabla u_n)\nabla u dx\Big]ds\\ &\leq \int_0^1 \Big[C_{p'}\int_\Omega \big|a(x,t,s\nabla u)\big|^{p'}dx+C_p\int_\Omega |\nabla u|^p dx\Big]ds\\ &\leq C_1+C'\int_0^1\int_\Omega |s\,\nabla u|^p\,dx\,ds+C_p\|u\|^p_{L^p(\Omega)}\\ &\leq C_1+C_2\int_\Omega |\nabla u|^p\,dx+C_p\|u\|^p_{1,p}\\ &\leq C\|u\|^p_{1,p}.\end{aligned}\tag{7}$$

By (7), then $\int_\Omega (A(x,t,\nabla u_n)dx$ is bounded.

As $\mathcal{M}$ is continuous, up to a subsequence there is $k\geq 0$ by

$$\mathcal{M}\big(\mathcal{K}(u_n)\big)\longrightarrow \mathcal{M}(k)\geq m_0 \qquad \text{as} \qquad n\to\infty.\tag{8}$$

In addition by appliying the assumption (M_0), we have

$$\lim_{n\to\infty} m_0\Big(\int_0^T \Big[\int_\Omega a(x,t,\nabla u_n)\nabla(u_n-u)dx+\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)dx\Big]dt$$
$$-\int_0^T \Big[\int_\Omega a(x,t,\nabla u)\nabla(u_n-u)dx+\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(u_n-u)dx\Big]dt\Big)\leq 0.$$

Using the compact embedding $\mathcal{H}\hookrightarrow\hookrightarrow L^p(Q)$, we have
$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n-\nabla u)dx=0.$ and $\lim_{n\to\infty}\int_\Omega |\nabla u|^{p-2}\nabla u(\nabla u_n-\nabla u)dx=0.$
Since $m_0\geq 0$ then, we have

$$\lim_{n\to\infty}\Big(\int_0^T\int_\Omega a(x,t,\nabla u_n)\nabla(u_n-u)dxdt-\int_0^T\int_\Omega a(x,t,\nabla u)\nabla(u_n-u)dxdt\Big)\leq 0$$

which means

$$\lim_{n\to\infty}\int_0^T\int_\Omega \Big[a(x,t,\nabla u_n)-a(x,t,\nabla u)\Big](\nabla u_n-\nabla u)dxdt\leq 0.\tag{9}$$

By combining (9) and (h_4), we deduce that

$$\lim_{n\to\infty}\int_0^T\int_\Omega\Big[a(x,t,\nabla u_n)-a(x,t,\nabla u)\Big](\nabla u_n-\nabla u)dxdt=0.$$

In light of Lemma 4.1, we obtain

$$u_n \longrightarrow u \qquad \text{in } \mathcal{H},$$

which implies that $\mathcal{G}'$ is of type (S_+). □

Let us consider the following operator L defined from the subset $D(L)$ of $\mathcal{H}$ into its dual $\mathcal{H}^*$, such that

$$D(L)=\Big\{\varphi\in\mathcal{H}\ :\ \varphi'\in\mathcal{H}^*,\ \varphi(0)=0\Big\},$$

by

$$\langle Lu,\varphi\rangle=-\int_Q u\varphi_t\,dxdt,\qquad \text{for all } u\in D(L),\ \ \varphi\in\mathcal{H}.$$

Consequently, the operator L is generated by $\partial/\partial t$ by means of the relation

$$\langle Lu,\varphi\rangle=\int_0^T\langle u'(t),\varphi(t)\rangle dt,\ \ \text{for all } u\in D(L),\ \varphi\in\mathcal{H}.$$

Lemma 4.3. *([29]). L is a linear maximal monotone densely defined map.*

Our main result is the following existence theorem:

Theorem 4.4. *Let $f\in\mathcal{H}^*$ and $u_0\in L^2(\Omega)$, there exists at least one weak solution $u\in D(L)$ of problem* (1) *in the following sense*

$$\begin{aligned}&-\int_Q u\varphi_t\,dxdt+\int_0^T\Big\{\mathcal{M}(\mathcal{K}(u))\Big[\int_\Omega a(x,t,\nabla u)\nabla\varphi|dx+\int_\Omega|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\Big]\Big\}dt\\&=\int_0^T\langle f,\varphi\rangle\,dt.\end{aligned}$$

Proof. On the one hand, from the Lemma 4.3, the operator

$$L:D(L)\subset\mathcal{H}\longrightarrow\mathcal{H},$$

$$\langle Lu,\varphi\rangle_{\mathcal{H}}=\int_0^T\langle u'(t),\varphi(t)\rangle dt,\ \ \text{for all } u\in D(L),\ \varphi\in\mathcal{H}.$$

is a densely defined maximal monotone operator.
By the monotonicity of L we have

$$\langle Lu,u\rangle\geq 0\ \ \text{for all } u\in D(L)\,,$$

then we obtain

$$\begin{aligned}\langle Lu+\mathcal{G}'u,u\rangle&\geq\langle\mathcal{G}'u,u\rangle\\&=\int_0^T\Big\{\mathcal{M}\Big(\int_\Omega A(x,t,\nabla u)+\tfrac{1}{p}|\nabla u|^p\Big)dx\Big)\int_\Omega a(x,t,\nabla u)\nabla u dx+\int_\Omega|\nabla u|^p\,dx\Big\}dt,\end{aligned}$$

by using th assumptions (h_2) and (M_0), we get

$$\begin{aligned}\langle Lu + \mathcal{G}'u, u\rangle &\geq \int_0^T m_0 \int_\Omega a(x,t,\nabla u)\nabla u dxdt + \int_0^T m_0 \int_\Omega |\nabla u|^p \, dxdt \\ &\geq m_0 \int_Q a(x,t,\nabla u)\nabla u dxdt + m_0 \int_Q |\nabla u|^p \, dxdt \\ &\geq m_0\alpha \int_Q |\nabla u|^p dxdt + m_0 \int_Q |\nabla u|^p \, dxdt \\ &\geq C_{min}\|u\|^p_{\mathcal{H}},\end{aligned} \tag{10}$$

for all $u \in \mathcal{H}$.
Since the right side of the above inequality (10) tends to ∞ as $\|u\|_{\mathcal{H}} \to \infty$, then for each $f \in \mathcal{G}^*$ there exists $R = R(f)$ such that

$$\langle Lu + \mathcal{G}'u - f, u\rangle > 0, \tag{11}$$

for all $u \in B_R(0) \cap D(L)$.
By appliying Lemma 3.5, we infer that the equation

$$Lu + \mathcal{G}'u = f,$$

is solvable in $D(L)$.
Which implies that the problem $(\mathcal{P})$ admits at least one-weak solution. This ends the proof. □

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