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On $\mathbb{Z}_2\mathbb{Z}_2[u]\mathbb{Z}_2[u,v]$-additive cyclic codes and their application in obtaining optimal codes

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Abstract. Let $\mathbb{Z}_2 = \{0,1\}$, $\mathfrak{R}_1 = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\mathfrak{R}_2 = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2$, where $u^2 = v^2 = 0 = uv = vu$. In this article, we study $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive cyclic, additive constacyclic and additive dual codes and find the structural properties of these codes. The additive cyclic codes are characterized as $\mathfrak{R}_2[y]$-submodules of a ring

$$\mathcal{S}_{\beta_1,\beta_2,\beta_3} = \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \mathfrak{R}_1[y]/\langle y^{\beta_2} - 1\rangle \times \mathfrak{R}_2[y]/\langle y^{\beta_3} - 1\rangle.$$

The extended Gray map is represented by

$$\Psi_1 : \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}_1^{\beta_2} \times \mathfrak{R}_2^{\beta_3} \longrightarrow \mathbb{Z}_2^{\beta_1+2\beta_2+4\beta_3}$$

and is utilized to construct the binary codes with good parameters. The minimal generating polynomials and smallest spanning sets of the above specified codes are obtained. We also establish the relationship between the minimal polynomials of additive cyclic codes and their duals. Further, we provide some examples that support our main results. Finally, the optimal binary codes are determined in Table.

1. Introduction

In the early history of the art of error-correcting codes, the codes had been studied over finite fields, but with the time, more general structures have been considered and implemented. The study of codes over rings has attracted many researchers.

In 1973, additive codes were first defined by Delsarte [13, 14] in terms of association schemes. Generally, an additive code is defined as a subgroup of the underlying abelian group. In the special case of a binary Hamming scheme, when the underlying abelian group is of order 2^n, the structure for the abelian groups are those which are of the form $\mathbb{Z}_2^{\beta_1} \times \mathbb{Z}_4^{\beta_2}$ with $\beta_1 + 2\beta_2 = n$. Therefore, the subgroup C of $\mathbb{Z}_2^{\beta_1} \times \mathbb{Z}_4^{\beta_2}$ is the only additive code in a binary Hamming scheme.

In 2015, $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes were another generalization of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes which was introduced by Aydogdu et al. [4]. In this article, they determined the $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive cyclic codes and defined a mixed code consisting of the binary part and non-binary part from the ring $\mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$. The

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$\mathbb{Z}_2\mathbb{Z}_4$-additive codes were further generalized to $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes by Aydogdu et al. [7]. Later on, Aydogdu et al. [8] generalized $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes to $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. In 2016, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes were studied in[10].

Note that in $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes, $\mathbb{Z}_2$ is considered as $\mathbb{Z}_4$-algebra and $\mathbb{Z}_{2^s}$-algebra respectively. Also in $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive code, $\mathbb{Z}_2$ is known as a $\mathbb{Z}_2[u]$-algebra and $\mathbb{Z}_{p^r}$ is a $\mathbb{Z}_{p^s}$-algebra in $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. In 2018, J. Gao et al. [15] gave the structural properties of additive cyclic codes over $\mathbb{Z}_p\mathbb{Z}_p[u]$, where $u^2 = 0$. They also found the minimal generating sets of additive cyclic codes. Moreover, they determined the relationship of generators between the additive codes and its dual code.

In 2019, Minjia Shi et al.[22] described $\mathbb{Z}_2\mathbb{Z}_2[u,v]$-additive cyclic codes, where $u^2 = v^2 = 0,\ uv = vu$, which was the generalization of previously introduced $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. Recently, Mahmoudi et al.[21] gave the structures of $\mathbb{Z}_2(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, where $u^3 = 0$, $\mathbb{Z}_2(\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2)$ and$(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, where $u^2 = v^2 = uv = uv = 0$. They determined additive codes, dual additive codes and found singleton bound.

Inspired by aforementioned work, we consider two rings $\Re_1 = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\Re_2 = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2$, where $u^2 = v^2 = 0 = uv = vu$ with characteristic 2. In this article, we determine $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes, constacyclic codes and dual of additive cyclic codes. We also find the optimal binary images from $\mathbb{Z}_2\Re_1\Re_2$-additive codes. It is to be noted that the additive code of length $(\beta_1, \beta_2, \beta_3)$ is the subgroup of the commutative group $\mathbb{Z}_2^{\beta_1}\times\Re_1^{\beta_2}\times\Re_2^{\beta_3}$. The $\mathbb{Z}_2\Re_1\Re_2$-additive code is a linear code over $\mathbb{Z}_2$ if $\beta_2 = 0$ and $\beta_3 = 0$, over $\Re_1$ if $\beta_1 = 0$ and $\beta_3 = 0$ and over $\Re_2$ if $\beta_1 = 0$ and $\beta_2 = 0$. Clearly, we observe that it is the generalization of linear code over $\mathbb{Z}_2$, $\Re_1$ and $\Re_2$. Furthermore, we obtain the generator polynomials and minimal spanning sets for $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic, additive constacyclic and additive dual codes. These codes are classified as $\Re_2[y]$-submodules of the ring $\mathcal{S}_{\beta_1,\beta_2,\beta_3} = \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \Re_1[y]/\langle y^{\beta_2} - 1\rangle \times \Re_2[y]/\langle y^{\beta_3} - 1\rangle$.

This paper is organized as follows: In Section 2, we present some basic notions and define some Gray maps and their extensions. Section 3 contains the cyclic structures of the rings $\Re_1 = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\Re_2 = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2$, where $u^2 = v^2 = 0 = uv = vu$. In Section 4, we study $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes and find the minimal generating sets when β_2 is odd and β_3 is even(or odd). In Section 5, we define the duality of $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes and their results . In Section 6, we determine the minimal spanning sets of $\mathbb{Z}_2\Re_1\Re_2$-additive constacyclic codes. In section 7, we give some examples and form a table of optimal binary images from $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes by MAGMA. Last section contains the conclusion and the suggestion for future work.

2. Preliminaries

Let $\Re_1 = \mathbb{Z}_2 + u\mathbb{Z}_2$, $u^2 = 0$ and $\Re_2 = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2$, where $u^2 = 0$, $v^2 = 0 = uv = vu$ be two rings with characteristic 2. Any element $z \in \Re_2$ can be written as $z = a + ub + vc$ for all $a, b, c \in \mathbb{Z}_2$. An element $z = a+ub+vc \in \Re_2$ is a unit if a is unit. The total number of ideals in $\Re_2$ are listed as $I_1 = \{0\}$, $I_2 = \langle u\rangle$, $I_3 = \langle v\rangle$, $I_4 = \langle u + v\rangle$ and $I_5 = \langle u, v\rangle$. Since $I_5 = \langle u, v\rangle$ is the unique maximal ideal in $\Re_2$, the finite commutative ring $\Re_2$ is a local ring. Let

$$\mathbb{Z}_2\Re_1\Re_2 = \{(c, c^{'}, c^{''}) \mid c \in \mathbb{Z}_2,\ c^{'} \in \Re_1,\ c^{''} \in \Re_2\}.$$

Define three maps $\theta_1 : \Re_2 \longrightarrow \mathbb{Z}_2$, $\theta_2 : \Re_2 \longrightarrow \Re_1$ and $\theta_3 : \Re_1 \longrightarrow \mathbb{Z}_2$ such that $\theta_1(a + ub + vc) = a$, $\theta_2(a + ub + vc) = a + ub$ and $\theta_3(a + ub) = a$, respectively. Clearly, θ_1, θ_2 and θ_3 are well-defined onto ring homomorphisms. Let $\mathbb{Z}_2^{\beta_1}$ be a β_1-tuples of $\mathbb{Z}_2$, $\Re_1^{\beta_2}$ be a β_2-tuples of $\Re_1$ and $\Re_2^{\beta_3}$ be a β_3-tuples of $\Re_2$, where β_1, β_2 and β_3 are positive integers. Let $\mathbf{y} = (y, y^{'}, y^{''}) \in \mathbb{Z}_2^{\beta_1}\times\Re_1^{\beta_2}\times\Re_2^{\beta_3}$ be a vector, where $y = (y_0, y_1, \ldots, y_{\beta_1-1})$, $y^{'} = (y^{'}_0, y^{'}_1, \ldots, y^{'}_{\beta_2-1})$ and $y^{''} = (y^{''}_0, y^{''}_1, \ldots, y^{''}_{\beta_3-1})$. For any $z = a + ub + vc \in \Re_2$, the $\Re_2$-scalar multiplication on $\mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$ is defined as follows:

$$z\mathbf{y} = (\theta_1(z)y_0, \ldots, \theta_1(z)y_{\beta_1-1}|\theta_2(z)y^{'}_0, \ldots, \theta_2(z)y^{'}_{\beta_2-1}|zy^{''}_0, \ldots, zy^{''}_{\beta_3-1}) \in \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}, \qquad (1)$$

where $\theta_1(z)y_i$, $\theta_2(z)y^{'}_j$ and $zy^{''}_k$ are performed mod2 for all $0\leq i \leq \beta_1 - 1$, $0 \leq j \leq \beta_2 - 1$ and $0 \leq k \leq \beta_3 - 1$. The structure $\mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$ forms a $\Re_2$-module under usual addition and multiplication defined in (2.1).

Let $\mathcal{S}_{\beta_1,\beta_2,\beta_3} = \mathbb{Z}_2[y]/\langle y^{\beta_1}-1\rangle \times \Re_1[y]/\langle y^{\beta_2}-1\rangle \times \Re_2[y]/\langle y^{\beta_3}-1\rangle$. Define a map

$$\Phi : \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3} \longrightarrow \mathcal{S}_{\beta_1,\beta_2,\beta_3}$$
$$d = (f|g|h) \longmapsto d(y) = (f(y)|g(y)|h(y)),$$

where $(f|g|h) = (f_0, f_1, \ldots, f_{\beta_1-1}|g_0, g_1, \ldots, g_{\beta_2-1}|h_0, h_1, \ldots, h_{\beta_3-1})$,
$f(y) = f_0 + f_1y + \cdots + f_{\beta_1-1}y^{\beta_1-1}$, $g(y)$=$g_0 + g_1y + \cdots + g_{\beta_2-1}y^{\beta_2-1}$ and
$h(y)$=$h_0+h_1y+\cdots+h_{\beta_3-1}y^{\beta_3-1}$. For any $\ell(y) = \ell_0+\ell_1y+\cdots+\ell_ry^r \in \Re_2[y]$ and $d(y) = (f(y)|g(y)|h(y)) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$, define the $\Re_2[y]$-scalar multiplication

$$\ell(y) \cdot d(y) = (\theta_1(\ell(y))f(y)|\theta_2(\ell(y))g(y)|\ell(y)h(y)), \tag{2}$$

where $\theta_1(\ell(y)) = \theta_1(\ell_0) + \theta_1(\ell_1)y + \cdots + \theta_1(\ell_r)y^r$ and
$\theta_2(\ell(y)) = \theta_2(\ell_0) + \theta_2(\ell_1)y + \cdots + \theta_2(\ell_r)y^r$. Then $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ forms a $\Re_2[y]$-module under usual addition and scalar multiplication of polynomials defined in (2.2).

Definition 2.1. *A non-empty subset C of $\mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$ is called a $\mathbb{Z}_2\Re_1\Re_2$-additive code if C is a subgroup of $\mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$, that is, C is isomorphic to $\mathbb{Z}_2^{n_1} \times \mathbb{Z}_2^{2n_2} \times \mathbb{Z}_2^{n_3} \times \mathbb{Z}_2^{3n_4} \times \mathbb{Z}_2^{2n_5} \times \mathbb{Z}_2^{n_6}$, for some positive integers n_1, n_2, n_3, n_4, n_5 and n_6.*

If C is a $\mathbb{Z}_2\Re_1\Re_2$-additive code, then C is of the type $(\beta_1, \beta_2, \beta_3, n_1, n_2, n_3, n_4, n_5, n_6)$. Let for any $\mathbf{z}_1 = (a_0, a_1, \ldots, a_{\beta_1-1}|b_0, b_1, \ldots, b_{\beta_2-1}|c_0, c_1, \ldots, c_{\beta_3-1})$ and
$\mathbf{z}_2 = (a_0^{'}, a_1^{'}, \ldots, a_{\beta_1-1}^{'}|b_0^{'}, b_1^{'}, \ldots, b_{\beta_2-1}^{'}|c_0^{'}, c_1^{'}, \ldots, c_{\beta_3-1}^{'})$, the inner product is defined as

$$\mathbf{z}_1 \cdot \mathbf{z}_2 \quad = \quad (v\sum_{i=0}^{\beta_1-1} a_ia_i^{'} + u\sum_{j=0}^{\beta_2-1} b_jb_j^{'} + \sum_{k=0}^{\beta_3-1} c_kc_k^{'})(\text{mod}2).$$

Definition 2.2. *A non-empty subset C of $\mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$ is called a $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code if*

(*i*) *C is additive code.*

(*ii*) *For any codeword $z = (a_0, a_1, \ldots, a_{\beta_1-1}|b_0, b_1, \ldots, b_{\beta_2-1}|c_0, c_1, \ldots, c_{\beta_3-1}) \in C$ its cyclic shift $T(z)$=$(a_{\beta_1-1}, a_0, \ldots, a_{\beta_1-2}|b_{\beta_2-1}, b_0, \ldots, b_{\beta_2-2}|c_{\beta_3-1}, c_0, \ldots, c_{\beta_3-2}) \in C$.*

Definition 2.3. *Let C be any $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code. Then the dual code of C with respect to the inner product is defined as*

$$C^{\perp} = \{z_2 \in \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3} |\, z_1 \cdot z_2 = 0 \textit{ for all } z_1 \in C\}.$$

Let us define Gray maps as follows:

$$\phi_1 : \Re_1 \longrightarrow \mathbb{Z}_2^2 \tag{3}$$

such that $\phi_1(e + uf) = (f, e + f)$ for all $e, f \in \mathbb{Z}_2$ and

$$\phi_2 : \Re_2 \longrightarrow \mathbb{Z}_2^4 \tag{4}$$

such that $\phi_2(a + ub + vc) = (a + b + c, a + c, b + c, c)$ for all $a, b, c, \in \mathbb{Z}_2$. Using (2.3) and (2.4), we can define the another Gray map

$$\Psi : \mathbb{Z}_2 \times \Re_1 \times \Re_2 :\longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_2^4 \tag{5}$$

as $\Psi(c|c^{'}|c^{''}) = (c, \phi_1(c^{'}), \phi_2(c^{''}))$. An extension of the map Ψ in (2.5) is defined by

$$\Psi_1 : \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3} :\longrightarrow \mathbb{Z}_2^{\beta_1+2\beta_2+4\beta_3} \tag{6}$$

such that $\Psi_1(\mathbf{y}$=$(y|y^{'}|y^{''}))$=$(y, \phi_1(y^{'}), \phi_2(y^{''}))$, where
$\mathbf{y}$=$(y_0, y_1, \ldots, y_{\beta_1-1}|y_0^{'}, y_1^{'}, \ldots, y_{\beta_2-1}^{'}|y_0^{''}, y_1^{''}, \ldots, y_{\beta_3-1}^{''}) \in \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$.

Definition 2.4. *Let* $\boldsymbol{y}=(y|y^{'}|y^{''}) \in \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$, *where* $y \in \mathbb{Z}_2^{\beta_1}$, $y^{'} \in \Re_1^{\beta_2}$ *and* $y^{''} \in \Re_2^{\beta_3}$. *Then the Gray weight of* $\boldsymbol{y}$ *is defined as*

$$w_G(\boldsymbol{y}) = w_H(y) + w_H(\phi_1(y^{'})) + w_H(\phi_2(y^{''})),$$

where w_H *denotes the Hamming weight.*

Definition 2.5. *Let* $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_2}$. *Then the Gray distance between* $\boldsymbol{y}$ *and* $\boldsymbol{z}$ *is defined as*

$$d_G(\boldsymbol{y}, \boldsymbol{z}) = w_G(\boldsymbol{y} - \boldsymbol{z}) = d_H((y|\phi_1(y^{'})|\phi_2(y^{''})), (z|\phi_1(z^{'})|\phi_2(z^{''}))).$$

3. Cyclic structure of codes over $\Re_1$ and $\Re_2$

In this section, we study the generating sets of cyclic codes over $\Re_1$ and $\Re_2$. We will use these structures to construct the composition $\mathbb{Z}_2\Re_1\Re_2$.

Lemma 3.1. *A code C of length* β_2 *over* $\Re_1$ *is cyclic code if and only if C is an* $\Re_1$*-submodule of* $\Re_{1\beta_2} = \Re_1[y]/\langle y^{\beta_2} - 1\rangle$.

Definition 3.2. *(Division Algorithm) If for any* $f(y), g(y) \in \Re_1$, *where* $g(y)$ *has unit as its leading coefficient, then*

$$f(y) = q(y)g(y) + r(y),$$

for some $q(y)$, $r(y) \in \Re_1$, *where* $r(y) = 0$ *or* $deg(r(y)) < deg(g(y))$.

Let C_1 be a cyclic code in $\Re_{1\beta_2}$. We can define a map $\theta_3 : \Re_1 \longrightarrow \mathbb{Z}_2$ by $\theta_3(a + ub) = a$. Clearly, θ_3 is a ring homomorphism in $\Re_1$. The extension of θ_3 can be expressed by

$$\eta_1 : C_1 \longrightarrow \Re_1[y]/\langle y^{\beta_2} - 1\rangle$$

such that $\eta_1(a_0 + a_1 y + \cdots + a_{\beta_2 - 1} y^{\beta_2 - 1}) = \theta_3(a_0) + \theta_3(a_1)y + \cdots + \theta_3(a_{\beta_2 - 1})y^{\beta_2 - 1}$. Now, we can easily obtain the kernel of η_1 as

$$ker(\eta_1) = \{ub(y) \mid b(y) \in \mathbb{Z}_2[y]/\langle y^{\beta_2} - 1\rangle\} = \langle ub_1(y)\rangle,$$

where $b_1(y)|(y^{\beta_2} - 1)(\text{mod}2)$. Since the image of η_1 is also an ideal in $\mathbb{Z}_2[y]/\langle y^{\beta_2} - 1\rangle\}$, a binary cyclic code is generated by $f(y)$ with $f(y)|y^{\beta_2} - 1$. So $C_1 = \langle f(y) + up(y), ub_1(y)\rangle$, for some binary polynomial $p(y)$ and $b_1(y)|p(y)\frac{y^{\beta_2}-1}{f(y)}$. Obviously, $uf(y) \in ker(\eta_1)$. This implies that $b_1(y)|f(y)$.

Lemma 3.3. *Let* C_1 *be a cyclic code in* $\Re_{1\beta_2} = \Re_1[y]/\langle y^{\beta_2} - 1\rangle$.

(1) *If* β_2 *is odd, then* $\Re_{1\beta_2}$ *is principal ideal ring and*

$$C_1 = \langle f(y), ub_1(y)\rangle = \langle f(y) + ub_1(y)\rangle,$$

where $f(y)$, $b_1(y) \in \mathbb{Z}_2[y]/(y^{\beta_2} - 1)$ *and* $b_1(y)|f(y)|(y^{\beta_2} - 1)$.

(2) *If* β_2 *is not odd, then*

- *(i)* $C_1 = \langle f(y) + up(y)\rangle$, *where* $f(y)|(y^{\beta_2} - 1)(\text{mod}2)$ *and* $f(y) + up(y)|(y^{\beta_2} - 1)$ *in* $\Re_1$.
- *(ii)* $C_1 = \langle f(y) + up(y), ub_1(y)\rangle$, *where* $f(y)$, $b_1(y)$ *and* $p(y)$ *are binary polynomials such that* $b_1(y)|f(y)|(y^{\beta_2} - 1)(\text{mod}2)$, $b_1(y)|p(y)\frac{y^{\beta_2}-1}{f(y)}$ *and* $deg(b_1(y)) > deg(p(y))$.

Proof. The proof directly follows from [1, Lemma 2]. □

Let C_2 be a cyclic code in $\Re_{2\beta_3} = \Re_2[y]/\langle y^{\beta_3} - 1\rangle$. We can define a map
$\theta_2 : \Re_2 \longrightarrow \mathbb{Z}_2 + u\mathbb{Z}_2$ such that $\theta_2(a + ub + vc) = a + ub$. Clearly, θ_2 is a ring homomorphism in $\Re_2$. The extension of θ_2 can be expressed by

$$\sigma_1 : C_2 \longrightarrow (\mathbb{Z}_2 + u\mathbb{Z}_2)[y]/\langle y^{\beta_3} - 1\rangle$$

such that $\sigma_1(\ell_0 + \ell_1 y + \cdots + \ell_{\beta_3-1}y^{\beta_3-1}) = \theta_2(\ell_0) + \theta_2(\ell_1)y + \cdots + \theta_2(\ell_{\beta_3-1})y^{\beta_3-1}$, where $\ell_i = a_i + ub_i + vc_i$, for $i = 0, 1, \ldots, \beta_3 - 1$ and $a_i, b_i, c_i \in \mathbb{Z}_2$. It is easy to obtain the kernel of σ_1 as $ker(\sigma_1) = \{v(a(y)) \mid a(y), \in (\mathbb{Z}_2 + u\mathbb{Z}_2)[y]/\langle y^{\beta_3} - 1\rangle\}$=$v\mathbb{Z}_2$. Since the image of σ_1 is also an ideal in $(\mathbb{Z}_2 + u\mathbb{Z}_2)[y]/\langle y^{\beta_3} - 1\rangle\}$, a cyclic code is generated by $g(y)$ with $g(y)|y^{\beta_3} - 1$. Hence, $C_2 = \langle g(y) + up_1(y) + vq_1(y), ua_1(y) + vq_2(y), va_2(y)\rangle$, where $a_i \mid g \mid (y^{\beta_3} - 1)$ for $1 \leq i \leq 2$.

Lemma 3.4. *[17, Theorem 2] Let C_2 be a cyclic code in $\Re_{2\beta_3} = \Re_2[y]/(y^{\beta_3} - 1)$.*

(1) *If β_3 is odd, then $C_2 = \langle g(y) + ua_1(y) + va_2(y)\rangle$, where $a_2(y)|a_1(y)|g(y)|(y^{\beta_3} - 1)$.*

(2) *If β_3 is not odd, then*
$$C_2 = \langle g(y) + up_1(y) + vq_1(y), ua_1(y) + vq_2(y), va_2(y)\rangle,$$
where $a_i \mid g \mid (y^{\beta_3} - 1)$ for $i = 1, 2$ and $a_1(y)|p_1(y)\frac{y^{\beta_3}-1}{g(y)}$ and $a_2(y)|q_2(y)\frac{y^{\beta_3}-1}{a_1(y)}$.

4. $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes

In this section, we obtain a set of generators for $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes as $\Re_2[y]$-submodules of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$. Here, C will always denote a $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code. Since C and $\Re_2[y]/\langle y^{\beta_3} - 1\rangle$ are $\Re_2[y]$-submodules of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$, we define a map

$$\eta : C \longrightarrow \Re_2[y]/\langle y^{\beta_3} - 1\rangle,$$

by $\eta(f(y)|g(y)|h(y)) = h(y)$. Clearly, η is a module homomorphism whose image is $\Re_2[y]$-submodule in $\Re_2[y]/\langle y^{\beta_3} - 1\rangle$ and $ker(\eta)$ is a submodule of C. Further, $\eta(C)$ can easily be identified as an ideal in the ring $\Re_2[y]/\langle y^{\beta_3} - 1\rangle$. Since $\eta(C)$ is an ideal in $\Re_2[y]/\langle y^{\beta_3} - 1\rangle$, $\eta(C) = \langle h(y) + up_1(y) + vq_1(y), ua_1(y) + vq_2(y), va_2(y)\rangle$ with $a_i \mid g_i \mid (y^{\beta_3} - 1)(\text{mod}2)$, for $i = 1, 2$. Let us define

$$ker(\eta) = \{(f(y)|g(y)|0) \in C|f(y) \in \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle, g(y) \in \Re_1[y]/\langle y^{\beta_2} - 1\rangle\},$$

$$J = \{(f(y), g(y)) \in \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \Re_1[y]/\langle y^{\beta_2} - 1\rangle|(f(y)|g(y)|0) \in ker(\eta)\}.$$

It is clear that J is an ideal in the ring $\mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \Re_1[y]/(y^{\beta_2} - 1)$ and hence a cyclic code. Therefore, by the well-known result on generators of binary cyclic codes, we have $J = \langle f(y), g(y)\rangle$. Now, for any element $(f_1(y)|g_1(y)|0) \in ker(\eta)$, we get $(f_1(y), g_1(y)) \in J = \langle f(y), g(y)\rangle$ and it can be written as $(f_1(y), g_1(y)) = m_1(y)(f(y), g(y))$ for some polynomial $m_1(y) \in \Re_1[y]/(y^{\beta_2} - 1)$. Thus, $(f_1(y), g_1(y), 0) = (\theta_3(m_1(y))f(y), m_1(y)g(y), 0)$. This implies that $ker(\eta)$ is a submodule of C generated by an element of the form $(f(y), g(y), 0)$, where $f(y)|(y^{\beta_1} - 1)\text{mod}2$ and $g(y)|(y^{\beta_2} - 1)\text{mod}2$. By the first isomorphism theorem for rings, we have

$$\frac{C}{ker(\eta)} \cong \langle h(y) + up_1(y) + vq_1(y), ua_1(y) + vq_2(y), va_2(y)\rangle.$$

This implies that any $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code can be generated as a $\Re_2[y]$-submodule of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ by $(f_1(y)|0|0)$, $(f_2(y)|g(y)+up_1(y)|0)$, $(f_3(y)|\ell_1(y)|h(y)+up_1(y)+vq_1(y))$, $(f_4(y)|\ell_2(y)|ua_1(y)+vq_2(y))$ and $(f_5|\ell_3(y)|va_2(y))$. Hence, any element in C can be expressed as

$$d_1(y) \times (f_1(y)|0|0) + d_2(y) \times (f_2(y)|g(y) + up_1(y)|0) + d_3(y) \times (f_3(y)|\ell_1(y)|h(y)+ up_2(y) + vq_1(y)) + d_4(y) \times (f_4(y)|\ell_2(y)|ua_1(y) + vq_2(y)) + d_5(y) \times (f_5|\ell_3(y)|va_2(y)),$$

where $d_1(y)$, $d_2(y)$, $d_3(y)$, $d_4(y)$ and $d_5(y)$ are polynomials in the ring $\Re_2[y]$.

Theorem 4.1. *Let C be any $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code. Then $C^{\perp}$ is also cyclic.*

Proof. Let C be any $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code and

$$z_2=(a_0^{'},a_1^{'},\ldots,a_{\beta_1-1}^{'}|b_0^{'},b_1^{'},\ldots,b_{\beta_2-1}^{'}|c_0^{'},c_1^{'},\ldots,c_{\beta_3-1}^{'})\in C^{\perp}.$$

We have to prove that $z_1\cdot T(z_2)=0$. Since C is cyclic, we have $T^{\ell}(z_1)$ also in C, where $\ell=\operatorname{lcm}(\beta_1,\beta_2,\beta_3)$. Now, we can write

$$\begin{aligned}
0 &= T^{\ell-1}(z_1)\cdot z_2\\
&= (a_1,\ldots,a_{\beta_1-1},a_0|b_1,\ldots,b_{\beta_2-1},b_0|c_1,\ldots,c_{\beta_3-1},c_0)\cdot(a_0^{'},a_1^{'},\ldots,a_{\beta_1-1}^{'}|b_0^{'},b_1^{'},\ldots,b_{\beta_2-1}^{'}|\\
&\quad (c_0^{'},c_1^{'},\ldots,c_{\beta_3-1}^{'})\\
&= u(a_1a_0^{'}+a_2a_1^{'}+\cdots+a_0a_{\beta_1-1}^{'})+v(b_1b_0^{'}+b_2b_1^{'}+\cdots+b_0b_{\beta_2-1}^{'})+(c_1c_0^{'}+c_2c_1^{'}\\
&\quad +\cdots+c_0c_{\beta_3-1}^{'})\\
&= u(a_0a_{\beta_1-1}^{'}+a_1a_0^{'}+\cdots+a_{\beta_1-1}a_{\beta_1-2}^{'})+v(b_0b_{\beta_2-1}^{'}+b_1b_0^{'}+\cdots+b_{\beta_2-1}b_{\beta_2-2}^{'})+\\
&\quad (c_0c_{\beta_3-1}^{'}+c_1c_0^{'}+\cdots+c_{\beta_3-1}c_{\beta_3-2}^{'})\\
0 &= z_1\cdot T(z_2).
\end{aligned}$$

This implies that $T(z_2)\in C^{\perp}$. Hence, $C^{\perp}$ is $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code. □

Definition 4.2. *A subset $C\subseteq\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ is called a $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code if and only if C is a subgroup of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ and for all*

$$\begin{aligned}
d(y) &= (f(y)|g(y)|h(y))\\
&= (f_0+f_1y+\cdots+f_{\beta_1-1}y^{\beta_1-1}|g_0+g_1y+\cdots+g_{\beta_2-1}y^{\beta_2-1}|h_0+h_1y\\
&\quad +\cdots+h_{\beta_3-1}y^{\beta_3-1})\in C,
\end{aligned}$$

we have

$$\begin{aligned}
y\cdot d(y) &= (f_{\beta_1-1}+f_0y+\cdots+f_{\beta_1-2}y^{\beta_1-1}|g_{\beta_2-1}+g_0y+\cdots+g_{\beta_2-2}y^{\beta_2-1}|h_{\beta_3-1}\\
&\quad +h_0y+\cdots+h_{\beta_3-2}y^{\beta_3-1})\in C.
\end{aligned}$$

Theorem 4.3. *A code C is a $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code if and only if C is a $\Re_2[y]$-submodule of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$.*

Proof. Let C be a $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code. Then we have to show that for any $d(y)\in C$ and $\ell(y)\in\Re_2[y]$, $\ell(y)d(y)\in C$. Assume that $d(y)=(f(y)|g(y)|h(y))\in C$, where $f(y)=(f_0+f_1y+\cdots+f_{\beta_1-1}y^{\beta_1-1})$, $g(y)=(g_0+g_1y+\cdots+g_{\beta_2-1}y^{\beta_2-1})$ and $h(y)=(h_0+h_1y+\cdots+h_{\beta_3-1}y^{\beta_3-1})$. The multiplication

$$\begin{aligned}
y\cdot d(y) &= (f_{\beta_1-1}+f_0y+\cdots+f_{\beta_1-2}y^{\beta_1-1}|g_{\beta_2-1}+g_0y+\cdots+g_{\beta_2-2}y^{\beta_2-1}|h_{\beta_3-1}\\
&\quad +h_0y+\cdots+h_{\beta_3-2}y^{\beta_3-1})
\end{aligned}$$

represents the cyclic shift $T(d(y))$ of $d(y)$. Since C is $\mathbb{Z}_2\Re_1\Re_2$- additive cyclic code, $y^id(y)\in C$ for all $i\in\mathbb{N}$. It follows that $l(y)\cdot d(y)\in C$. This implies that C is $\Re_2[y]$- submodule of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$. Converse part is directly followed by the Definition 4.1. □

Theorem 4.4. *Let*

$$C=\left\langle\begin{array}{c}(f_1(y)|0|0),(f_2(y)|g(y)+up_1(y)|0),\\ (f_3(y)|\ell_1(y)|h(y)+up_1(y)+vq_1(y)),(f_4(y)|\ell_2(y)|ua_1(y)+vq_2(y)),\\ (f_5(y)|\ell_3(y)|va_2(y))\end{array}\right\rangle$$

be a $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code. Then $deg(f_i(y))<deg(f_1)$, where $2\leq i\leq 5$ and $deg(\ell_j(y))<deg(\ell_1(y))$, where $j=2,3$.

Proof. Let $deg(f_i(y)) \geq deg(f(y))$ for $2 \leq i \leq 5$. Then we can assume that $deg(f_i(y)) - deg(f(y)) = t$. In particular $i = 2$, we define a code with generators of the form

$$\begin{aligned} C^{'} &= \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y) + up_1(y)|0) + y^t \cdot (f_1(y)|0|0), \\ (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)), (f_4(y)|\ell_2(y)|ua_1(y) + vq_2(y)), \\ (f_5(y)|\ell_3(y)|va_2(y)) \end{array} \right\rangle \\ &= \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y) + y^t f_1(y)|g(y) + up_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)), (f_4(y)|\ell_2(y)|ua_1(y) + vq_2(y)), \\ (f_5(y)|\ell_3(y)|va_2(y)) \end{array} \right\rangle. \end{aligned}$$

This implies that $C^{'} \subseteq C$. Now, suppose that $(f_2(y)|g(y) + up_1(y)|0) \in C$. Then it can be written as

$$(f_2(y)|g(y) + up_1(y)|0) = (f_2(y) + y^t f_1(y)|g(y) + up_1(y)|0) - y^t \cdot (f_1(y)|0|0).$$

This shows that $C \subseteq C^{'}$. Hence $C = C^{'}$. Similarly, other cases can be easily proved. □

Theorem 4.5. *Assume that*

$$C = \langle (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), (f_3(y)|l_1(y)|h(y) + ua_1(y) + va_2(y)) \rangle$$

be a $\mathbb{Z}_2\Re_1\Re_2$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$*, where* β_2 *and* β_3 *are odd integers. If* $m_g(y) = \frac{(y^{\beta_2}-1)}{g(y)+ub_1(y)}$ *and* $m_h(y) = \frac{(y^{\beta_3}-1)}{h(y)+ua_1(y)+va_2(y)}$*, then* $f_1(y) \mid m_g(y)f_2(y)$ *and* $g(y) + ub_1(y) \mid m_h(y)\ell_1(y)$.

Proof. Let $\eta(m_g(y)(f_2(y)|g(y) + ub_1(y)|0)) = \eta(m_g(y)f_2(y)|0|0)$. It gives that $(m_g(y)f_2(y)|0|0) \in ker(\eta)$ and hence $f_1(y) \mid m_g(y)f_2(y)$. Similarly, we consider that $\eta(m_h(y)(f_3(y)|\ell_1(y)|h(y) + ua_1(y) + va_2(y))) = \eta(m_h(y)f_3(y)|m_h(y)\ell_1(y)|0))$, we get $(m_h(y)f_3(y)|m_h(y)\ell_1(y)|0)) \in ker(\eta)$. Therefore, $g(y) + ub_1(y) \mid m_h(y)\ell_1(y)$. □

Theorem 4.6. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)), \\ (f_4(y)|\ell_2(y)|ua_1(y) + vq_2(y)), (f_5|\ell_3(y)|va_2(y)) \end{array} \right\rangle$$

be a $\mathbb{Z}_2\Re_1\Re_2$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$*, where* β_2 *is an odd integer and* β_3 *is an even integer and* $b_1(y)|g(y)|(y^{\beta_2} - 1)$*,* $a_i(y) \mid h(y) \mid (y^{\beta_3} - 1)$ *for* $i = 1, 2$*. If* $m_h(y) = \frac{(y^{\beta_3}-1)}{h}$*,* $m_1(y) = \gcd(m_h(y)p_1(y), m_h(y)q_1(y), (y^{\beta_3} - 1))$*,* $m_2(y) = \frac{(y^{\beta_3}-1)}{m_1(y)}$*,* $m_{a_1}(y) = \frac{(y^{\beta_3}-1)}{a_1(y)}$*,*
$s_1(y) = \gcd(m_{a_1}(y)q_2(y), (y^{\beta_3} - 1))$*,* $s_2(y) = \frac{(y^{\beta_3}-1)}{s_1(y)}$*,* $h_{a_2}(y) = \frac{(y^{\beta_3}-1)}{a_2(y)}$*, then*

(*i*) $f_1(y)|m_2(y)m_h(y)f_3(y)$*,* $g(y) + ub_1(y) \mid m_2(y)m_h(y)\ell_1(y)$*,*

(*ii*) $f_1(y)|s_2(y)m_{a_1}(y)f_4(y)$*,* $g(y) + ub_1(y) \mid s_2(y)m_{a_1}(y)\ell_2(y)$*,*

(*iii*) $f_1(y)|m_{a_2}(y)f_5(y)$*,* $g(y) + ub_1(y) \mid m_{a_2}(y)\ell_3(y)$.

Proof. (*i*) Since $m_1(y) \mid m_h(y)p_1(y)$ and $m_1(y) \mid m_h(y)q_1(y)$, $m_h(y)p_1(y) = r_1(y)m_1(y)$ and $m_h(y)q_1(y) = r_2(y)m_1(y)$ for some polynomials $r_1(y), r_2(y) \in \Re_2[y]$, we have

$$\begin{aligned} &\eta(m_2(y)m_h(y)(f_3(y)|\ell_1(y)|h(y) + vp_1(y) + vq_1(y)) \\ &= \eta(m_2(y)m_h(y)f_3(y)|m_2(y)m_h(y)\ell_1(y)|m_2(y)m_h(y)(h(y) + vp_1(y) + vq_1(y))) \\ &= \eta(m_2(y)m_h(y)f_3(y)|m_2(y)m_h(y)\ell_1(y)|0) \\ &= 0. \end{aligned}$$

This implies that $(m_2(y)m_h(y)f_3(y)|m_2(y)m_h(y)\ell_1(y)|0) \in ker(\eta) = \langle (f_1(y)|g(y) + ub_1(y), 0) \rangle$. Therefore, $f_1(y)|m_2(y)m_h(y)f_3(y)$ and $g(y) + ub_1(y) \mid m_2(y)m_h(y)l_1(y)$. We can analogously prove other cases also. □

Theorem 4.7. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y)+ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y)+up_1(y)+vq_1(y)), \\ (f_4(y)|\ell_2(y)|ua_1(y)+vq_2(y)), (f_5|\ell_3(y)|va_2(y)) \end{array} \right\rangle$$

be a $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic code of length $(\beta_1, \beta_2, \beta_3)$, where β_2 is an odd integer and β_3 is an even integer and $b_1(y)|g(y)|(y^{\beta_2}-1)$, $a_i(y) \mid h(y) \mid (y^{\beta_3}-1)$ for $i = 1, 2$. If $m_g(y)=\frac{(y^{\beta_2}-1)}{g(y)}$, $t_1 = \gcd(m_g(y)b_1(y), (y^{\beta_2}-1))$, $t_2(y) = \frac{(y^{\beta_2}-1)}{t_1(y)}$, $m_h(y)=\frac{(y^{\beta_3}-1)}{h}$, $m_1(y)=\gcd(m_h(y)p_1(y), m_h(y)q_1(y), (y^{\beta_3}-1))$, $m_2(y) = \frac{(y^{\beta_3}-1)}{m_1(y)}$, $m_{a_1}(y) = \frac{(y^{\beta_3}-1)}{a_1(y)}$, $m_{a_2}(y) = \frac{(y^{\beta_3}-1)}{a_2(y)}$, $s_1(y) = \gcd(m_{a_1}(y)q_2(y), (y^{\beta_3}-1))$, $s_2(y) = \frac{(y^{\beta_3}-1)}{s_1(y)}$,

$$S_1 = \bigcup_{i=0}^{\beta_1-deg(f_1(y))-1} \{y^i \cdot (f_1(y)|0|0)\};$$

$$S_2 = \bigcup_{i=0}^{\beta_2-deg(g(y))-1} \{y^i \cdot (f_2(y)|g(y)+ub_1(y)|0)\};$$

$$S_3 = \bigcup_{i=0}^{deg(g(y))-deg(b_1(y))-1} \{y^i \cdot (m_g(y)f_2(y)|um_g(y)b_1(y)|0)\};$$

$$S_4 = \bigcup_{i=0}^{\beta_3-deg(h(y))-1} \{y^i \cdot (f_3(y)|\ell_1(y)|h(y)+up_1(y)+vq_1(y))\};$$

$$S_5 = \bigcup_{i=0}^{\beta_3-deg(m_1(y))-1} \{y^i \cdot (m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)p_1(y)+vm_h(y)q_1(y)\};$$

$$S_6 = \bigcup_{i=0}^{deg(h)-deg(a_1)-1} \{y^i \cdot (f_4(y)|\ell_2(y)|ua_1(y)+vq_2(y)\};$$

$$S_7 = \bigcup_{i=0}^{\beta_3-deg(s_1(y))-1} \{y^i \cdot (m_{a_1}f_4(y)|m_{a_1}(y)\ell_2(y)|vm_{a_1}(y)q_2(y)\};$$

$$S_8 = \bigcup_{i=0}^{deg(a_1(y))-deg(a_2(y))-1} \{y^i \cdot (f_5(y)|\ell_3(y)|va_2(y)\},$$

then $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8$ is a minimal generating set for the code C and

$$|C| = 2^{(\beta_1-deg(f_1))}2^{2\beta_2-deg(g)-deg(b_1)}2^{6\beta_3-deg(h)-2deg(m_1)-deg(a_1)-deg(a_2)-deg(s_1)}.$$

Proof. Let $c \in C$ be a codeword and $c_i \in \Re_2[y]$, $1 \le i \le 5$. Then

$$\begin{aligned} c &= c_1 \cdot (f_1(y)|0|0) + c_2 \cdot (f_2(y)|g(y)+ub_1(y)|0) + c_3 \cdot (f_3(y)|l_1(y)|h(y)+up_1(y)+ \\ &\quad vq_1(y)) + c_4 \cdot (f_4(y)|l_2(y)|ua_1(y)+vq_2(y)) + c_5 \cdot (f_5(y)|l_3(y)|va_2(y)) \\ &= (\theta_1(c_1)f_1(y)|0|0) + c_2 \cdot (f_2(y)|g(y)+ub_1(y)|0) + c_3 \cdot (f_3(y)|l_1(y)|h(y)+ \\ &\quad up_1(y)+vq_1(y)) + c_4 \cdot (f_4(y)|l_2(y)|ua_1(y)+vq_2(y)) + c_5 \cdot (f_5|l_3(y)|va_2(y)). \end{aligned}$$

If $deg(\theta(c_1)) \le \beta_1 - deg(f_1) - 1$, then $(f_1(y)|0|0) \in span(S_1)$. Otherwise, by division algorithm, we have

$$deg(\theta(c_1))=\frac{(y^{\beta_1}-1)}{f_1(y)}d_1 + e_1,$$

where $deg(e_1) \leq \beta_1 - deg(f_1) - 1$. Therefore,

$$\begin{aligned}(\theta(c_1)(f_1(y)|0|0) &= ((\frac{(y^{\beta_1}-1)}{f_1(y)}d_1 + e_1)(f_1(y))|0|0)\\ &= (e_1(f_1(y))|0|0) = e_1(f_1(y)|0|0).\end{aligned}$$

This implies that $(\theta(c_1)(f_1(y)), 0) \in span(S_1)$. Now, we have to show that

$$c_2 \cdot (f_2(y)|g(y) + ub_1|0) \in span(S_1 \cup S_2 \cup S_3) \subset span(S).$$

Suppose that m_g divides c_2, that is, $c_2 = d_2m_g + e_2$, where $e_2 = 0$ or $deg(e_2) \leq deg(m_g) - 1$, we get

$$\begin{aligned}c_2 \cdot (f_2(y)|g(y) + ub_1(y)|0) &= (d_2m_g + e_2) \cdot (f_2(y)|g(y) + ub_1(y)|0)\\ &= d_2(m_gf_2|um_gb_1(y)|0) + e_2(f_2|g(y) + ub_1(y)|0).\end{aligned}$$

Cleraly, $e_2(f_2|g(y) + ub_1(y)|0) \in span(S_2)$. It remains to show that $d_2(m_gf_2, um_gb_1(y)|0) \in span(S_1 \cup S_3)$. Since $t_1 \mid m_gb_1$, we obtain $m_gb_1 = r_1t_1$. Hence, $m_gb_1t_2 = 0$. By division algorithm, we have $d_2 = d_2^{'}t_2 + e_2^{'}$, where $e_2^{'} = 0$ or $deg(e_2^{'}) \leq deg(t_2) - 1$. The expression $d_2(m_gf_2|um_gb_1(y)|0)$ can be written as

$$\begin{aligned}d_2(m_gf_2|um_gb_1(y)|0) &= (d_2^{'}t_2 + e_2^{'})(m_gf_2|um_gb_1(y)|0)\\ &= d_2^{'}(t_2m_gf_2, ut_2m_gb_1(y)|0) + e_2^{'}(m_gf_2, um_gb_1(y)|0).\\ &= d_2^{'}(t_2m_gf_2|0|0) + e_2^{'}(m_gf_2, um_gb_1(y)|0).\end{aligned}$$

By Theorem 4.4, $f_1|t_2m_gf_2$ and $d_2^{'}(t_2m_gf_2|0|0) \in span(S_1)$. Since $(m_gf_2, um_gb_1(y)|0) \in span(S_3)$, $c_2 \cdot (f_2(y)|g(y) + ub_1|0) \in span(S_1 \cup S_2 \cup S_3)$. Next, we need to show
$c_3 \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)) \in span(S_2 \cup S_4 \cup S_5) \subset span(S)$. Let us assume that c_3 is divisible by m_h. This means that $c_3 = d_3m_h + e_3$, where $e_3 = 0$ or $deg(e_3) \leq deg(m_h) - 1$. Therefore,

$$\begin{aligned}&c_3 \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y))\\ &= (d_3m_h + e_3) \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y))\\ &= d_3(m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)p_1(y) + vm_h(y)q_1(y))\\ &\quad +e_3(f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)).\end{aligned}$$

Obviously, $e_3(f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)) \in span(S_4)$. Further, we prove that

$$d_3(m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)p_1(y) + vm_h(y)q_2(y)) \in span(S_2 \cup S_5).$$

Since $m_1 \mid m_hp_1$ and $m_1 \mid m_hq_1$, so $m_h(y)p_1(y) = r_2m_1, m_h(y)q_1(y) = r_3m_1$. Hence, $m_h(y)p_1(y)m_2 = m_h(y)q_1(y)m_2 = 0$. Again, by division algorithm, we have $d_3 = d_4m_2 + e_4$, where $d_4 = 0$ or $deg(d_4) \leq deg(m_2) - 1$. Now,

$$\begin{aligned}&d_3(m_h(y)f_3(y)|m_h(y)l_1(y)|um_h(y)p_1(y) + vm_h(y)q_1(y))\\ &= (d_4m_2 + e_4)(m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)p_1(y) + vm_h(y)q_1(y))\\ &= d_4(m_2(y)m_h(y)f_3(y)|m_2(y)m_h(y)\ell_1(y)|0) + e_4(m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)p_1(y)\\ &\quad +vm_h(y)q_1(y)).\end{aligned}$$

Again by Theorem 4.4, $d_4(m_2(y)m_h(y)f_3(y)|m_2(y)m_h(y)\ell_1(y)|0) \in span(S_2)$. Also,
$e_4(m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)p_1(y) + vm_h(y)q_1(y)) \in span(S_5)$. Hence

$$c_3 \cdot (f_3, ua_1(y) + vq_2(y) + uvr_2(y)) \in span(S_2 \cup S_4 \cup S_5).$$

Similarly, we can prove that

$$c_4 \cdot (f_4(y)|\ell_2(y)|ua_1(y) + vq_2(y)) \in span(S_2 \cup S_6 \cup S_7)$$

and $c_5 \cdot (f_5(y)|\ell_3(y)|va_2(y)) \in span(S_2 \cup S_8)$. Finally, we conclude that $c \in span(S)$. Thus, S is the minimal spanning set for C because none of the element of S is a linear combination of the other and

$$|C| = 2^{(\beta_1 - deg(f_1))} 2^{2\beta_2 - deg(g) - deg(b_1)} 2^{6\beta_3 - deg(h) - 2deg(m_1) - deg(a_1) - deg(a_2) - deg(s_1)}.$$

□

From Theorem 4.6, the following results follow immediately.

Corollary 4.8. *Let* $C = \langle (f_1(y)|0|0), (f_2|g(y) + ub_1(y)|0) \rangle$ *be a* $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$*, where* β_2 *is an odd integer and* $g(y) + ub_1(y)) \mid y^{\beta_2} - 1$*. If*

$$\begin{aligned}
S_1 &= \bigcup_{i=0}^{\beta_1 - deg(f_1(y)) - 1} \{y^i \cdot (f_1(y)|0|0)\};\\
S_2 &= \bigcup_{i=0}^{\beta_2 - deg(g(y)) - 1} \{y^i \cdot (f_2(y)|g(y) + ub_1(y)|0)\};\\
S_3 &= \bigcup_{i=0}^{deg(g(y)) - deg(b_1(y)) - 1} \{y^i \cdot (m_g(y)f_2(y)|um_g(y)b_1(y)|0)\},
\end{aligned}$$

then $S_1 \cup S_2 \cup S_3$ *forms a basis for* C *with* $|C| = 2^{(\beta_1 - deg(f_1))} 2^{(2\beta_2 - deg(g_1) - deg(b_1))}$.

Corollary 4.9. *Let* $C = \langle (f_1(y)(y)|0|0) \rangle$ *be a* $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$ *and* $f_1(y) \mid y^{\beta_1} - 1$*. If*

$$S_1 = \bigcup_{i=0}^{\beta_1 - deg(f_1) - 1} \{y^i \cdot (f_1(y)|0|0)\},$$

then S_1 *forms a basis for* C *with* $|C| = 2^{(\beta_1 - deg(f_1))}$.

Theorem 4.10. *Let*

$$C = \langle (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)), (f_4(y)|\ell_2(y)|ua_1(y) + vq_2(y)), (f_5|\ell_3(y)|va_2(y)) \rangle$$

be a $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$*, where* β_3 *is an even integer and* $a_2(y)|a_1(y)|h(y)|(y^{\beta_3} - 1)$*. If* $m_h(y) = \frac{(y^{\beta_3}-1)}{h}$, $m_1(y) = \gcd(m_h(y)p_1(y), m_h(y)q_1(y), (y^{\beta_3}-1))$, $m_2(y) = \frac{(y^{\beta_3}-1)}{m_1(y)}$, $m_{a_1}(y) = \frac{(y^{\beta_3}-1)}{a_1(y)}$, $s_1(y) = \gcd(m_{a_1}(y)q_2(y), (y^{\beta_3} - 1))$, $s_2(y) = \frac{(y^{\beta_3}-1)}{s_1(y)}$, $m_{a_2}(y) = \frac{(y^{\beta_3}-1)}{a_2(y)}$,

$$\begin{aligned}
S_1 &= \bigcup_{i=0}^{\beta_3 - deg(h(y)) - 1} \{y^i \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y))\};\\
S_2 &= \bigcup_{i=0}^{\beta_3 - deg(m_1(y)) - 1} \{y^i \cdot (m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)p_1(y) + vm_h(y)q_1(y)\};\\
S_3 &= \bigcup_{i=0}^{deg(h) - deg(a_1) - 1} \{y^i \cdot (f_4(y)|\ell_2(y)|ua_1(y) + vq_2(y)\};\\
S_4 &= \bigcup_{i=0}^{\beta_3 - deg(s_1(y)) - 1} \{y^i \cdot (m_{a_1}f_4(y)|m_{a_1}(y)\ell_2(y)|vm_{a_1}(y)q_2(y)\};\\
S_5 &= \bigcup_{i=0}^{deg(a_1(y)) - deg(a_2(y)) - 1} \{y^i \cdot (f_5(y)|\ell_3(y)|va_2(y)\},
\end{aligned}$$

then $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ is a minimal generating set for the code C and

$$|C| = 2^{6\beta_3 - deg(h) - 2deg(m_1) - deg(a_1) - deg(a_2) - deg(s_1)}.$$

5. Duality of $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes

In this section, we give the relationship between the generator polynomial of C and its dual code. Let $f(y) \in \Re_2[y]$ and $deg(f(y)) = t$. Then its reciprocal polynomial can be defined as $f^*(y) = y^{deg(f(y))} f(\frac{1}{y})$. Assume that $\omega_m(y) = \sum_{i=0}^{m-1} y^i$ is a polynomial. Now, let $m = \mathrm{lcm}\{\beta_1, \beta_2, \beta_3\}$ and $\mathbf{f}(y) = (f(y)|f^{'}(y)|f^{''}(y))$, $\mathbf{g}(y) = (g(y)|g^{'}(y)|g^{''}(y) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$. Define a map

$$\zeta : \mathcal{S}_{\beta_1,\beta_2,\beta_3} \times \mathcal{S}_{\beta_1,\beta_2,\beta_3} \longrightarrow \frac{\Re_2[y]}{\langle y^m - 1 \rangle}$$

such that

$$\begin{aligned} \zeta(\mathbf{f}(y), \mathbf{g}(y)) &= v f(y) \omega_{\frac{m}{\beta_1}}(y^{\beta_1}) y^{m-1-deg(g(y))} g^*(y) \\ &\quad + u f^{'}(y) \omega_{\frac{m}{\beta_2}}(y^{\beta_2}) y^{m-1-deg(g^{'}(y))} g^{'*}(y) \\ &\quad + f^{''}(y) \omega_{\frac{m}{\beta_3}}(y^{\beta_3}) y^{m-1-deg(g^{''}(y))} g^{''*}(y) \end{aligned}$$

Now, we state the relevant lemmas that will be used to demonstrate the continuing results.

Lemma 5.1. *Let $n_1, n_2 \in \mathbb{N}$. Then*

$$y^{n_1 n_2} - 1 = (y^{n_1} - 1)\omega_{n_2}(y^{n_1}).$$

Proof. Let $x^{n_2} - 1 = (x-1)(x^{n_2-1} + x^{n_2-2} + \cdots + x + 1) = (x-1)\omega_{n_2}(x)$. Putting $x = y^{n_1}$, we get the desired result. □

Lemma 5.2. *[16, Lemma 6.5] Let $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{Z}_2^{\beta_1} \times \Re_1^{\beta_2} \times \Re_2^{\beta_3}$ with associated polynomials $\boldsymbol{f}(y) = (f(y)|f^{'}(y)|f^{''}(y))$, $\boldsymbol{g}(y) = (g(y)|g^{'}(y)|g^{''}(y)) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$. Then $\boldsymbol{f}$ is orthogonal to $\boldsymbol{g}$ and all its shifts if and only if*

$$\zeta(\boldsymbol{f}(y), \boldsymbol{g}(y)) = 0.$$

Theorem 5.3. *Let $\boldsymbol{f}(y) = (f(y)|f^{'}(y)|f^{''}(y))$, $\boldsymbol{g}(y) = (g(y)|g^{'}(y)|g^{''}(y)) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$ such that $\zeta(\boldsymbol{f}(y), \boldsymbol{g}(y)) = 0$.*

(*i*) *If $f^{'}(y)=0$ or $g^{'}(y)=0$ and $f^{''}(y)=0$ or $g^{''}(y)=0$, then $f(y)g^*(y) = 0 \bmod (y^{\beta_1} - 1)$.*

(*ii*) *If $f(y)=0$ or $g(y)=0$ and $f^{''}(y)=0$ or $g^{''}(y)=0$, then $f^{'}(y)g^{'*}(y) = 0 \bmod (y^{\beta_2} - 1)$.*

(*iii*) *If $f(y)=0$ or $g(y)=0$ and $f^{'}(y)=0$ or $g^{'}(y)=0$, then $f^{''}(y)g^{''*}(y) = 0 \bmod (y^{\beta_3} - 1)$.*

Proof. (*i*) Suppose that either $f^{'}(y)=0$ or $g^{'}(y)=0$ and $f^{''}(y)=0$ or $g^{''}(y)=0$. Then we need to show that $f(y)g^*(y) = 0 \bmod (y^{\beta_2} - 1)$. By Lemma 5.2, we have

$$\begin{aligned} 0 &= \zeta(\mathbf{f}(y), \mathbf{g}(y)) \\ &= f(y)\omega_{\frac{m}{\beta_1}}(y^{\beta_1}) y^{m-1-deg(g(y))} g^*(y) \mathrm{mod}(\mathrm{y}^{\mathrm{m}} - 1). \end{aligned}$$

We find that there exists a polynomial $h(y) \in \mathbb{Z}_2[y]$ such that

$$\begin{aligned} f(y)\omega_{\frac{m}{\beta_1}}(y^{\beta_1}) y^{m-1-deg(g(y))} g^*(y) &= h(y)\mathrm{mod}(\mathrm{y}^{\mathrm{m}} - 1) \\ &= h(y)(y^m - 1). \end{aligned}$$

By Lemma 5.1, $y^{m\beta_1} - 1 = (y^{\beta_1} - 1)\omega_m(y^{\beta_1})$, we get

$$\begin{aligned} f(y) y^m g^*(y) &= h^{'}(y)(y^{\beta_1} - 1) \\ f(y)g^*(y) &= 0 \bmod (y^{\beta_1} - 1). \end{aligned}$$

Similarly, we can prove other cases. □

Theorem 5.4. *Let*

$$C = \langle (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)) \rangle$$

be a $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive cyclic code of length $(\beta_1, \beta_2, \beta_3)$. If $C^\perp = \langle (\bar{f_1}(y)|0|0), (\bar{f_2}|\bar{g}(y) + u\bar{b}_1(y)|0), (\bar{f_3}|\bar{\ell}_1(y)|\bar{h}(y) + u\bar{p}_1(y) + v\bar{q}_1(y)) \rangle$ is the dual of C, then

(*i*) $\bar{f_1}^*(y)\gcd(f_1(y), f_2(y), f_3(y)) = h_1(y)(y^{\beta_1} - 1)$,

(*ii*) $u\frac{f_1(y)\ell_1(y)(f_3(y)g(y)+f_2(y)\ell_1(y))}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))} \cdot (\bar{g}(y) + u\bar{b}_1(y))^* = h_2(y)(y^{\beta_2} - 1)$.

Proof. (*i*) Since $(f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)) \in C$ and $(\bar{f_1}(y), 0) \in C^\perp$, by Lemma 5.2, we get

$$\zeta((f_1(y), 0), (\bar{f_1}(y), 0)) = 0,$$

$$\zeta((f_2(y)|g(y) + ub_1(y)|0), (\bar{f_1}(y), 0)) = 0$$

and

$$\zeta((f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y), (\bar{f_1}(y), 0)) = 0.$$

Using Theorem 5.1, we obtain $f_1(y)\bar{f}_1^*(y) = 0$, $f_2(y)\bar{f}_1^*(y) = 0$ and $f_3(y)\bar{f}_1^*(y) = 0$. It is obvious that $\bar{f}_1^*(y)\gcd(f_1(y), f_2(y), f_3(y)) = 0 \bmod (y^{\beta_1} - 1)$. This implies that there exits a polynomial $h_1(y) \in \mathbb{Z}_2[y]$ such that

$$\bar{f}_1^*(y)\gcd(f_1(y), f_2(y), f_2(y)) = h_1(y)(y^{\beta_1} - 1).$$

(*ii*) We know that

$$(f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)) \in C.$$

Then any element $c(y) \in C$ can be expressed as

$$\begin{aligned}
c(y) &= \frac{f_2(y)f_3(y)\ell_1(y)}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))} \times (f_1(y)|0|0) \\
&\quad + u\frac{f_1(y)f_3(y)\ell_1(y)}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))} \times (f_2(y)|g(y) + ub_1(y)|0) \\
&\quad + u\frac{f_1(y)f_2(y)\ell_1(y)}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))} \times (f_3(y)|\ell_1(y)|h(y) + up_1(y) + vq_1(y)) \\
&= (0|u\frac{f_1(y)f_3(y)\ell_1(y)}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))}g(y)|0) \\
&\quad + (0|u\frac{f_1(y)f_2(y)\ell_1^2(y)}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))}|u\frac{f_1(y)f_2(y)\ell_1(y)}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))}h(y)) \\
&= (0|u\frac{f_1(y)\ell_1(y)(f_3(y)g(y) + f_2(y)\ell_1(y))}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))}|u\frac{f_1(y)f_2(y)\ell_1(y)}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))}h(y)).
\end{aligned}$$

This implies that $\zeta\Big((0|u\frac{f_1(y)\ell_1(y)(f_3(y)g(y)+f_2(y)\ell_1(y))}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))}|u\frac{f_1(y)f_2(y)\ell_1(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))}h(y)), (\bar{f_2}, \bar{g}(y) + u\bar{b}_1(y)|0) = 0\Big)$. By Theorem 5.1, we get

$$u\frac{f_1(y)\ell_1(y)(f_3(y)g(y) + f_2(y)\ell_1(y))}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))} \cdot (\bar{g}(y) + u\bar{b}_1(y))^* = 0.$$

This means that there exists a polynomial $h_2(y) \in \mathfrak{R}_1[y]$ such that

$$\frac{f_1(y)\ell_1(y)(f_3(y)g(y) + f_2(y)\ell_1(y))}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))} \cdot (\bar{g}(y) + u\bar{b}_1(y))^* = h_2(y)(y^{\beta_2} - 1).$$

□

6. $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive constacyclic codes

Definition 6.1. *Let λ be a unit in $\mathfrak{R}_2$. A non-empty subset C of $\mathbb{Z}_2^{\beta_1} \times \mathfrak{R}_1^{\beta_2} \times \mathfrak{R}_2^{\beta_3}$ is called a $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive λ-constacyclic code of length $(\beta_1, \beta_2, \beta_3)$ if*

(*i*) *C is additive code;*

(*ii*) *For any codeword $z = (a_0, a_1, \ldots, a_{\beta_1-1}|b_0, b_1, \ldots, b_{\beta_2-1}|c_0, c_1, \ldots, c_{\beta_3-1}) \in C$ its cyclic shift*

$$T_\lambda(z) = (a_{\beta_1-1}, a_0, \ldots, a_{\beta_1-2}|b_{\beta_2-1}, b_0, \ldots, b_{\beta_2-2}|\lambda c_{\beta_3-1}, c_0, \ldots, c_{\beta_3-2}) \in C.$$

Let $\mathcal{S}_{\beta_1,\beta_2,\beta_3,\lambda} = \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \mathfrak{R}_1[y]/\langle y^{\beta_2} - \lambda\rangle \times \mathfrak{R}_2[y]/\langle y^{\beta_3} - \lambda\rangle$. Then $\mathcal{S}_{\beta_1,\beta_2,\beta_3,\lambda}$ forms a $\mathfrak{R}_2[y]$-module under usual addition and scalar multiplication defined in (2.1).

Theorem 6.2. *A code C is a $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive constacyclic code of length $(\beta_1, \beta_2, \beta_3)$ if and only if C is a $\mathfrak{R}_2[y]$-submodule of $\mathcal{S}_{\beta_1,\beta_2,\beta_3,\lambda}$.*

Proof. Proof directly follows the from Theorem 4.2 □

Let β_3 be any odd number. Let C be any $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive constacyclic code. Both C and $\mathfrak{R}_2[y]/\langle y^{\beta_3} - \lambda\rangle$ are $\mathfrak{R}_2[y]$-submodules of $\mathcal{S}_{\beta_1,\beta_2,\beta_3,\lambda}$, we define a mapping

$$\eta_2 : C \longrightarrow \mathfrak{R}_2[y]/\langle y^{\beta_3} - \lambda\rangle,$$

where $\eta_2(f(y)|g(y)|h(y)) = h(y)$. Clearly, η_2 is a module homomorphism whose image is $\mathfrak{R}_2[y]$-submodule of $\mathfrak{R}_2[y]/\langle y^{\beta_3} - \lambda\rangle$ and $ker(\eta_2)$ is a submodule of C. Further, $\eta_2(C)$ can easily be identified as an ideal in the ring $\mathfrak{R}_2[y]/\langle y^{\beta_3} - \lambda\rangle$. Since $\eta_2(C)$ is an ideal in $\mathfrak{R}_2[y]/\langle y^{\beta_3} - \lambda\rangle$, $\eta_2(C)$ is an additive λ-constacyclic code over $\mathfrak{R}_2$ of length β_3. Now, we give a map Ψ_2 which gives the relationship between cyclic codes and constacyclic codes over $\mathfrak{R}_2$ as follows:

$$\Psi_2 : \mathfrak{R}_2[y]/\langle y^{\beta_3} - 1\rangle \longrightarrow \mathfrak{R}_2[y]/\langle y^{\beta_3} - \lambda\rangle$$

$$h(y) \longmapsto h(\lambda y).$$

Since β_3 is odd integer, Ψ_2 is ring isomorphism. Also, J is an ideal in $\mathfrak{R}_2[y]/\langle y^{\beta_3} - 1\rangle$ if and only if $\Psi_2(J)$ is an ideal $\mathfrak{R}_2[y]/\langle y^{\beta_3} - 1\rangle$ (see for reference [20]). Using the above assumption , we have the following result.

Theorem 6.3. *Let*

$$C = \langle (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + va_2(y)\rangle$$

be a $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive constacyclic code of length $(\beta_1, \beta_2, \beta_3)$, where β_2 and β_3 are odd integers and $b_1(y)|g(y)|(y^{\beta_2} - 1)$, $a_i(y) \mid h(y) \mid (y^{\beta_3} - \lambda)$ for $i = 1, 2$. If $m_g(y) = \frac{(y^{\beta_2}-1)}{g(y)}$, $t_1 = \gcd(m_g(y)b_1(y), (y^{\beta_2} - 1))$, $t_2(y) = \frac{(y^{\beta_2}-1)}{m_1(y)}$, $m_h(y) = \frac{(y^{\beta_3}-\lambda)}{h}$, $m_1(y) = \gcd(m_h(y)a_1(y),$

$m_h(y)a_2(y),(y^{\beta_3}-\lambda))$, $m_2(y)=\frac{(y^{\beta_3}-\lambda)}{m_1(y)}$, $m_{a_1}(y)=\frac{(y^{\beta_3}-\lambda)}{a_1(y)}$, $m_{a_2}(y)=\frac{(y^{\beta_3}-\lambda)}{a_2(y)}$. *If*

$$\begin{aligned}
S_1 &= \bigcup_{i=0}^{\beta_1-deg(f_1(y))-1} \{y^i\cdot(f_1(y)|0|0)\};\\
S_2 &= \bigcup_{i=0}^{\beta_2-deg(g(y))-1} \{y^i\cdot(f_2(y)|g(y)+ub_1(y)|0)\};\\
S_3 &= \bigcup_{i=0}^{deg(g(y))-deg(b_1(y))-1} \{y^i\cdot(m_g(y)f_2(y)|um_g(y)b_1(y)|0)\};\\
S_4 &= \bigcup_{i=0}^{\beta_3-deg(h(y))-1} \{y^i\cdot(f_3(y)|\ell_1(y)|h(y)+ua_1(y)+va_2(y))\};\\
S_5 &= \bigcup_{i=0}^{deg(h(y))-deg(a_1(y))} \{y^i\cdot(m_h(y)f_3(y)|m_h(y)\ell_1(y)|um_h(y)a_1(y)+vm_h(y)a_2(y)\};\\
S_6 &= \bigcup_{i=0}^{deg(a_1(y))-deg(a_2(y))} \{y^i\cdot(f_5(y)|l_3(y)|va_2(y)\},
\end{aligned}$$

then $S=S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6$ is a minimal generating set for the code C and

$$|C| = 2^{\beta_1-deg(f)}2^{2\beta_2-deg(g)-deg(b_1)}2^{3\beta_3-deg(h)-deg(a_1)-deg(a_2)}.$$

Proof. Proof directly follows from Theorem 4.6 □

7. Examples & Table

Example 7.1. *Let C be a $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive cyclic code of length $(2,3,4)$. Then C is a $\mathfrak{R}_2$-submodule of $\mathcal{S}_{2,3,4}=\mathbb{Z}_2[y]/\langle y^2-1\rangle\times\mathfrak{R}_1[y]/\langle y^3-1\rangle\times\mathfrak{R}_2[y]/\langle y^4-1\rangle$ generated by*

$$\langle (f_3(y)|\ell_1(y)|h(y)+up_1(y)+vq_1(y)),(f_4(y)|\ell_2(y)|ua_1(y)+vq_2(y)),(f_5|\ell_3(y)|va_2(y))\rangle$$

as in Theorem 4.8. Suppose that $f_3(y)=f_4=f_5=y+1$, $\ell_1(y)=\ell_2(y)=\ell_3(y)=y^2+y+1$, $h(y)=y^2+1=a_1(y)=a_2(y)$, $p_1(y)=q_1(y)=q_2(y)=0$. Then $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive cyclic code C with parameters $[24,8,7]$ is near-optimal code.

Example 7.2. *Let C be a $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive cyclic code of length $(2,3,6)$. Then C is a $\mathfrak{R}_2$-submodule of $\mathcal{S}_{2,3,6}=\mathbb{Z}_2[y]/\langle y^2-1\rangle\times\mathfrak{R}_1[y]/\langle y^3-1\rangle\times\mathfrak{R}_2[y]/\langle y^6-1\rangle$ generated by*

$$\langle (f_3(y)|\ell_1(y)|h(y)+up_1(y)+vq_1(y)),(f_4(y)|\ell_2(y)|ua_1(y)+vq_2(y)),(f_5|\ell_3(y)|va_2(y))\rangle$$

as in Theorem 4.8. Suppose that $f_3(y)=f_4=f_5=y+1$, $\ell_1(y)=\ell_2(y)=\ell_3(y)=y^2+y+1$, $h(y)=y^6-1=a_1(y)$, $a_2(y)=y^4+y^2+y+1$, $p_1(y)=q_1(y)=q_2(y)=0$. Then $\mathbb{Z}_2\mathfrak{R}_1\mathfrak{R}_2$-additive cyclic code C with parameters $[32,2,21]$ is optimal code.

Table: Optimal binary images from $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes.

$(\beta_1, \beta_2, \beta_3)$	Generators	Binary Image
$(1,1,3)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y)=\ell_1(y) = 1$, $h(y)=y^3-1 = a_1(y)$, $a_2(y) = y-1$	$[15,2,10]$
$(1,1,3)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y) = 0$, $\ell_1(y)=a_2(y) = 1$, $h(y)=a_1(y) = y^2+y+1$,	$[15,5,7]$
$(1,3,3)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y^3-1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = y^2+y+1$, $a_1(y) = y^2+y+1$, $a_2(y) = 1$	$[19,5,8]$
$(1,1,6)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y) = 0$, $\ell_1(y) = 0$, $\ell_2(y) = 1$, $h(y) = y^6-1$, $a_1(y) = y^4+y^3+y+1$, $a_2(y) = y^2+1$ $f_4(y) = 1$, $f_5(y) = 1$, $\ell_3(y) = 1$	$[27,8,10]$
$(1,1,7)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = y^7-1$, $a_1(y) = y^4+y^2+y+1$, $a_2(y) = y^3+y^2+1$	$[31,4,16]$
$(2,1,7)$	$f_1(y) = y^2-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = a_1(y) = y^7-1$, $a_2(y) = y^4+y^2+y+1$	$[32,3,18]$
$(2,1,7)$	$f_1(y) = y^2-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = y^7-1$, $a_1(y) = y^4+y^2+y+1$, $a_2(y) = y^3+y^2+1$	$[32,4,16]$
$(1,5,6)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = y^5-1 = b_1(y)$, $f_3(y) = 0$, $\ell_1(y) = 0$, $h(y) = y^6-1$, $f_4(y) = 1$, $\ell_2(y) = y^4+y^3+y^2+y+1$ $a_1(y) = y^3+y^2+y+1$, $a_2(y) = y^2+1$ $f_5(y) = 1$, $\ell_3(y) = y^4+y^3+y^2+y+1$	$[35,8,14]$
$(1,7,6)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = y^7-1 = b_1(y)$, $f_3(y) = 0$, $\ell_1(y) = 0$, $h(y) = y^6-1$, $f_4(y) = 1$, $\ell_2(y) = y^6+y^5+y^4+y^3+y^2+y+1$ $a_1(y) = y^3+y^2+y+1$, $a_2(y) = y^2+1$ $f_5(y) = 1$, $\ell_3(y) = y^6+y^5+y^4+y^3+y^2+y+1$	$[39,8,16]$
$(1,1,9)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = y^9-1$, $a_1(y) = y^9-1$, $a_2(y) = y^7+y^6+y^4+y^3+y+1$	$[39,2,26]$
$(9,1,9)$	$f_1(y) = y-1$, $f_2(y) = 0$, $g(y) = b_1(y) = y-1$, $f_3(y) = y^7+y^6+y^4+y^3+y+1$, $\ell_1(y) = 1$, $h(y) = y^9-1$, $a_1(y) = y^9-1$, $a_2(y) = y^7+y^6+y^4+y^3+y+1$	$[45,2,30]$
$(9,5,6)$	$f_1(y) = y^9-1$, $f_2(y) = 0$, $g(y) = y^5-1 = b_1(y)$, $f_3(y) = 0$, $\ell_1(y) = 0$, $h(y) = y^6-1$, $f_4(y) = y^7+y^6+y^4+y^3+y+1$, $\ell_2(y) = y^4+y^3+y^2+y+1$ $a_1(y) = y^3+y^2+y+1$, $a_2(y) = y^2+1$ $f_5(y) = y^7+y^6+y^4+y^3+y+1$, $\ell_3(y) = y^4+y^3+y^2+y+1$	$[43,8,20]$

8. Conclusion

In this article, we have described the structures $\Re_1 = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\Re_2 = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2$, where $u^2 = v^2 = 0 = uv = vu$ with characteristic 2. The characterization of $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes, additive constacyclic codes and the duality of additive cyclic codes are discussed . The structural attributes of $\mathbb{Z}_2\Re_1\Re_2$-additive codes have been given. Also, we establish the relationships between the minimal generating polynomials of additive cyclic codes and their duals. Furthermore, the minimal generating sets for even and odd length of $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes are determined. We have also obtained optimal binary images from $\mathbb{Z}_2\Re_1\Re_2$-additive cyclic codes that have a number of advantages over linear codes, including the fact that they are more efficient. In future work, it will be an interesting problem to generalize this over $\mathbb{Z}_2\mathbb{Z}_2[u]\mathbb{Z}_2[u_1,\dots,u_k]$, where $u^2=0$ and $u_i^2=0=u_iu_j=u_ju_i$ for all $i,j\in\{1,2,\dots,k\}$.

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