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Basic hypergeometric formulas and identities for negative degree q-Bernstein bases

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Abstract. We utilize formulas for basic hypergeometric series to derive identities and formulas for negative degree q-Bernstein bases, including the Marsden identity, the partition of unity property, the monomial representation formula, the reparametrization formula, and the degree reduction formula. We show that all these identities are just special forms of the q-analogue of Gauss' formula. We also provide a new proof for the q-analogue of Gauss' formula by using the Marsden identity for negative degree q-Bernstein bases together with the identity theorem for analytic functions.

1. Introduction

The Bernstein bases lie at the core of the theory of Bézier curves and surfaces. These polynomial curves and surfaces play a fundamental role in the field of Computer Aided Geometric Design (CAGD) [8]. The Bernstein basis functions of degree $n \geq 0$ on the interval $[0, 1]$ are defined by

$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k}, \qquad k = 0, \ldots, n. \tag{1.1}$$

Many properties, algorithms, and identities for Bernstein bases and Bézier curves and surfaces have been studied by using a powerful technique called *blossoming* [8].

Today there is also a q-form of the classical Bernstein bases (1.1). The q-Bernstein basis functions of degree $n \geq 0$ on the interval $[0, 1]$ are defined by

$$B_k^n(t;q) = \begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{i=0}^{n-k-1} (1 - q^i t), \qquad k = 0, \ldots, n. \tag{1.2}$$

In (1.2), $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient defined by [10]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad k = 0, \ldots, n, \tag{1.3}$$

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and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $k > n$, where the q-integers $[n]_q$ are defined by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and the q-factorials are given by

$$[0]_q! = 1, \qquad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad n \geq 1.$$

Notice that when $q = 1$, the q-Bernstein basis functions reduce to the classical Bernstein basis functions (1.1). These q-Bernstein bases and the corresponding q-Bernstein operators introduced by Phillips [14] have been widely studied by [1, 12, 13, 15–17] for the interval $[0, 1]$ and extended to arbitrary parameter intervals $[a, b]$ by Lewanowicz and Woźny [11]. Simeonov et al. [19] derive many important identities, formulas, and algorithms for the q-Bernstein bases and q-Bézier curves by introducing *q-blossoming*. Recently two of these fundamental identities for the q-Bernstein bases, the partition of unity property and the q-Marsden identity, have been shown to be intimately related to formulas for basic hypergeometric series [24]. Zürnacı et al. [24] show that the partition of unity property for the q-Bernstein bases on the interval $[a, b]$ is equivalent to the q-Chu-Vandermonde formula for basic hypergeometric series, and the q-Marsden identity for the q-Bernstein bases is equivalent to the q-Pfaff-Saalschütz formula for basic hypergeometric series.

As well as positive degree Bernstein bases (1.1), there are also Bernstein bases of negative degree (also known as the Baskakov bases [5]). Algebraic and geometric properties of these negative degree Bernstein bases have been investigated using *multirational blossoming* [7, 20]. As in the case of Bernstein bases of positive degree, a q-form of the Bernstein bases of negative degree exists. These q-Baskakov bases (or negative degree q-Bernstein bases) and the corresponding q-Baskakov operators have been studied in the context of Approximation Theory by [2, 3, 6, 9, 23], and in the context of CAGD by introducing *multirational q-blossoming* [21, 22].

The goal of this paper is to show how some fundamental identities for the negative degree q-Bernstein bases are related to basic hypergeometric series.

This paper is organized as follows. In Section 2 we introduce the basic definitions, notation, and results for q-shifted factorials and basic hypergeometric series. In Section 3 we first provide an alternative representation for the negative degree q-Bernstein basis functions in terms of the q-shifted factorials. We then use formulas for basic hypergeometric series to prove some important identities for negative degree q-Bernstein bases, including the Marsden identity, the partition of unity property, the monomial representation formula, the reparametrization formula, and the degree reduction formula. We show that all these identities are just special forms of the q-analogue of Gauss' theorem. We also give a new proof for this q-analogue of Gauss' theorem by using the Marsden identity for negative degree q-Bernstein bases first established in [22], together with the identity theorem for analytic functions. We close in Section 4 with a short summary of this work.

2. Preliminaries

Throughout this paper, we shall adopt the standard definitions and notation from [10] provided in the following two subsections.

2.1. *q-shifted factorials and q-binomial coefficients*

From now on we shall assume that $0 < q < 1$. The q-shifted factorials are defined by

$$(a; q)_0 = 1, \qquad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n = 1, 2, \ldots. \tag{2.1}$$

Note that for $n = \infty$, $(a;q)_\infty = \prod_{j=0}^{\infty}(1-q^j a)$ is well-defined. These formulas imply that

$$(a;q)_{n+k} = (a;q)_n(q^n a;q)_k, \tag{2.2}$$
$$(a;q)_\infty = (a;q)_n(q^n a;q)_\infty. \tag{2.3}$$

The multiple q-shifted factorials are defined by

$$(a_1,\ldots,a_m;q)_n = \prod_{j=0}^{m}(a_j;q)_n, \qquad n = 0,1,\ldots. \tag{2.4}$$

We will use the following straightforward identity.

$$(a;1/q)_n = (-1)^n q^{-\binom{n}{2}} a^n (1/a;q)_n. \tag{2.5}$$

The q-binomial coefficients can be expressed in terms of the q-shifted factorials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \qquad k = 0,\ldots,n. \tag{2.6}$$

The following property of the q-binomial coefficients will be useful in this paper

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = q^{k(m-k)} \begin{bmatrix} m \\ k \end{bmatrix}_{1/q}. \tag{2.7}$$

2.2. Basic hypergeometric series

The ${}_r\phi_s$ basic hypergeometric series is defined by [10, (12.1.6)]

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} \,\middle|\, q,\, z \right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k} \left(-q^{(k-1)/2}\right)^{k(s+1-r)} z^k. \tag{2.8}$$

Theorem 2.1 (The q-analogue of Gauss' theorem [10, (12.2.18)]).

$${}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix} \,\middle|\, q,\, \frac{c}{ab} \right) = \frac{(c/a,c/b;q)_\infty}{(c,c/(ab);q)_\infty}, \qquad \left|\frac{c}{ab}\right| < 1. \tag{2.9}$$

The following corollary is the special case $b \to \infty$ of (2.9).

Corollary 2.2.

$${}_1\phi_1\left(\begin{matrix} a \\ c \end{matrix} \,\middle|\, q,\, \frac{c}{a} \right) = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{2.10}$$

3. Identities for negative degree q-Bernstein bases and basic hypergeometric series

The q-Bernstein basis functions of negative degree or q-Baskakov basis functions [4] on the interval $(-\infty, 1)$ are defined by

$$B_k^{-n}(t;q) = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{t^k}{\prod_{i=0}^{n+k-1}(1-q^i t)}, \qquad k \geq 0,\ n \geq 0. \tag{3.1}$$

We begin with providing an alternative representation for the negative degree q-Bernstein basis functions (3.1). Using (2.1) and applying (2.2) with $a = q$ and $a = t$, we can rewrite (3.1) as

$$B_k^{-n}(t;q) = (-1)^k q^{\binom{k}{2}} \frac{(q^n;q)_k}{(q;q)_k} \frac{t^k}{(t;q)_n (q^n t;q)_k}. \tag{3.2}$$

In the following subsections, we show how to use basic hypergeometric series to derive identities and formulas for negative degree q-Bernstein bases, including the Marsden identity, the partition of unity property, the monomial representation formula, the reparametrization formula, and the degree reduction formula. All these identities and properties except the degree reduction formula were first derived in [22] using the multirational q-blossom.

3.1. The Marsden Identity

Lemma 3.1.

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (x;q)_k}{x^k} B_k^{-n}(t;q) = \frac{1}{(t;q)_n} {}_2\phi_1\left(\begin{matrix} q^n, x \\ q^n t \end{matrix} \middle| q, \frac{t}{x} \right), \qquad \left|\frac{t}{x}\right| < 1. \tag{3.3}$$

Proof. This result follows directly from (3.2) and (2.8). □

Theorem 3.2 (The Marsden identity, [22, Theorem 5.1]).

$$\frac{1}{(t/x;q)_n} = \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (x;q)_k}{x^k} B_k^{-n}(t;q), \qquad \left|\frac{t}{x}\right| < 1. \tag{3.4}$$

Proof. Using (3.3) and (2.3) and applying (2.9) with $a = q^n$, $b = x$ and $c = q^n t$, the right-hand side of (3.4) becomes

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (x;q)_k}{x^k} B_k^{-n}(t;q) &= \frac{1}{(t;q)_n} {}_2\phi_1\left(\begin{matrix} q^n, x \\ q^n t \end{matrix} \middle| q, \frac{t}{x} \right) \\ &= \frac{1}{(t;q)_n} \frac{(t, q^n t/x;q)_\infty}{(q^n t, t/x;q)_\infty} \\ &= \frac{1}{(t/x;q)_n}. \end{aligned}$$

□

Corollary 3.3. *The basic hypergeometric form of the Marsden identity is*

$${}_2\phi_1\left(\begin{matrix} q^n, x \\ q^n t \end{matrix} \middle| q, \frac{t}{x} \right) = \frac{(t;q)_n}{(t/x;q)_n}, \qquad \left|\frac{t}{x}\right| < 1. \tag{3.5}$$

The Marsden identity, in turn, can be used to give an alternative proof for the q-analogue of Gauss' theorem. To show how, we first recall the identity theorem for analytic functions.

Theorem 3.4 (Identity theorem [18]). *If $f(z)$ and $g(z)$ are analytic functions in a domain D and if $f(z_n) = g(z_n)$ $\forall n \in \mathbb{N}$ for some sequence $\{z_n\} \subset D$ such that $\lim_{n\to\infty} z_n = z_0 \in D$, then $f(z) = g(z)$ $\forall z \in D$.*

Proof of Theorem 2.1. To derive the q-analogue of Gauss' theorem starting from the Marsden identity, observe that (3.5) is equivalent to (2.9) with $a = q^n$, $b = x$, and $c = q^n t$, provided $|t/x| < 1$. Equivalently (3.5) is

$${}_2\phi_1\left(\begin{matrix} z, x \\ zt \end{matrix} \middle| q, \frac{t}{x} \right) = \frac{(t, zt/x;q)_\infty}{(zt, t/x;q)_\infty}, \qquad \forall z = q^n, \quad n = 0, 1, 2, \ldots.$$

Since ${}_2\phi_1\left(\begin{matrix} z, x \\ zt \end{matrix} \middle| q, \frac{t}{x} \right)$ and $(t, zt/x;q)_\infty/(zt, t/x;q)_\infty$ are both analytic in an open neighborhood of $z = 0$, this equation is true for all $z \in \mathbb{C}$ (except at poles) by Theorem 3.4. □

3.2. The Partition of Unity Property

Theorem 3.5 (Partition of unity, [22, Theorem 5.2]).

$$1 = \sum_{k=0}^{\infty} B_k^{-n}(t;q). \tag{3.6}$$

Proof. From (3.2), (2.8), (2.10) and (2.3) it follows that

$$\begin{aligned}\sum_{k=0}^{\infty} B_k^{-n}(t;q) &= \frac{1}{(t;q)_n}\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}}\frac{(q^n;q)_k}{(q;q)_k}\frac{t^k}{(q^nt;q)_k} \\ &= \frac{1}{(t;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^n \\ q^nt\end{matrix}\,\middle|\, q,\, t\right) = \frac{1}{(t;q)_n}\frac{(t;q)_\infty}{(q^nt;q)_\infty} = 1.\end{aligned}$$

□

Corollary 3.6. *The basic hypergeometric form of the partition of unity property is*

$${}_1\phi_1\left(\begin{matrix} q^n \\ q^nt\end{matrix}\,\middle|\, q,\, t\right) = (t;q)_n.$$

3.3. The Monomial Representation Formula

Theorem 3.7 (Monomial representation, [22, Theorem 5.3]).

$$t^m = \sum_{k=m}^{\infty}\frac{\begin{bmatrix} k \\ m\end{bmatrix}_q}{\begin{bmatrix} n+m-1 \\ m\end{bmatrix}_q}(-1)^m q^{\binom{k-m}{2}-\binom{k}{2}} B_k^{-n}(t;q). \tag{3.7}$$

Proof. From (3.2), (2.2) and (2.6), we find that

$$\sum_{k=m}^{\infty}\frac{\begin{bmatrix} k \\ m\end{bmatrix}_q}{\begin{bmatrix} n+m-1 \\ m\end{bmatrix}_q}(-1)^m q^{\binom{k-m}{2}-\binom{k}{2}} B_k^{-n}(t;q) = \sum_{k=m}^{\infty}\frac{(q^n;q)_k}{(q;q)_{k-m}(q^n;q)_m}(-1)^{m+k}q^{\binom{k-m}{2}}\frac{t^k}{(t;q)_n(q^nt;q)_k}.$$

Setting $k-m=j$, and using (2.2), (2.8), (2.10), and (2.3) yields

$$\begin{aligned}\sum_{k=m}^{\infty}\frac{\begin{bmatrix} k \\ m\end{bmatrix}_q}{\begin{bmatrix} n+m-1 \\ m\end{bmatrix}_q}(-1)^m q^{\binom{k-m}{2}-\binom{k}{2}} B_k^{-n}(t;q) &= \frac{t^m}{(t;q)_n(q^nt;q)_m}\sum_{j=0}^{\infty}\frac{(q^{n+m};q)_j}{(q,q^{n+m}t;q)_j}(-1)^j q^{\binom{j}{2}}t^j \\ &= \frac{t^m}{(t;q)_{n+m}}\,{}_1\phi_1\left(\begin{matrix} q^{n+m} \\ q^{n+m}t\end{matrix}\,\middle|\, q,\, t\right) \\ &= \frac{t^m}{(t;q)_{n+m}}\frac{(t;q)_\infty}{(q^{n+m}t;q)_\infty} \\ &= t^m.\end{aligned}$$

□

Corollary 3.8. *The basic hypergeometric form of the monomial representation formula is*

$${}_1\phi_1\left(\begin{matrix} q^{n+m} \\ q^{n+m}t\end{matrix}\,\middle|\, q,\, t\right) = (t;q)_{n+m}.$$

3.4. The Reparametrization Formula

Theorem 3.9 (Reparametrization formula, [22, Theorem 5.5]).

$$B_i^{-n}(rt;q) = \sum_{k=i}^{\infty} B_i^k(r;1/q)B_k^{-n}(t;q), \qquad |rt| < 1. \tag{3.8}$$

Proof. It follows from (1.2), (2.6), and (2.7) that

$$B_i^k(r;1/q) = q^{-i(k-i)}\frac{(q;q)_k}{(q;q)_i(q;q)_{k-i}}r^i(r;1/q)_{k-i}.$$

Using this equation, (3.2), (2.2), (2.5) and (2.8), and setting $k-i=j$, the right-hand side of (3.8) becomes

$$\begin{aligned}\sum_{k=i}^{\infty} B_i^k(r;1/q)B_k^{-n}(t;q) &= \frac{(-1)^i(q^n;q)_i}{(q;q)_i}q^{\binom{i}{2}}\frac{(rt)^i}{(t;q)_n(q^nt;q)_i}\sum_{j=0}^{\infty}\frac{(1/r,q^{n+i};q)_j}{(q,q^{n+i}t;q)_j}(rt)^j\\ &= \frac{(-1)^i(q^n;q)_i}{(q;q)_i}q^{\binom{i}{2}}\frac{(rt)^i}{(t;q)_{n+i}}\,{}_2\phi_1\left(\begin{matrix}1/r,q^{n+i}\\ q^{n+i}t\end{matrix}\,\middle|\,q,\,rt\right).\end{aligned}$$

Hence from (2.9), (2.3), (2.2), and (3.2)

$$\begin{aligned}\sum_{k=i}^{\infty} B_i^k(r;1/q)B_k^{-n}(t;q) &= \frac{(-1)^i(q^n;q)_i}{(q;q)_i}q^{\binom{i}{2}}\frac{(rt)^i}{(t;q)_{n+i}}\frac{(q^{n+i}rt,t;q)_\infty}{(q^{n+i}t,rt;q)_\infty}\\ &= (-1)^iq^{\binom{i}{2}}\frac{(q^n;q)_i}{(q;q)_i}\frac{(rt)^i}{(rt;q)_{n+i}}\\ &= B_i^{-n}(rt;q).\end{aligned}$$

□

Corollary 3.10. *The basic hypergeometric form of the reparametrization formula is*

$${}_2\phi_1\left(\begin{matrix}1/r,q^{n+i}\\ q^{n+i}t\end{matrix}\,\middle|\,q,\,rt\right) = \frac{(t;q)_{n+i}}{(rt;q)_{n+i}}.$$

3.5. The Degree Reduction Formula

Theorem 3.11 (Degree reduction formula, [22, Proposition 3.4]).

$$B_k^{-n}(t;q) = \sum_{j=0}^{\infty} q^{jn}\left\{\begin{bmatrix}n+k-1\\ k\end{bmatrix}_q\Big/\begin{bmatrix}n+k+j\\ k+j\end{bmatrix}_q\right\}B_{k+j}^{-(n+1)}(t;q). \tag{3.9}$$

Proof. Using (3.1), (2.2), (2.8), and (2.10) yields

$$\begin{aligned}&\sum_{j=0}^{\infty} q^{jn}\left\{\begin{bmatrix}n+k-1\\ k\end{bmatrix}_q\Big/\begin{bmatrix}n+k+j\\ k+j\end{bmatrix}_q\right\}B_{k+j}^{-(n+1)}(t;q)\\ &\quad= (-1)^kq^{\binom{k}{2}}\begin{bmatrix}n+k-1\\ k\end{bmatrix}_q\frac{t^k}{(t;q)_{n+k+1}}\sum_{j=0}^{\infty}\frac{1}{(q^{n+k+1}t;q)_j}(-1)^jq^{\binom{j}{2}}(q^{n+k}t)^j\\ &\quad= B_k^{-n}(t;q)\frac{1}{1-q^{n+k}t}\,{}_1\phi_1\left(\begin{matrix}q\\ q^{n+k+1}t\end{matrix}\,\middle|\,q,\,q^{n+k}t\right)\\ &\quad= B_k^{-n}(t;q).\end{aligned}$$

□

Corollary 3.12. *The basic hypergeometric form of the degree reduction formula is*

$${}_1\phi_1\left(\begin{matrix}q\\ q^{n+k+1}t\end{matrix}\,\middle|\,q,\,q^{n+k}t\right) = 1-q^{n+k}t.$$

4. Conclusions

Motivated by [24], we have shown that the theories of negative degree q-Bernstein bases and basic hypergeometric series are intimately related. We have used formulas for basic hypergeometric series to give new proofs of some fundamental identities and formulas for negative degree q-Bernstein bases, including the Marsden identity, the partition of unity property, the monomial representation formula, the reparametrization formula, and the degree reduction formula and we have provided basic hypergeometric forms of these identities. All of these identities are just special cases of the q-analogue of Gauss' theorem (Theorem 2.1) with certain particular values of a, b and c:

1. The Marsden identity for negative degree q-Bernstein bases is a special form of a q-analogue of Gauss' theorem with $a = q^n$, $b = x$, and $c = q^n t$.
2. The partition of unity property for negative degree q-Bernstein bases is a special form of a q-analogue of Gauss' theorem with $a = q^n$, $b \to \infty$, and $c = q^n t$.
3. The monomial representation formula for negative degree q-Bernstein bases is a special form of a q-analogue of Gauss' theorem with $a = q^{n+m}$, $b \to \infty$, and $c = q^{n+m} t$.
4. The reparametrization formula for negative degree q-Bernstein bases is a special form of a q-analogue of Gauss' theorem with $a = 1/r$, $b = q^{n+i}$, and $c = q^{n+i} t$.
5. The degree reduction formula for negative degree q-Bernstein bases is a special form of a q-analogue of Gauss' theorem with $a = q$, $b \to \infty$, and $c = q^{n+k+1} t$.

We have also given a new proof of the q-analogue of Gauss' theorem for basic hypergeometric series by using the Marsden identity for negative degree q-Bernstein bases together with the identity theorem for analytic functions.

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