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Some results associated with hyperbolic Tangent function

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Abstract. In this paper, we examine some properties of analytic functions associated with hyperbolic Tangent functions. That is, the behaviour of the hyperbolic Tangent function inside and at the boundary of the unit disk has been examined. The sharpness some of these results is also proved. Furthermore, an example for our results is considered.

1. Introduction

Let f be an analytic function in the unit disc $D = \{z : |z| < 1\}$, $f(0) = 0$ and $f : D \to D$. In accordance with the classical Schwarz lemma, for any point z in the unit disc D, we have $\left|f(z)\right| \leq |z|$ for all $z \in D$ and $\left|f'(0)\right| \leq 1$. In addition, if the equality $\left|f(z)\right| = |z|$ holds for any $z \neq 0$, or $\left|f'(0)\right| = 1$, then f is a rotation; that is $f(z) = ze^{i\beta}$, β real [6]. Schwarz lemma and its generalizations have important applications in geometric functions theory [1, 11–14]. In this study, the Shwarz lemma will be obtained for the following class $\mathcal{S}$ which will be given. That is, the behavior of the hyperbolic Tangent function inside the unit disk is examined. Similar studies related to this class are given in [8]. In addition, in this research, the author provided a class of analytic functions that depend on the $1 + \tanh z$ function and provided various conditions for the functions which belongs in this class.

To prove our main results, we need the the following lemma [7].

Lemma 1.1 (Jack's lemma). *Let $f(z)$ be a non-constant anaytic function in D with $f(0) = 0$. If*

$$\left|f(z_0)\right| = \max\left\{\left|f(z)\right| : |z| \leq |z_0|\right\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + ...$ that are analytic in D. Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions $p(z)$ satisfying

$$1 + \beta z p'(z) \prec \frac{1+z}{1-z}, \quad z \in D \tag{1.1}$$

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and

$$\left|\beta\right| \operatorname{sech}^2(1) \geq 2 + \left|\beta\right| \sec^2(1), \ \beta \in \mathbb{R}. \tag{1.2}$$

Let $p \in \mathcal{S}$ and consider the following function

$$\Theta(z) = \operatorname{arctanh}(\mathrm{p(z)} - 1),$$

where we have chosen the principle branches of the square root and logarithmic functions. Since arctanhz functions is defined as

$$\operatorname{arctanhz} = \frac{1}{2}\ln\left(\frac{1+\mathrm{z}}{1-\mathrm{z}}\right),$$

so $\Theta(z)$ is an analytic function in D and $\Theta(0) = 0$. To prove our result, let us show that $|\Theta(z)| < 1$ for $z \in D$. From the definition for $\Theta(z)$, we have

$$p(z) = 1 + \tanh(\Theta(z)),$$

$$d(z) = 1 + \beta z p'(z) = 1 + \beta z \Theta'(z)\operatorname{sech}^2(\Theta(\mathrm{z}))$$

and

$$\left|\frac{d(z)-1}{d(z)+1}\right| = \left|\frac{\beta z \Theta'(z)\operatorname{sech}^2(\Theta(\mathrm{z}))}{2 + \beta z \Theta'(z)\operatorname{sech}^2(\Theta(\mathrm{z}))}\right|$$

We suppose that there exists a point $z_0 \in D$ such that

$$\max_{|z|\leq|z_0|} |\Theta(z)| = |\Theta(z_0)| = 1.$$

From Jack's lemma, we have

$$\Theta(z_0) = e^{i\theta} \ \text{ and } \ \frac{z_0\Theta'(z_0)}{\Theta(z_0)} = k,$$

for $\theta \in [-\pi, \pi]$. Therefore, we take

$$\begin{aligned}
\left|\frac{d(z_0)-1}{d(z_0)+1}\right| &= \left|\frac{\beta z \Theta'(z_0)\operatorname{sech}^2(\Theta(\mathrm{z}_0))}{2 + \beta z_0 \Theta'(z_0)\operatorname{sech}^2(\Theta(\mathrm{z}_0))}\right| \\
&= \left|\frac{\beta k e^{i\theta}\operatorname{sech}^2\left(\mathrm{e}^{\mathrm{i}\theta}\right)}{2 + \beta k e^{i\theta}\operatorname{sech}^2(\mathrm{e}^{\mathrm{i}\theta})}\right| \\
&\geq \frac{\left|\beta\right| k \left|\operatorname{sech}^2\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|}{2 + \left|\beta\right| k \left|\operatorname{sech}^2(\mathrm{e}^{\mathrm{i}\theta})\right|}.
\end{aligned}$$

With the simple calculations, we take

$$\left|\operatorname{sech}^2\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2 = \frac{1}{\left|\cosh^2(e^{i\theta})\right|^2} = \Upsilon(\theta),$$

where

$$\begin{aligned}
\left|\cosh^2\left(e^{i\theta}\right)\right|^2 &= \cosh^4(\cos\theta) - 2\cos^2(\sin\theta) - 2\cosh^2(\cos\theta) \\
&\quad +2\cosh^2(\cos\theta)\cos^2(\sin\theta) + \cos^4(\sin\theta) + 1
\end{aligned}$$

The $\Upsilon(\theta)$ equation have critical points of $0, \pm\pi, \pm\frac{\pi}{2}$ in the interval $[-\pi, \pi]$. That is, we can easily get that 0, $\mp\pi$ and $\mp\frac{\pi}{2}$ are the roots of $\Upsilon'(\theta) = 0$ in $[-\pi, \pi]$. Also, $\Upsilon(\theta)$ are even functions in this interval. Therefore, for $\theta \in [0, \pi]$, we obtain

$$\max\left(\Upsilon(\theta)\right) = \Upsilon\left(\frac{\pi}{2}\right) = \operatorname{sech}^4(1)$$

and

$$\min\left(\Upsilon(\theta)\right) = \Upsilon(0) = \Upsilon(\pi) = \sec^4(1).$$

From these expressions, we take

$$\operatorname{sech}^2(1) \leq \left|\operatorname{sech}^2\left(e^{i\theta}\right)\right| \leq \sec^2(1).$$

Thus, we obtain

$$\left|\frac{d(z_0)-1}{d(z_0)+1}\right| \geq \frac{\left|\beta\right| k \operatorname{sech}^2(1)}{2+\left|\beta\right| k \sec^2(1)}.$$

Let

$$\lambda(k) = \frac{\left|\beta\right| k \operatorname{sech}^2(1)}{2+\left|\beta\right| k \sec^2(1)}.$$

Therefore, we take

$$\begin{aligned}
\lambda'(k) &= \frac{\left|\beta\right| \operatorname{sech}^2(1)\left(2+\left|\beta\right| \mathrm{k} \sec^2(1)\right)}{\left(2+\left|\beta\right| k \sec^2(1)\right)^2} \\
&\quad - \frac{\left|\beta\right| \sec^2(1)\left|\beta\right| k \operatorname{sech}^2(1)}{\left(2+\left|\beta\right| k \sec^2(1)\right)^2} \\
&> 0.
\end{aligned}$$

Here, $\lambda(k)$ is an increasing function and hence it will have its minimum value at $k = 1$. Thus we obtain

$$\left|\frac{d(z_0)-1}{d(z_0)+1}\right| \geq \frac{\left|\beta\right| \operatorname{sech}^2(1)}{2+\left|\beta\right| \sec^2(1)}$$

and from (1.2)

$$\left|\frac{d(z_0)-1}{d(z_0)+1}\right| \geq 1.$$

This contradicts $p(z) \in \mathcal{S}$. This means that there is no point $z_0 \in D$ such that $\max\limits_{|z|\leq|z_0|} |\Theta(z)| = |\Theta(z_0)| = 1$. Hence, we take $|\Theta(z)| < 1$ in D. From the Schwarz lemma, we take $|\Theta'(0)| \leq 1$. Therefore, we have

$$\Theta(z) = \operatorname{arctanh}\left(\mathrm{p(z)}-1\right),$$

$$1+\tanh\left(\Theta(z)\right) = p(z) = 1 + c_1 z + c_2 z^2 + ...,$$

$$\tanh\left(\Theta(z)\right) = c_1 z + c_2 z^2 + ...$$

and

$$c_1 + c_2 z + ... = \frac{\tanh(\Theta(z))}{z}.$$

Passing to limit as z tends to 0 in the last equality, we obtain

$$c_1 = \Theta'(0)\mathrm{sech}^2(\Theta(0)),$$

$$\Theta'(0) = \frac{c_1}{\mathrm{sech}^2(\Theta(0))}$$

and

$$|c_1| \leq 1.$$

Now, let us show the sharpness of this inequality. Let

$$p(z) = 1 + \tanh(z).$$

Then

$$p'(z) = \mathrm{sech}^2(\mathrm{z}),$$

$$p'(0) = \mathrm{sech}^2(0) = 1$$

and

$$|c_1| = 1.$$

We thus obtain the following lemma.

Lemma 1.2. *If $p \in \mathcal{S}$, then we have the inequality*

$$|p'(0)| \leq 1.$$

This result is sharp with equality for the function

$$p(z) = 1 + \tanh(z).$$

Since the area of applicability of Schwarz lemma is quite wide, there exist many studies about it. Some of these studies, which are called the boundary version of Schwarz Lemma, are about being estimated from the below modulus of the derivative of the function at some boundary point of the unit disc [2–5, 9, 10, 15–17, 19, 20]. The boundary version of Schwarz Lemma is given as follows [18, 22]:

Lemma 1.3. *Let f be an analytic function in D, $f(0) = 0$ and $f(D) \subset D$. If $f(z)$ extends continuously to boundary point $1 \in \partial D = \{z : |z| = 1\}$, and if $|f(1)| = 1$ and $f'(1)$ exists, then*

$$|f'(1)| \geq \frac{2}{1 + |f'(0)|} \tag{1.3}$$

and

$$|f'(1)| \geq 1. \tag{1.4}$$

Moreover, the equality in (1.3) *holds if and only if*

$$f(z) = z\frac{z-a}{1-az}$$

for some $a \in (-1, 0]$. Also, the equality in (1.4) *holds if and only if $f(z) = ze^{i\beta}$, β real.*

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel [21].

Lemma 1.4 (Julia-Wolff lemma). *Let f be an analytic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(1)$ at $1 \in \partial D$, $|f(1)| = 1$, then the angular derivative $f'(1)$ exists and $1 \leq |f'(1)| \leq \infty$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{S}$ class.

Theorem 2.1. *Let $p \in \mathcal{S}$. Assume that, for $1 \in \partial D$, p has an angular limit $p(1)$ at the points 1, $p(1) = 1 + \tanh(1)$. Then we have the inequality*

$$\left|p'(1)\right| \geq \operatorname{sech}^2(1). \tag{2.1}$$

This result is sharp with the function

$$p(z) = 1 + \tanh(z).$$

Proof. Consider the function

$$\Theta(z) = \operatorname{arctanh}(\mathrm{p(z)} - 1).$$

With simple edits, we have

$$p(z) = 1 + \tanh(\Theta(z))$$

and $|\Theta(1)| = 1$ for $p(1) = 1 + \tanh(1)$. Therefore, from the Schwarz lemma at the boundary, we take $|\Theta'(1)| \geq 1$. With the simple calculations, we obtain

$$p'(1) = \Theta'(1)\operatorname{sech}^2(\Theta(1)) = \Theta'(1)\operatorname{sech}^2(1),$$

$$1 \leq |\Theta'(1)| = \frac{\left|p'(1)\right|}{\operatorname{sech}^2(1)}$$

and

$$\left|p'(1)\right| \geq \operatorname{sech}^2(1).$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$p(z) = 1 + \tanh(z).$$

Then

$$p'(1) = \operatorname{sech}^2(1).$$

□

The inequality (2.1) can be strengthened from below by taking into account, $c_1 = p'(0)$, the second coefficient of the expansion of the function $p(z) = 1 + c_1 z + c_2 z^2 + ...$

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|p'(1)\right| \geq \frac{2\operatorname{sech}^2(1)}{1 + |c_1|}. \tag{2.2}$$

The equality in (2.2) *occurs for the function*

$$p(z) = 1 + \tanh(z).$$

Proof. If we apply the inequality (1.3) to the analytic function $\Theta(z)$ given in Theorem 2.1, we obtain

$$\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(1)| = \frac{\left|p'(1)\right|}{\operatorname{sech}^2(1)}.$$

Since $|\Theta'(0)| = |c_1|$, we take

$$\left|p'(1)\right| \geq \frac{2\operatorname{sech}^2(1)}{1+|c_1|}.$$

To prove the sharpness of the inequality (2.2), let

$$p(z) = 1 + \tanh(z).$$

Then

$$p'(1) = \operatorname{sech}^2(1).$$

On the other hand, we take

$$\tanh(z) = c_1 z + c_2 z^2 + ...$$

and passing to limit in the last equality yields

$$|c_1| = 1.$$

Therefore, we obtain

$$\frac{2\operatorname{sech}^2(1)}{1+|c_1|} = \frac{2\operatorname{sech}^2(1)}{1+1} = \operatorname{sech}^2(1).$$

□

An interesting special case of Theorem 2.2 is when $c_1 = 0$, in which case inequality (2.2) implies $\left|p'(1)\right| \geq 2\operatorname{sech}^2(1)$.

The inequality (2.2) can be strengthened as below by taking into account $c_2 = \frac{p''(0)}{2!}$ which is the coefficient in the expansion of the function $p(z) = 1 + c_1 z + c_2 z^2 + ...$

Theorem 2.3. *Let $p \in \mathcal{S}$. Assume that, for $1 \in \partial D$, p has an angular limit $p(1)$ at the points 1, $p(1) = 1 + \tanh 1$. Then we have the inequality*

$$\left|p'(1)\right| \geq \operatorname{sech}^2(1)\left(1 + \frac{2(1-|c_1|)^2}{1-|c_1|^2+|c_2|}\right). \tag{2.3}$$

Proof. Let $\Theta(z)$ be the same as in the proof of Theorem 2.1 and $l(z) = z$. By the maximum principle, for each $z \in D$, we have the inequality $|\Theta(z)| \leq |l(z)|$. Therefore, we have

$$\begin{aligned} m(z) &= \frac{\Theta(z)}{l(z)} = \frac{\operatorname{arctanh}(p(z)-1)}{z} \\ &= \frac{\operatorname{arctanh}\left(c_1 z + c_2 z^2 + ...\right)}{z}. \end{aligned}$$

Since

$$\operatorname{arctanh}\left(c_2 z + c_3 z^2 + ...\right) = c_1 z + c_2 z^2 + ... + \frac{1}{3}\left(c_1 z + c_2 z^2 + ...\right)^3 + ...,$$

we take

$$\begin{aligned} m(z) &= \frac{c_1 z + c_2 z^2 + ... + \frac{1}{3}\left(c_1 z + c_2 z^2 + ...\right)^3 + ...}{z} \\ &= c_1 + c_2 z + ... + \frac{1}{3} z^2 \left(c_1 + c_2 z + ...\right)^3 + ... \end{aligned}$$

Here, $m(z)$ is an analytic function in D and $|m(z)| \leq 1$ for $z \in D$. In particular, we have

$$|m(0)| = |c_1| \leq 1 \tag{2.4}$$

and

$$|m'(0)| = |c_2| .$$

The auxiliary function

$$w(z) = \frac{m(z) - m(0)}{1 - \overline{m(0)} m(z)}$$

is analytic in D, $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $|w(1)| = 1$ for $1 \in \partial D$. From (1.3), we obtain

$$\begin{aligned} \frac{2}{1 + |w'(0)|} &\leq |w'(1)| = \frac{1 - |m(0)|^2}{\left|1 - \overline{m(0)} m(1)\right|^2} |m'(1)| \\ &\leq \frac{1 + |m(0)|}{1 - |m(0)|} |m'(1)| \\ &= \frac{1 + |m(0)|}{1 - |m(0)|} \left(|\Theta'(1)| - |l'(1)|\right) . \end{aligned}$$

Also, since

$$|w'(0)| = \frac{|m'(0)|}{1 - |m(0)|^2} = \frac{|c_2|}{1 - |c_1|^2},$$

we take

$$\frac{2}{1 + \frac{|c_2|}{1 - |c_1|^2}} \leq \frac{1 + |c_1|}{1 - |c_1|} \left(\frac{\left|p'(1)\right|}{\operatorname{sech}^2(1)} - 1 \right),$$

$$\frac{2\left(1 - |c_1|^2\right)}{1 - |c_1|^2 + |c_2|} \frac{1 - |c_1|}{1 + |c_1|} + 1 \leq \frac{\left|p'(1)\right|}{\operatorname{sech}^2(1)}$$

and

$$\left|p'(1)\right| \geq \operatorname{sech}^2(1) \left(1 + \frac{2\left(1 - |c_1|\right)^2}{1 - |c_1|^2 + |c_2|} \right).$$

□

If $p(z) - 1$ has zeros different from $z = 0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

Theorem 2.4. *Let $p \in \mathcal{S}$ and $a_1, a_2, ..., a_n$ be zeros of the function $p(z) - 1$ in D that are different from zero. Assume that, for $1 \in \partial D$, p has an angular limit $p(1)$ at the points 1, $p(1) = 1 + \tanh(1)$. Then we have the inequality*

$$\left|p'(1)\right| \geq \operatorname{sech}^2(1)\left(1 + \sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2} + \frac{2\left(\prod_{i=1}^{n}|a_i| - |c_1|\right)^2}{\left(\prod_{i=1}^{n}|a_i|\right)^2 - |c_1|^2 + \prod_{i=1}^{n}|a_i|\left|c_2 + c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}\right). \tag{2.5}$$

Proof. Let $\Theta(z)$ be the same as in the proof of Theorem 2.1 and $a_1, a_2, ..., a_n$ be zeros of the function $p(z) - 1$ in D that are different from zero. Also, consider the function

$$B(z) = z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}.$$

By the maximum principle for each $z \in D$, we have

$$|\Theta(z)| \leq |B(z)|.$$

Consider the function

$$\begin{aligned} g(z) &= \frac{\Theta(z)}{B(z)} = \frac{\operatorname{arctanh}(p(z)-1)}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}} \\ &= \frac{c_1 + c_2 z + ... + \frac{1}{3}z^2\left(c_1 + c_2 z + ...\right)^3 + ...}{\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}. \end{aligned}$$

$g(z)$ is analytic in D and $\left|g(z)\right| < 1$ for $|z| < 1$. In particular, we have

$$\left|g(0)\right| = \frac{|c_1|}{\prod_{i=1}^{n}|a_i|}$$

and

$$\left|g'(0)\right| = \frac{\left|c_2 + c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\prod_{i=1}^{n}|a_i|}.$$

The auxiliary function

$$\varphi(z) = \frac{g(z) - g(0)}{1 - \overline{g(0)}g(z)}$$

is analytic in D, $\left|\varphi(z)\right| < 1$ for $|z| < 1$ and $\varphi(0) = 0$. For $1 \in \partial D$ and $p(1) = 1 + \tanh(1)$, we take $\left|\varphi(1)\right| = 1$.

From (1.3), we obtain

$$\begin{aligned} \frac{2}{1 + \left|\varphi'(0)\right|} &\leq \left|\varphi'(1)\right| = \frac{1 - \left|g(0)\right|^2}{\left|1 - \overline{g(0)}g(1)\right|}\left|g'(1)\right| \\ &\leq \frac{1 + \left|g(0)\right|}{1 - \left|g(0)\right|}\left(|\Theta'(1)| - |B'(1)|\right). \end{aligned}$$

It can be seen that

$$\left|\varphi'(0)\right| = \frac{\left|g'(0)\right|}{1-\left|g(0)\right|^2}$$

and

$$\left|\varphi'(0)\right| = \frac{\frac{\left|c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\prod_{i=1}^{n}|a_i|}}{1-\left(\frac{|c_1|}{\prod_{i=1}^{n}|a_i|}\right)^2} = \prod_{i=1}^{n}|a_i|\frac{\left|c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2}.$$

Also,we have

$$|B'(1)| = 1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2},\ 1\in\partial D.$$

Therefore, we obtain

$$\frac{2}{1+\prod_{i=1}^{n}|a_i|\frac{\left|c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2}} \le \frac{\prod_{i=1}^{n}|a_i|+|c_1|}{\prod_{i=1}^{n}|a_i|-|c_1|}\left(\frac{\left|p'(1)\right|}{\operatorname{sech}^2(1)}-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right)$$

and

$$\left|p'(1)\right| \ge \operatorname{sech}^2(1)\left(1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}+\frac{2\left(\prod_{i=1}^{n}|a_i|-|c_1|\right)^2}{\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2+\prod_{i=1}^{n}|a_i|\left|c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}\right).$$

□

If $p(z)-1$ has no zeros different from $z=0$ in Theorem 3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.5. *Let $p\in\mathcal{S}$, $p(z)-1$ has no zeros in D except $z=0$ and $c_1>0$. Assume that, for $1\in\partial D$, p has an angular limit $p(1)$ at the points 1, $p(1)=1+\tanh(1)$. Then we have the inequality*

$$\left|p'(1)\right| \ge \operatorname{sech}^2(1)\left(1-\frac{2c_1\ln^2(c_1)}{2c_1\ln(c_1)-|c_1|}\right).$$

Proof. Let $c_1>0$ in the expression of the function $p(z)$. Having in mind inequality (2.4) and the function $p(z)-1$ has no zeros in D except $z=0$, we use $\ln m(z)$ to denote the analytic branch of the logarithm normed by the condition

$$\ln m(0) = \ln(c_1) < 0.$$

The auxiliary function

$$q(z) = \frac{\ln m(z)-\ln m(0)}{\ln m(z)+\ln m(0)}$$

is analytic in the unit disc D, $\left|q(z)\right| < 1$, $q(0) = 0$ and $\left|q(1)\right| = 1$ for $1 \in \partial D$. From (1.3), we obtain

$$\begin{aligned}\frac{2}{1+\left|q'(0)\right|} &\leq \left|q'(1)\right| = \frac{|2\ln m(0)|}{|\ln m(1)+\ln m(0)|^2}\left|\frac{m'(1)}{m(1)}\right| \\ &= \frac{-2\ln m(0)}{\ln^2 m(0)+\arg^2 m(1)}\left\{|\Theta'(1)|-1\right\}.\end{aligned}$$

Replacing $\arg^2 m(1)$ by zero, then

$$\frac{1}{1-\frac{|c_2|}{2c_1\ln(c_1)}} \leq \frac{-1}{\ln(c_1)}\left\{\frac{\left|p'(1)\right|}{\operatorname{sech}^2(1)}-1\right\}$$

and

$$\left|p'(1)\right| \geq \operatorname{sech}^2(1)\left(1-\frac{2\mathrm{c}_1\ln^2(\mathrm{c}_1)}{2\mathrm{c}_1\ln(\mathrm{c}_1)-|\mathrm{c}_1|}\right).$$

□

The following theorem shows the relationship between the coeffcients c_1 and c_2 in the Maclaurin expansion of $p(z) = 1 + c_1z + c_2z^2 + \ldots$.

Theorem 2.6. *Let $p \in \mathcal{S}$, $p(z) - 1$ has no zeros in D except $z = 0$ and $c_1 > 0$. Then we have the inequality*

$$|c_2| \leq 2\,|c_1\ln(c_1)|. \tag{2.6}$$

Proof. The $q(z)$ function we have expressed in Theorem 2.5 satisfies the conditions for the Schwarz lemma. Thus, if we apply the Schwarz lemma to the function $q(z)$, we obtain

$$1 \geq \left|q'(0)\right| = -\frac{|c_2|}{2c_1\ln(c_1)}$$

and

$$|c_2| \leq 2\,|c_1\ln(c_1)|.$$

□

3. Examples

Example 3.1. *Let us consider the function $p(z)$ defined by*

$$p(z) = 1 + \tanh(z).$$

From here, we have

$$d(z) = 1 + \beta zp'(z) = 1 + \beta z\operatorname{sech}^2(z).$$

Now let's show that property (1.1) *is provided. Let*

$$d(z) = \frac{1+\omega(z)}{1-\omega(z)}.$$

Therefore, we have

$$\omega(z) = \frac{d(z)-1}{d(z)+1} = \frac{\beta zp'(z)}{2+\beta zp'(z)} = \frac{\beta z\operatorname{sech}^2(z)}{2+\beta z\operatorname{sech}^2(z)}.$$

Here, $\omega(0) = 0$ and $|\omega(z)| \leq 1$. Thus,

$$1 + \beta zp'(z) \prec \frac{1+z}{1-z}.$$

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