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Solution of certain stochastic differential equations: Pseudo $\mathcal{S}$-asymptotically omega periodic solution with measures

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Abstract. This research paper discusses a mathematical concept known as the ω-periodic process, and displays a new type of function called the doubly measure pseudo $\mathcal{S}$-asymptotically ω-periodic function. It explores the properties of these functions and uses them to examine the solution of a stochastic differential equation guided by Brownian motion. The main target of the current work is to establish the existence and uniqueness of this solution.

1. Introduction

The concept of periodicity is highly significant in probability and exhibits outstanding applications in various fields, such as engineering and statistics. Recently, several research works have been particularly oriented towards investigating periodic solution for stochastic evolution equation, including almost periodic, pseudo-almost periodic, measure pseudo-almost periodic, almost automorphic, asymptotically almost periodic, etc, (see [2, 3, 14]).
The study of asymptotically ω-periodic solution is an intrinsic area of research in the qualitative theory, yielding pertinent applications in mathematical biology, control theory, and physics. Asymptotically periodic functions are a type of approximately periodic functions, and systems described by them are often more realistic than those that are strictly periodic. Further information on this topic can be found in references [1, 22].
There are multiple concepts related to asymptotically ω-periodic functions, including asymptotically ω-periodic functions in the Stepanov sense [19], $\mathcal{S}$-asymptotically ω-periodic functions [9, 10, 16], and $\mathcal{S}$-asymptotically ω-periodic functions in the Stepanov sense [8, 18].
The $\mathcal{S}$-asymptotically periodicity is a significant generalization of asymptotic periodicity that was first introduced by Henriquez et al. in [10, 11]. While much attention has been devoted to this concept in the deterministic case, with many authors contribution to its development, there has been relatively scarce interest dedicated to the stochastic case, see [5, 9] and the references therein.
In this respect, S. Zhao and M. Song were the first to investigate an $\mathcal{S}$-asymptotically ω-periodic solution for a certain class of stochastic fraction evolution equation driven by Levy noise. They revealed the existence

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of square-mean $\mathcal{S}$-asymptotically ω-periodic solution in their works ([20, 21]).
In [7], the concept of $\mathcal{S}$-asymptotically ω-periodic in the Stepanov sense was elaborated and the application to semilinear first-order abstract differential equations was tackled. In [8], the authors demonstrated the existence of a function which is not $\mathcal{S}$-asymptotically ω-periodic, but rather $\mathcal{S}$-asymptotically ω-periodic in the Stepanov sense. In [19], Xie and Zhang characterize the asymptotically ω-periodic functions in the Stepanov sense. They applied a criteria obtained to investigate nthe existence and uniqueness of asymptotically ω-periodic mild solution to semilinear fractional integro-differential equations with Stepanov asymptotically ω-periodic coefficients. Recently, N'Guérékata and Valmorin have set forward the concept of asymptotically antiperiodic functions and explored their propositionositionerties in [15].

In this work, the following stochastic equation driven by Brownian motion in a separable Hilbert space $\mathbb{H}$ is considered :

$$\begin{cases} d\xi(t) = A\xi(t)dt + F(t,\xi(t))dt + G(t,\xi(t))dB(t), & t \geqslant 0 \\ \xi(0) = c_0, \end{cases} \tag{1}$$

where A refers to a closed linear operator and

$$F, G : \mathbb{R}^+ \times \mathbb{L}^p(\Omega,\mathbb{H}) \to \mathbb{L}^p(\Omega,\mathbb{H})$$

$(B(t))_t$ stands for a two-sided one-dimensional Brownian motion with values in $\mathbb{H}$.
In [12], Solym Manou-Abi and William Dimbour addressed the existence of the square-mean asymptotically ω-periodic solution in equation (1) and in [13] they handled the existence, uniqueness and asymptotic stability of the p-th mean $\mathcal{S}$-asymptotically ω-periodic solution for the same equation. The derivation method used in our paper is the usual derivative. There are other methods, for example with distributions (See [6]).
Inspired from the above mentioned works, we will integrate the concept of doubly measure pseudo $\mathcal{S}$-asymptotically ω-periodic functions. We will equally provide fundamental propositionositionerties and investigate the existence, uniqueness of (m_1, m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic solution for equation (1).
The paper is organized as follows. In Section 2, several notions and preliminary results are presented. In Section 3, we introduce a new class of function called (m_1, m_2)-S^p-pseudo $\mathcal{S}$-asymptotically ω-periodic functions, explore its propositionositionerties and establish its composition theorems. Section 4 is devoted to corroborate existence and uniqueness of (m_1, m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic solution for equation (1). In Section 5, display certain expressive examples to illustrate our main results.

2. (m_1, m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic processes

Let's start by introducing the following notions. $(\Omega, \mathcal{F}, \mathbb{P})$: the complete probability space.
$\mathbb{L}^p(\Omega,\mathbb{H})$: indicates the space of all measurable p-th integrable random variables $\xi : \Omega \to \mathbb{H}$ such that

$$\mathbb{E}\|\xi\|^p = \int_\Omega \|\xi(\omega)\|^p d\mathbb{P}(\omega) < \infty.$$

$C_b(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}))$: the set of all bounded continuous functions from $\mathbb{R}^+$ to $\mathbb{L}^p(\Omega,\mathbb{H})$.
$C_0(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H})) = \{\xi \in C_b(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H})) : \lim_{t \longrightarrow +\infty} \mathbb{E}\|\xi(t)\|^p = 0\}$.

Definition 2.1. *[12]*
i) A stochastic process $\xi : \mathbb{R}^+ \to \mathbb{L}^p(\Omega,\mathbb{H})$ *is called continuous in p-th mean sense, whenever*

$$\lim_{t \to s} \mathbb{E}\|\xi(t) - \xi(s)\|^p = 0, \ \forall\ t, s \in \mathbb{R}^+.$$

ii) A stochastic process $\xi : \mathbb{R}^+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ is called bounded in p-th mean sense, if there exists a constant $K > 0$ such that

$$\mathbb{E}\|\xi(t)\|^p \leqslant K, \qquad \forall t \geqslant 0.$$

We indicate by $\mathcal{L}_b$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{P}_m$ the set of all positive measures m on $\mathcal{L}_b$ satisfying $m(\mathbb{R}) = +\infty$ and $m([a, b]) < +\infty$ for all $a, b \in \mathbb{R}, \quad a < b$.

Definition 2.2. *[3] Let $m_1, m_2 \in \mathcal{P}_m$. We state that m_1 and m_2 are equivalent ($m_1 \sim m_2$), if there exist positive constants α, β and a bounded interval I (eventually $I = \emptyset$) such that*

$$\alpha m_1(A) \leqslant m_2(A) \leqslant \beta m_1(A),$$

for $A \in \mathcal{L}_b$ satisfying $A \cap I = \emptyset$.

Definition 2.3. *For $m \in \mathcal{P}_m$ and $\lambda \in \mathbb{R}$, we define the positive measure m_λ on $(\mathbb{R}, \mathcal{L}_b)$ by*

$$m_\lambda(A) = m(a + \lambda : a \in A), A \in \mathcal{L}_b.$$

In this research, the following assumption is needed :
For m, m_1 and $m_2 \in \mathcal{P}_m$, and for all $\lambda \in \mathbb{R}$, there exist $\beta > 0$ such that
(H_1) $\quad \limsup_{T\to\infty} \frac{m_2([0,T])}{m_1([0,T])} = \alpha < \infty.$
(H_2) $\quad m_\lambda(A) \leqslant \beta m(A),$ where $A \in \mathcal{L}_b$.

Lemma 2.4. *[3] Departing from hypothesis (H_2), it follows that*

$$\forall \varsigma > 0, \quad \limsup_{T\to\infty} \frac{m([\varsigma, T + \varsigma])}{m([0, T])} < \infty.$$

Definition 2.5. *Let $m_1, m_2 \in \mathcal{P}_m$ and ξ be a stochastic process in $C_b(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}))$. ξ is called (m_1, m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic in p-th mean sense, if there exists $\omega > 0$ such that*

$$\lim_{T\to\infty} \frac{1}{m_1([0, T])} \int_0^T \mathbb{E}\|\xi(t + \omega) - \xi(t)\|^p dm_2(t) = 0.$$

We denote the set of such functions by $PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$.

Proposition 2.6. *Departing from (H_1), the space $(PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2), \|.\|_\infty)$ is a Banach space, with $\|\xi\|_\infty = \sup_{t\in\mathbb{R}^+}(\mathbb{E}\|\xi(t)\|^p)^{1/p}$.*

Proof. To corroborate that the space $(PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2), \|.\|_\infty)$ is a Banach space, it is sufficient to prove that $PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$ is closed in $C_b(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}))$. Let $(\xi_n)_n$ be a sequence in $PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$ such that $\lim_{n\to+\infty} \|\xi_n - \xi\|_\infty = 0$.
Therefore $T > 0$, $\omega > 0$, we have

$$\begin{aligned}\frac{1}{m_1([0, T])} \int_0^T \mathbb{E}\|\xi(t + \omega) - \xi(t)\|^p dm_2(t) \leqslant{}& \frac{3^{p-1}}{m_1([0, T])} \int_0^T \mathbb{E}\|\xi_n(t + \omega) - \xi(t + \omega)\|^p dm_2(t)\\ &+ \frac{3^{p-1}}{m_1([0, T])} \int_0^T \mathbb{E}\|\xi_n(t + \omega) - \xi_n(t)\|^p dm_2(t)\\ &+ \frac{3^{p-1}}{m_1([0, T])} \int_0^T \mathbb{E}\|\xi_n(t) - \xi(t)\|^p dm_2(t).\end{aligned}$$

It follows that

$$\limsup_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|\xi(t+\omega)-\xi(t)\|^p dm_2(t)$$
$$\leqslant 3^{p-1}\limsup_{T\to\infty}\frac{m_2([0,T])}{m_1([0,T])}\sup_{t\in\mathbb{R}^+}\mathbb{E}\|\xi_n(t+\omega)-\xi(t+\omega)\|^p$$
$$+\,3^{p-1}\limsup_{T\to\infty}\frac{m_2([0,T])}{m_1([0,T])}\sup_{t\in\mathbb{R}^+}\mathbb{E}\|\xi_n(t)-\xi(t)\|^p.$$

Departing from (H_1), we obtain

$$\limsup_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|\xi(t+\omega)-\xi(t)\|^p dm_2(t)\leqslant 2\alpha.3^{p-1}\|\xi_n-\xi\|_\infty^p.$$

Since $\lim_{n\to+\infty}\|\xi_n-\xi\|_\infty = 0$, we infer that

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|\xi(t+\omega)-\xi(t)\|^p dm_2(t)=0,$$

which implies that $(PSAP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2),\|.\|_\infty)$ is a Banach space. ■

Definition 2.7. *Let $m_1,m_2\in\mathcal{P}_m$. A continuous bounded stochastic process $F:\mathbb{R}^+\times\mathbb{L}^p(\Omega,\mathbb{H})\to\mathbb{L}^p(\Omega,\mathbb{H})$ is called uniformly (m_1,m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic in $\xi\in K$, where $K\subset\mathbb{L}^p(\Omega,\mathbb{H})$ is bounded subset, if there exists $\omega>0$ such that*

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|F(t+\omega,\xi)-F(t,\xi)\|^p dm_2(t)=0.$$

We designate the set of such functions by :

$$PSAP_\omega(\mathbb{R}^+\times\mathbb{L}^p(\Omega,\mathbb{H}),\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)=$$
$$\{F(.,\xi)\in PSAP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2),\ \xi\in\mathbb{L}^p(\Omega,\mathbb{H})\}.$$

Lemma 2.8. *Let $F\in PSAP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$, and $m_1,m_2\in\mathcal{P}_m$ satisfy (H_2). Then, $F(\cdot+\varsigma)\in PSAP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$ for all $\varsigma>0$.*

Proof. Let $F\in PSAP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$. We hence have :

$$\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|F(t+\varsigma+\omega)-F(t+\varsigma)\|^p dm_2(t)$$
$$=\frac{1}{m_1([0,T])}\int_\varsigma^{T+\varsigma}\mathbb{E}\|F(t+\omega)-F(t)\|^p dm_2(t-\varsigma)$$
$$\leqslant\frac{m_1([0,T+\varsigma])}{m_1([0,T])m_1([0,T+\varsigma])}\int_0^{T+\varsigma}\mathbb{E}\|F(t+\omega)-F(t)\|^p dm_2(t-\varsigma),$$

let's note that

$$m_1([0,T+\varsigma])=\int_0^{T+\varsigma}dm_1(t)=\int_0^{\varsigma}dm_1(t)+\int_\varsigma^{T+\varsigma}dm_1(t)=m_1([0,\varsigma])+m_1([\varsigma,T+\varsigma]).$$

We get that

$$\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|F(t+\varsigma+\omega)-F(t+\varsigma)\|^p dm_2(t)$$
$$\leqslant\frac{m_1([0,\varsigma])}{m_1([0,T])}.\frac{1}{m_1([0,T+\varsigma])}\int_0^{T+\varsigma}\mathbb{E}\|F(t+\omega)-F(t)\|^p dm_2(t-\varsigma)$$
$$+\frac{m_1([\varsigma,T+\varsigma])}{m_1([0,T])}.\frac{1}{m_1([0,T+\varsigma])}\int_0^{T+\varsigma}\mathbb{E}\|F(t+\omega)-F(t)\|^p dm_2(t-\varsigma).$$

Thus m_1, m_2 satisfy (H_2) and referring to Lemma 2.4, we obtain

$$\lim_{T\to\infty} \frac{1}{m_1([0,T])} \int_0^T \mathbb{E}\|F(t+\varsigma+\omega) - F(t+\varsigma)\|^p dm_2(t) = 0.$$

Hence, for all $\varsigma > 0$, $F(\cdot + \varsigma) \in P\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}(\Omega, \mathbb{H}), m_1, m_2)$. ∎

Theorem 2.9. *Let $F \in P\mathcal{S}AP_\omega(\mathbb{R}^+ \times \mathbb{L}^p(\Omega, \mathbb{H}), \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$ such that F satisfies the Lipschitz condition, i.e, if there exists a constant $L_F > 0$ such that*

$$\mathbb{E}\|F(t,\xi_1) - F(t,\xi_2)\|^p \leqslant L_F \mathbb{E}\|\xi_1 - \xi_2\|^p, \quad t \in \mathbb{R}^+, \xi_1, \xi_2 \in \mathbb{L}^p(\Omega, \mathbb{H}),$$

then $\zeta(\cdot) = F(\cdot, \xi(\cdot)) \in P\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$ if $\xi(\cdot) \in P\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$.

Proof. We have the range $\mathcal{R}(\xi)$ of $\xi(\cdot)$ which is a bounded set. Thus, ζ is a bounded function. On the other side, for $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that, for every $T \geqslant L_\varepsilon, X \in \mathcal{R}$

$$\frac{1}{m_1([0,T])} \int_0^T \mathbb{E}\|F(t+\omega, X) - F(t, X)\|^p dm_2(t) \leqslant \varepsilon,$$

$$\frac{L_F}{m_1([0,T])} \int_0^T \mathbb{E}\|\xi(t+\omega) - \xi(t)\|^p dm_2(t) \leqslant \varepsilon.$$

For $T \geqslant L_\varepsilon$, we have

$$\begin{aligned}
&\frac{1}{m_1([0,T])} \int_0^T \mathbb{E}\|F(t+\omega, \xi(t+\omega)) - F(t, \xi(t))\|^p dm_2(t) \\
&\qquad \leqslant \frac{2^{p-1}}{m_1([0,T])} \int_0^T \mathbb{E}\|F(t+\omega, \xi(t+\omega)) - F(t, \xi(t+\omega))\|^p dm_2(t) \\
&\qquad\quad + \frac{2^{p-1}}{m_1([0,T])} \int_0^T \mathbb{E}\|F(t, \xi(t+\omega)) - F(t, \xi(t))\|^p dm_2(t) \\
&\qquad \leqslant \frac{2^{p-1}}{m_1([0,T])} \int_0^T \mathbb{E}\|F(t+\omega, \xi(t+\omega)) - F(t, \xi(t+\omega))\|^p dm_2(t) \\
&\qquad\quad + \frac{2^{p-1} L_F}{m_1([0,T])} \int_0^T \mathbb{E}\|\xi(t+\omega) - \xi(t)\|^p dm_2(t) \\
&\qquad \leqslant 2^p \varepsilon,
\end{aligned}$$

so $\zeta(\cdot) \in P\mathcal{S}AP_\omega\,(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$. ∎

3. (m_1, m_2)-S^p-pseudo $\mathcal{S}$-asymptotically ω-periodic processes

In this section, we incorporate a new class of functions called (m_1, m_2)-S^p-pseudo $\mathcal{S}$-asymptotically ω-periodic function, which generalize the concept of asymptotically periodic function.

Definition 3.1. *[4]*
i) The Bochner transform $\xi^b(t,s), t \in \mathbb{R}^+, s \in [0,1]$, of a stochastic process $\xi : \mathbb{R}^+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ is defined by

$$\xi^b(t,s) := \xi(t+s).$$

ii) The Bochner transform $F^b(t,s,x)$, $t \in \mathbb{R}^+$, $s \in [0,1]$, $x \in \mathbb{L}^p(\Omega, \mathbb{H})$, of a function $F : \mathbb{R}^+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, \mathbb{H})$ is defined by

$$F^b(t,s,x) := F(t+s, x)$$

for each $x \in \mathbb{L}^p(\Omega, \mathbb{H})$.
iii) The space $BS^p(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}))$ *of all Stepanov bounded stochastic process consists of all measurable stochastic process* $\xi : \mathbb{R}^+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ *such that* $\xi^b \in \mathbb{L}^\infty(\mathbb{R}^+, \mathbb{L}^p([0,1], \mathbb{L}^p(\Omega, \mathbb{H})))$. *This is a Banach space with the norm*

$$\|\xi\|_{S^p} = \|\xi^b\|_{\mathbb{L}^\infty(\mathbb{R}^+,\mathbb{L}^p)} = \sup_{t\in\mathbb{R}^+}\left(\int_0^1 \mathbb{E}\|\xi(t+s)\|^p ds\right)^{1/p} = \sup_{t\in\mathbb{R}^+}\left(\int_t^{t+1} \mathbb{E}\|\xi(\tau)\|^p d\tau\right)^{1/p}.$$

Definition 3.2. *Let* $m_1, m_2 \in \mathcal{P}_m$ *and* ξ *be a stochastic process in* $BS^p(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}))$. ξ *is called* (m_1, m_2)*-pseudo* $\mathcal{S}$*-asymptotically* ω*-periodic in the Stepanove sense if there exists* $\omega > 0$ *such that*

$$\lim_{T\to\infty} \frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|\xi(s+\omega) - \xi(s)\|^p ds\right)^{1/p} dm_2(t) = 0.$$

We denote the set of such functions by $S^pP\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$.

Lemma 3.3. *For* $m_1, m_2 \in \mathcal{P}_m$ *satisfying* (H_1). *As a result,*

$$P\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2) \subset S^pP\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2),\ 1 \leqslant p < \infty.$$

Proof. Let $\xi \in P\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$. Therefore, grounded on lemma 2.8,
$\xi(\cdot + s) \in P\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$ for $s \in [0.1]$. Referring to Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|\xi(s+\omega) - \xi(s)\|^p ds\right)^{1/p} dm_2(t)\\
&\quad\leqslant \frac{1}{m_1([0,T])}\left(\int_0^T \int_t^{t+1} \mathbb{E}\|\xi(s+\omega) - \xi(s)\|^p ds dm_2(t)\right)^{1/p}\left(\int_0^T dm_2(t)\right)^{1/q}\\
&\quad\leqslant \frac{m_2([0,T])^{1/q}}{m_1([0,T])^{1/q}}\left(\int_0^1 \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|\xi(t+s+\omega) - \xi(t+s)\|^p dm_2(t) ds\right)^{1/p},
\end{aligned}$$

and relying on (H_1), we obtain

$$\frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|\xi(s+\omega) - \xi(s)\|^p ds\right)^{1/p} dm_2(t) \leqslant \alpha^{1/q}\varepsilon^{1/p},$$

where $1/p + 1/q = 1$. Hence, $\xi \in S^pP\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$, which completes the proof.

■

Definition 3.4. *Let* $m_1, m_2 \in \mathcal{P}_m$ *and* $F \in BS^p(\mathbb{R}^+ \times \mathbb{L}^p(\Omega, \mathbb{H}), \mathbb{L}^p(\Omega, \mathbb{H}))$. F *is called uniformly* (m_1, m_2)*-pseudo* $\mathcal{S}$*-asymptotically* ω*-periodic on bounded sets in the Stepanov sense if for every bounded subset* $K \subset \mathbb{L}^p(\Omega, \mathbb{H})$, *there exists a positive function* $g_K \in BS^p(\mathbb{R}^+, \mathbb{R}^+)$ *such that, for* $t \in \mathbb{R}^+$, $\xi \in \mathbb{L}^p(\Omega, \mathbb{H})$, $\mathbb{E}\|F(t,\xi)\|^p \leqslant g_K(t)^p$ *and*

$$\lim_{T\to\infty} \frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \sup_{\|\xi\|\leqslant r} \mathbb{E}\|F(s+\omega,\xi) - F(s,\xi)\|^p ds\right)^{1/p} dm_2(t) = 0$$

for all $r > 0, s \geqslant 0$.
We denote the set of such functions by $S^pP\mathcal{S}AP_\omega(\mathbb{R}^+ \times \mathbb{L}^p(\Omega, \mathbb{H}), \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$.

Theorem 3.5. *Let* $F \in S^pP\mathcal{S}AP_\omega(\mathbb{R}^+ \times \mathbb{L}^p(\Omega, \mathbb{H}), \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$, $p \geqslant 1$ *satisfying the Lipschitz condition, i.e, if there exists a constant* $L_F > 0$ *such that*

$$\mathbb{E}\|F(t,\xi_1) - F(t,\xi_2)\|^p \leqslant L_F\mathbb{E}\|\xi_1 - \xi_2\|^p, \quad \xi_1, \xi_2 \in \mathbb{L}^p(\Omega, \mathbb{H}), t \in \mathbb{R}^+.$$

If $\xi \in S^pP\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$ *and* $\mathcal{R}(\xi)$ *is a bounded set, then*

$$\zeta(\cdot) = F(\cdot, \xi(\cdot)) \in S^pP\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2).$$

Proof. Assume that $F \in S^pPSAP_\omega(\mathbb{R}^+ \times \mathbb{L}^p(\Omega,\mathbb{H}),\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$. With reference on Definition 3.4, for $K = \mathcal{R}(\xi)$, there exists $g_K \in BS^p(\mathbb{R}^+,\mathbb{R}^+)$ such that $\mathbb{E}\|F(t,\xi)\|^p \leqslant g_K(t)^p$, $t \in \mathbb{R}^+, \xi \in K$. We have

$$\left(\int_t^{t+1} \mathbb{E}\|\zeta(s)\|^p ds\right)^{1/p} = \left(\int_t^{t+1} \mathbb{E}\|F(s,\xi(s))\|^p ds\right)^{1/p} \leqslant \left(\int_t^{t+1} g_K(s)^p ds\right)^{1/p} \leqslant \|g_K\|_{S^p}, \quad t \in \mathbb{R}^+.$$

Therefore $\zeta(\cdot) \in BS^p(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}))$. Additionally, for $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that

$$\frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \sup_{\|X\|\leqslant R} \mathbb{E}\|F(s+\omega,X) - F(s,X)\|^p ds\right)^{1/p} dm_2(t) \leqslant \varepsilon,$$
$$\frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|\xi(s+\omega) - \xi(s)\|^p ds\right)^{1/p} dm_2(t) \leqslant \varepsilon$$

for every $T \geqslant L_\varepsilon, R > 0$. Relying on the Minkowski inequality, for $T \geqslant L_\varepsilon$, we have

$$\begin{aligned}
&\frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|F(s+\omega,\xi(s+\omega)) - F(s,\xi(s))\|^p ds\right)^{1/p} dm_2(t)\\
&\leqslant \frac{2^{p-1}}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|F(s+\omega,\xi(s+\omega)) - F(s,\xi(s+\omega))\|^p ds\right)^{1/p} dm_2(t)\\
&+ \frac{2^{p-1}}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|F(s,\xi(s+\omega)) - F(s,\xi(s))\|^p ds\right)^{1/p} dm_2(t)\\
&\leqslant \frac{2^{p-1}}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \sup_{\|X\|\leqslant R} \mathbb{E}\|F(s+\omega,X) - F(s,X)\|^p ds\right)^{1/p} dm_2(t)\\
&+ \frac{2^{p-1}L_F}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|\xi(s+\omega) - \xi(s)\|^p ds\right)^{1/p} dm_2(t)\\
&\leqslant (1+L_F)\varepsilon.2^{p-1},
\end{aligned}$$

then $\zeta(\cdot) \in S^pPSAP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$. ∎

4. (m_1, m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic solution

The basic aim of this section is to investigate the existence and the uniqueness of (m_1,m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic solution for the following stochastic differential equation :

$$\begin{cases} d\xi(t) = A\xi(t)dt + F(t,\xi(t))dt + G(t,\xi(t))dB(t), & t \geqslant 0\\ \xi(0) = c_0, \end{cases} \tag{2}$$

where : $(B(t))_t$ corresponds to two-sided one-dimensional Brownian motion $\mathcal{F}_t$-adapted with value in $\mathbb{H}$, where $\mathcal{F}_t = \sigma\{B(u) - B(v)/u, v \leqslant t\}$ and $c_0 \in \mathbb{L}^p(\Omega,\mathbb{H})$.
To examine (2), the following assumptions are considered :
(H_3) $A : D(A) \subset \mathbb{L}^p(\Omega,\mathbb{H}) \to \mathbb{L}^p(\Omega,\mathbb{H})$ refers to the infinitesimal generator of an exponentially stable C_0-semi-group $(S_g(t))_{t\geqslant 0}$ such that there exist constants $M > 0$ and $\theta > 0$ with

$$\|S_g(t)\| \leqslant Me^{-\theta t}, \quad t \geqslant 0.$$

(H_4) $F \in S^pPSAP_\omega(\mathbb{R}^+ \times \mathbb{L}^p(\Omega,\mathbb{H}),\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$, $m_1,m_2 \in \mathcal{P}_m$, and satisfies the Lipschitz condition, i.e, if there exists a constant $L_F > 0$ such that

$$\mathbb{E}\|F(t,\xi_1) - F(t,\xi_2)\|^p \leqslant L_F\mathbb{E}\|\xi_1 - \xi_2\|^p, \quad \xi_1,\xi_2 \in \mathbb{L}^p(\Omega,\mathbb{H}), t \in \mathbb{R}^+.$$

(H_5) $G \in S^pPSAP_\omega(\mathbb{R}^+ \times \mathbb{L}^p(\Omega, \mathbb{H}), \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$, $m_1, m_2 \in \mathcal{P}_m$, and satisfies the Lipschitz condition, i.e, if there exists a constant $L_G > 0$ such that

$$\mathbb{E}\|G(t,\xi_1) - G(t,\xi_2)\|^p \leqslant L_G \mathbb{E}\|\xi_1 - \xi_2\|^p, \quad \xi_1, \xi_2 \in \mathbb{L}^p(\Omega, \mathbb{H}), t \in \mathbb{R}^+.$$

(H_6) $\qquad \varpi := \int_0^\infty e^{-p\theta t} dm_2(t) < \infty.$

Definition 4.1. *Let $\{\xi(t), t \geqslant 0\}$ be a $\mathcal{F}_t$-progressively measurable stochastic process. $\xi(t)$ is said to be a mild solution of equation* (2), *if it satisfies the following stochastic integral equation for $t \in \mathbb{R}^+$:*

$$\xi(t) = S_g(t)c_0 + \int_0^t S_g(t-s)F(s,\xi(s))ds + \int_0^t S_g(t-s)G(s,\xi(s))dB(s).$$

Remark:

We recall a function $u \in C(\mathbb{R}^+, L^p(\Omega, H)$ is called a mild solution of equation (4.1), if it satisfies the equation (4.1), that's to say:

$$du(t) = Au(t) + F(t, u(t))dt + G(t, u(t))dB(t), \ t \geq 0.$$

Let $\xi \in C(\mathbb{R}^+, L^p(\Omega, H)$ a mild solution of equation (4.1), we pose:
$h(s) = S_g(t-s)\xi(s)$. Then for all $t \in \mathbb{R}^+$, h is of class C^1 on $[0,t]$, and for all $s \in [0,t]$ we have:
$\frac{dh}{ds}(s) = -AS_g(t-s)\xi(s) + S_g(t-s)\frac{d}{ds}\xi(s)$.
Since ξ is the mild solution for equation (4.1), then we have:

$$\begin{aligned} \frac{dh}{ds}(s) &= -AS_g(t-s)\xi(s) + S_g(t-s)[A\xi(s) + F(s,\xi(s)) + G(s,\xi(s))\frac{dB(s)}{ds}] \\ &= S_g(t-s)[F(s,\xi(s)) + G(s,\xi(s))\frac{dB(s)}{ds}]. \end{aligned}$$

We integrate on $[0,t]$, then we obtain:

$$\begin{aligned} h(t) - h(0) &= \xi(t) - S_g(t)c_0 \\ &= \int_0^t S_g(t-s)[F(s,\xi(s)) + G(s,\xi(s))\frac{dB(s)}{ds}]ds \\ &= \int_0^t S_g(t-s)F(s,\xi(s))ds + \int_0^t S_g(t-s)G(s,\xi(s))dB(s). \end{aligned}$$

Therefore, we deduce the Definition 4.1:

$$\xi(t) = S_g(t)c_0 + \int_0^t S_g(t-s)F(s,\xi(s))ds + \int_0^t S_g(t-s)G(s,\xi(s))dB(s).$$

In order to demonstrate the relevance of our results, we need the following lemmas :

Lemma 4.2. *[17] Let $\varphi : [0,T] \times \Omega \to l(\mathbb{L}^p(\Omega, \mathbb{H}))$ be an $\mathcal{F}_t$−adapted measurable stochastic process satisfying*

$$\int_0^T \mathbb{E}\|\varphi(t)\|^2 dt < \infty \quad a.s,$$

where $l(\mathbb{L}^p(\Omega, \mathbb{H}))$ stands for the space of all continuous linear operators from $\mathbb{L}^p(\Omega, \mathbb{H})$ to itself. From this perspective, $\forall\ p \geqslant 1$ and there exists a constant $C_p > 0$ such that

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^T \varphi(s) dB(s) \right\|^p \leqslant C_p \mathbb{E} \left(\int_0^T \|\varphi(s)\|^2 ds \right)^{p/2}, \quad T > 0.$$

Lemma 4.3. *Assuming that* $(H_3), (H_6)$ *hold, if* $\phi \in S^pP\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$, $m_1, m_2 \in \mathcal{P}_m$, *then*

$$(\Lambda_1\phi)(t) = \int_0^t S_g(t-s)\phi(s)ds$$

lies in $P\mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$.

Proof. For $t \in [n, n+1]$, we get

$$\begin{aligned}
\mathbb{E}\|(\Lambda_1\phi)(t)\|^p &\leqslant \mathbb{E}\left[\int_0^t Me^{-\theta(t-s)}\|\phi(s)\|ds\right]^p \\
&\leqslant \mathbb{E}\left[\sum_{k=0}^{n}\int_k^{k+1} Me^{-\theta(n-s)}\|\phi(s)\|ds\right]^p \\
&\leqslant \left[\sum_{k=0}^{n} Me^{-\theta(n-k-1)}\right]^p \int_k^{k+1} \mathbb{E}\|\phi(s)\|^p ds \\
&\leqslant \left(\frac{Me^{\theta}}{1-e^{-\theta}}\right)^p \|\phi\|_{S^p}^p.
\end{aligned}$$

Hence, $\Lambda_1\phi$ is bounded. In the same regard, we state that

$$\begin{aligned}
\mathbb{E}\|(\Lambda_1\phi)(t+\varepsilon) - (\Lambda_1\phi)(t)\|^p &= \mathbb{E}\left\|\int_0^{t+\varepsilon} S_g(t+\varepsilon-s)\phi(s)ds - \int_0^t S_g(t-s)\phi(s)ds\right\|^p \\
&= \mathbb{E}\left\|\int_0^{\varepsilon} S_g(t+\varepsilon-s)\phi(s)ds + \int_0^t S_g(t-s)[\phi(s+\varepsilon)-\phi(s)]ds\right\|^p \\
&\leqslant 2^{p-1}\mathbb{E}\left(\int_0^{\varepsilon} \|S_g(t+\varepsilon-s)\|\|\phi(s)\|ds\right)^p \\
&+ 2^{p-1}\mathbb{E}\left(\int_0^t Me^{-\theta(t-s)}\|\phi(s+\varepsilon)-\phi(s)\|ds\right)^p \\
&\leqslant 2^{p-1}\mathbb{E}\left(\int_0^{\varepsilon} \|S_g(t+\varepsilon-s)\|\|\phi(s)\|ds\right)^p \\
&+ 2^{p-1}M^p\left(\int_0^t e^{-q\theta(t-s)}ds\right)^{\frac{p}{q}} \times \int_0^t \mathbb{E}\|\phi(s+\varepsilon)-\phi(s)\|^p ds \\
&\to 0, \quad \varepsilon \to 0.
\end{aligned}$$

Therefore, $\Lambda_1\phi \in C_b(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}))$. Further more,

$$\begin{aligned}
\mathbb{E}\|(\Lambda_1\phi)(t+\omega) - (\Lambda_1\phi)(t)\|^p &= \mathbb{E}\left\|\int_0^{\omega} S_g(t+\omega-s)\phi(s)ds + \int_0^t S_g(t-s)[\phi(s+\omega)-\phi(s)]ds\right\|^p \\
&= \mathbb{E}\|I(t)+J(t)\|^p \leqslant 2^{p-1}\mathbb{E}\|I(t)\|^p + 2^{p-1}\mathbb{E}\|J(t)\|^p,
\end{aligned}$$

where

$$\|I(t)\|^p = \left\|\int_0^{\omega} S_g(t+\omega-s)\phi(s)ds\right\|^p, \quad \|J(t)\|^p = \left\|\int_0^t S_g(t-s)[\phi(s+\omega)-\phi(s)]ds\right\|^p.$$

As

$$\begin{aligned}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|I(t)\|^p dm_2(t) &\leqslant \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\left\|\int_0^\omega S_g(t+\omega-s)\phi(s)ds\right\|^p dm_2(t)\\ &\leqslant \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\left(\int_0^\omega Me^{-\theta(t+\omega-s)}\|\phi(s)\|ds\right)^p dm_2(t)\\ &\leqslant \frac{M^p e^{-p\theta\omega}}{m_1([0,T])}\int_0^T e^{-p\theta t}\mathbb{E}\left(\int_0^\omega e^{\theta s}\|\phi(s)\|ds\right)^p dm_2(t)\\ &\leqslant \frac{\varpi M^p e^{-p\theta\omega}}{m_1([0,T])}\mathbb{E}\left(\int_0^\omega e^{\theta s}\|\phi(s)\|ds\right)^p \to 0, \quad T\to\infty,\end{aligned}$$

then

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|I(t)\|^p dm_2(t) = 0. \tag{3}$$

In the same respect, since $\phi \in S^pPSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2)$, there exists $\varepsilon > 0, l \in \mathbb{N}$ such that

$$\frac{1}{m_1([0,T])}\int_0^T\left(\int_t^{t+1}\mathbb{E}\|\phi(s+\omega)-\phi(s)\|^p ds\right)^{1/p} dm_2(t) < \varepsilon \quad \text{for } T \geqslant l.$$

Since $l \leqslant n \leqslant T \leqslant n+1$, then $0 \leqslant t \leqslant T \leqslant n+1$. Let $K > 0$ be a constant such that $\mathbb{E}\|\phi(t+\omega)-\phi(t)\|^p \leqslant K$ for all $t \geqslant 0$. We therefore have

$$\begin{aligned}&\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|J(t)\|^p dm_2(t) \leqslant \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\left(\int_0^t \|S_g(s)\|\|\phi(t-s+\omega)-\phi(t-s)\|ds\right)^p dm_2(t)\\ &\leqslant \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\left(\int_0^t \|S_g(s)\|^{\frac{p-1}{p}}\|S_g(s)\|^{\frac{1}{p}}\|\phi(t-s+\omega)-\phi(t-s)\|ds\right)^p dm_2(t)\\ &\leqslant \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\left(\left[\int_0^t \|S_g(s)\|ds\right]^{\frac{p-1}{p}}\times\left[\int_0^t\left(\|S_g(s)\|^{\frac{1}{p}}\|\phi(t-s+\omega)-\phi(t-s)\|\right)^p ds\right]^{\frac{1}{p}}\right)^p dm_2(t)\\ &\leqslant \frac{1}{m_1([0,T])}\int_0^T \left[\int_0^t Me^{-\theta s}ds\right]^{p-1}\times\int_0^t Me^{-\theta s}\mathbb{E}\|\phi(t-s+\omega)-\phi(t-s)\|^p ds dm_2(t)\\ &\leqslant \frac{1}{m_1([0,T])}\int_0^T \left[\int_0^{n+1} Me^{-\theta s}ds\right]^{p-1}\times\int_0^{n+1} Me^{-\theta s}\mathbb{E}\|\phi(t-s+\omega)-\phi(t-s)\|^p ds dm_2(t)\\ &\leqslant \frac{1}{m_1([0,T])}\int_0^T \left[\int_0^{n+1} Me^{-\theta s}ds\right]^{p-1}\times\sum_{k=0}^n\int_k^{k+1} Me^{-\theta s}\mathbb{E}\|\phi(t-s+\omega)-\phi(t-s)\|^p ds dm_2(t)\\ &\leqslant \frac{1}{m_1([0,T])}\int_0^T \left[\int_0^{n+1} Me^{-\theta s}ds\right]^{p-1}\times\int_0^1\sum_{k=0}^n Me^{-\theta k}\mathbb{E}\|\phi(t-s-k+\omega)-\phi(t-s-k)\|^p ds dm_2(t)\\ &\leqslant \left[\int_0^{n+1} Me^{-\theta s}ds\right]^{p-1}\times\sum_{k=0}^n Me^{-\theta k}\frac{1}{m_1([0,T])}\int_0^T\int_0^1 \mathbb{E}\|\phi(t-s-k+\omega)-\phi(t-s-k)\|^p ds dm_2(t)\\ &\leqslant \left[\int_0^{n+1} Me^{-\theta s}ds\right]^{p-1}\times\sum_{k=0}^n Me^{-\theta k}\frac{K^{\frac{p-1}{p}}}{m_1([0,T])}\int_0^T\left(\int_{t-k-1}^{t-k}\mathbb{E}\|\phi(s+\omega)-\phi(s)\|^p ds\right)^{1/p} dm_2(t)\\ &\leqslant \frac{1}{\theta^{p-1}}\cdot\frac{M^pK^{\frac{p-1}{p}}}{1-e^{-\theta}}\varepsilon.\end{aligned}$$

Thus,

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|J(t)\|^p dm_2(t)=0. \tag{4}$$

Referring to (3), (4), we have

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|(\Lambda_1\phi)(t+\omega)-(\Lambda_1\phi)(t)\|^p dm_2(t)=0.$$

Hence, $\Lambda_1\phi \in P\mathcal{S}AP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$. The proof is therefore complete. ■

Lemma 4.4. *Assume that* $(H_3),(H_6)$ *hold, if* $\phi \in S^pP\mathcal{S}AP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$, $m_1,m_2\in\mathcal{P}_m$. *As a matter of fact,*

$$(\Lambda_2\phi)(t)=\int_0^t S_g(t-s)\phi(s)dB(s)$$

lies in $P\mathcal{S}AP_\omega(\mathbb{R}^+,\mathbb{L}^p(\Omega,\mathbb{H}),m_1,m_2)$.

Proof. For $t\in[n,n+1], n\in\mathbb{N}$ and according to Lemma 4.2, we obtain

$$\begin{aligned}
\mathbb{E}\|(\Lambda_2\phi)(t)\|^p &= \mathbb{E}\left\|\int_0^t S_g(t-s)\phi(s)dB(s)\right\|^p\\
&\leqslant C_p\mathbb{E}\left(\int_0^t \|S_g(t-s)\|^2\|\phi(s)\|^2ds\right)^{p/2}\\
&\leqslant C_p\mathbb{E}\left(\sum_{k=0}^n M^2e^{-2\theta(n-k-1)}\int_k^{k+1}\|\phi(s)\|^2ds\right)^{p/2}\\
&\leqslant \frac{C_pM^pe^{p\theta}}{(1-e^{-2\theta})^{p/2}}\|\phi\|^p_{S^2}.
\end{aligned}$$

Thus, $\Lambda_2\phi$ is bounded. Besides, let $s'=s-\varepsilon$ and $\tilde{B}(s')=B(s'+\varepsilon)-B(\varepsilon)$. We therefore have

$$\begin{aligned}
&\mathbb{E}\|(\Lambda_2\phi)(t+\varepsilon)-(\Lambda_2\phi)(t)\|^p =\\
&=\mathbb{E}\|\int_0^{t+\varepsilon} S_g(t+\varepsilon-s)\phi(s)dB(s)-\int_0^t S_g(t-s)\phi(s)dB(s)\|^p\\
&=\mathbb{E}\|\int_0^{\varepsilon} S_g(t+\varepsilon-s)\phi(s)dB(s)+\int_\varepsilon^{t+\varepsilon} S_g(t+\varepsilon-s)\phi(s)dB(s)-\int_0^t S_g(t-s)\phi(s)dB(s)\|^p\\
&=\mathbb{E}\|\int_0^{\varepsilon} S_g(t+\varepsilon-s)\phi(s)dB(s)+\int_0^{t} S_g(t-s')\phi(s'+\varepsilon)dB(s'+\varepsilon)-\int_0^t S_g(t-s)\phi(s)dB(s)\|^p\\
&=\mathbb{E}\|\int_0^{\varepsilon} S_g(t+\varepsilon-s)\phi(s)dB(s)+\int_0^{t} S_g(t-s)\phi(s+\varepsilon)d\tilde{B}(s)-\int_0^t S_g(t-s)\phi(s)d\tilde{B}(s)\|^p\\
&=\mathbb{E}\|\int_0^{\varepsilon} S_g(t+\varepsilon-s)\phi(s)dB(s)+\int_0^{t} S_g(t-s)[\phi(s+\varepsilon)-\phi(s)]d\tilde{B}(s)\|^p\\
&\leqslant C_p2^{p-1}\mathbb{E}\left(\int_0^\varepsilon \|S_g(t+\varepsilon-s)\|^2\|\phi(s)\|^2ds\right)^{p/2}+C_p2^{p-1}\mathbb{E}\left(\int_0^t M^2e^{-2\theta(t-s)}\|\phi(s+\varepsilon)-\phi(s)\|^2ds\right)^{p/2}\\
&\leqslant C_p2^{p-1}\mathbb{E}\left(\int_0^\varepsilon \|S_g(t+\varepsilon-s)\|^2\|\phi(s)\|^2ds\right)^{p/2}\\
&\qquad +M^pC_p2^{p-1}\left(\int_0^t e^{-2\theta(t-s)}ds\right)^{\frac{p-2}{2}}\times\int_0^t e^{-2\theta(t-s)}\mathbb{E}\|\phi(s+\varepsilon)-\phi(s)\|^pds \qquad \to 0,\varepsilon\to 0.
\end{aligned}$$

As a result, $\Lambda_2\phi \in C_b(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}))$. In addition, we obtain

$$(\Lambda_2\phi)(t+\omega) - (\Lambda_2\phi)(t) = \int_0^{t+\omega} S_g(t+\omega-s)\phi(s)dB(s) - \int_0^t S_g(t-s)\phi(s)dB(s) = I'(t) + J'(t),$$

where

$$I'(t) = \int_0^{\omega} S_g(t+\omega-s)\phi(s)dB(s), \qquad J'(t) = \int_0^t S_g(t-s)[\phi(s+\omega)-\phi(s)]d\tilde{B}(s).$$

As

$$\begin{aligned}
\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|I'(t)\|^p dm_2(t) &\leqslant \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|\int_0^{\omega} S_g(t+\omega-s)\phi(s)dB(s)\|^p dm_2(t)\\
&\leqslant \frac{C_p}{m_1([0,T])}\int_0^T \mathbb{E}\left(\int_0^{\omega} M^2 e^{-2\theta(t+\omega-s)}\|\phi(s)\|^2 ds\right)^{p/2} dm_2(t)\\
&\leqslant \frac{M^p C_p e^{-p\theta\omega}}{m_1([0,T])}\int_0^T e^{-p\theta t}\mathbb{E}\left(\int_0^{\omega} e^{2\theta s}\|\phi(s)\|^2 ds\right)^{p/2} dm_2(t)\\
&\leqslant \frac{\varpi C_p M^p e^{-p\theta\omega}}{m_1([0,T])}\mathbb{E}\left(\int_0^{\omega} e^{2\theta s}\|\phi(s)\|^2 ds\right)^{p/2} \to 0, \quad T\to\infty,
\end{aligned}$$

then

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|I'(t)\|^p dm_2(t) = 0. \tag{5}$$

In this vein, as $\phi \in S^pP\mathcal{S}AP_{\omega}(\mathbb{R}^+, \mathbb{L}^p(\Omega, \mathbb{H}), m_1, m_2)$, there exists $\varepsilon > 0, l \in \mathbb{N}$, such that

$$\frac{1}{m_1([0,T])}\int_0^T \left(\int_t^{t+1} \mathbb{E}\|\phi(s+\omega)-\phi(s)\|^p ds\right)^{1/p} dm_2(t) < \varepsilon \quad \text{ for } T \geqslant l.$$

Since $l \leqslant n \leqslant T \leqslant n+1$, then $0 \leqslant t \leqslant T \leqslant n+1$. Let $K > 0$ be a constant such that $\mathbb{E}\|\phi(t+\omega)-\phi(t)\|^p \leqslant K$ for all $t \geqslant 0$. It follows that

$$\begin{aligned}
&\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|J'(t)\|^p dm_2(t)\\
&\leqslant \frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|\int_0^t S_g(t-s)[\phi(s+\omega)-\phi(s)]d\tilde{B}(s)\|^p dm_2(t)\\
&\leqslant \frac{C_p}{m_1([0,T])}\int_0^T \mathbb{E}\left(\int_0^t \|S_g(s)\|^2\|\phi(t-s+\omega)-\phi(t-s)\|^2 ds\right)^{\frac{p}{2}} dm_2(t)
\end{aligned}$$

$$\leqslant \frac{C_p}{m_1([0,T])}\int_0^T\left(\int_0^t \|S_g(s)\|^2 ds\right)^{\frac{p-2}{2}} \times \int_0^t \|S_g(s)\|^2 \mathbb{E}\|\phi(t-s+\omega)-\phi(t-s)\|^p ds dm_2(t)$$
$$\leqslant \frac{C_p}{m_1([0,T])}\int_0^T\left(\int_0^{n+1} M^2 e^{-2\theta s} ds\right)^{\frac{p-2}{2}} \times \int_0^{n+1} M^2 e^{-2\theta s}\mathbb{E}\|\phi(t-s+\omega)-\phi(t-s)\|^p ds dm_2(t)$$
$$\leqslant \frac{M^2 C_p}{m_1([0,T])}\int_0^T\left(\int_0^{n+1} M^2 e^{-2\theta s} ds\right)^{\frac{p-2}{2}} \times \sum_{k=0}^{n}\int_k^{k+1} e^{-2\theta s}\mathbb{E}\|\phi(t-s+\omega)-\phi(t-s)\|^p ds dm_2(t)$$
$$\leqslant \frac{M^2 C_p}{m_1([0,T])}\int_0^T\left(\int_0^{n+1} M^2 e^{-2\theta s} ds\right)^{\frac{p-2}{2}} \times \int_0^{1}\sum_{k=0}^{n} e^{-2\theta k}\mathbb{E}\|\phi(t-s-k+\omega)-\phi(t-s-k)\|^p ds dm_2(t)$$
$$\leqslant \left(\int_0^{n+1} M^2 e^{-2\theta s} ds\right)^{\frac{p-2}{2}} \times \sum_{k=0}^{n} e^{-2\theta k}\frac{M^2 C_p K^{\frac{p-1}{p}}}{m_1([0,T])}\int_0^T\left(\int_{t-k-1}^{t-k}\mathbb{E}\|\phi(s+\omega)-\phi(s)\|^p ds\right)^{\frac{1}{p}} dm_2(t)$$
$$\leqslant \frac{1}{(2\theta)^{\frac{p-2}{2}}}.\frac{M^p C_p K^{\frac{p-1}{p}}}{1-e^{-2\theta}}\varepsilon,$$

so

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|J'(t)\|^p dm_2(t) = 0. \tag{6}$$

With reference to (5), (6), we get

$$\lim_{T\to\infty}\frac{1}{m_1([0,T])}\int_0^T \mathbb{E}\|(\Lambda_2\phi)(t+\omega)-(\Lambda_2\phi)(t)\|^p dm_2(t) = 0.$$

Thus, $\Lambda_2\phi \in PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2)$. ∎

Theorem 4.5. *Let $m_1, m_2 \in \mathcal{P}_m$ satisfying (H_1), (H_2). Assuming that $(H_1), \ldots, (H_6)$ hold, then if*

$$2^{p-1}M^p\left(\frac{L_F}{\theta^p}+\frac{C_p L_G}{(2\theta)^{\frac{p}{2}}}\right) < 1,$$

(2) has a unique solution $\xi(t) \in PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2)$.

Proof. Define the operator $\mathcal{F} : PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2) \to PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2)$ by

$$(\mathcal{F}\xi)(t) = S_g(t)c_0 + \int_0^t S_g(t-s)F(s,\xi(s))ds + \int_0^t S_g(t-s)G(s,\xi(s))dB(s).$$

For $\xi(t) \in PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2)$, resting upon Theorems 2.9, 3.5 and Lemma 3.3, we get $F(\cdot,\xi(\cdot)), G(\cdot,\xi(\cdot)) \in S^p PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2)$. Thus, $\mathcal{F}$ is well defined by Lemmas 4.3 and 4.4. In the same line, assuming that $\xi_1, \xi_2 \in PSAP_\omega(\mathbb{R}^+, \mathbb{L}^p(\Omega,\mathbb{H}), m_1, m_2)$, it follows that

$$\begin{aligned}\mathbb{E}\|(\mathcal{F}\xi_1)(t)-(\mathcal{F}\xi_2)(t)\| &\leqslant 2^{p-1}\mathbb{E}\|\int_0^t S_g(t-s)[F(s,\xi_1(s))-F(s,\xi_2(s))]ds\|^p \\ &\quad + 2^{p-1}\mathbb{E}\|\int_0^t S_g(t-s)[G(s,\xi_1(s))-G(s,\xi_2(s))]dB(s)\|^p \\ &\leqslant 2^{p-1}[I_1+I_2].\end{aligned}$$

Firstly, departing from Lipschitz conditions and Holder's inequality, we get

$$\begin{aligned}
I_1 &= \mathbb{E}\|\int_0^t S_g(t-s)[F(s,\xi_1(s)) - F(s,\xi_2(s))]ds\|^p \\
&\leqslant \mathbb{E}\left(\int_0^t \|S_g(t-s)\|^{\frac{p-1}{p}}\|S_g(t-s)\|^{\frac{1}{p}}\|F(s,\xi_1(s)) - F(s,\xi_2(s))\|ds\right)^p \\
&\leqslant \mathbb{E}\left[\left(\int_0^t \left(\|S_g(t-s)\|^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}} \times \left(\int_0^t \left(\|S_g(t-s)\|^{\frac{1}{p}}\|F(s,\xi_1(s)) - F(s,\xi_2(s))\|\right)^p ds\right)^{\frac{1}{p}}\right]^p \\
&\leqslant M^{p-1}\left(\int_0^t e^{-\theta(t-s)}ds\right)^{p-1} \times ML_F\int_0^t e^{-\theta(t-s)}\mathbb{E}\|\xi_1(s) - \xi_2(s)\|^p ds \\
&\leqslant \frac{M^p}{\theta^p}L_F \sup_{t\in\mathbb{R}^+}\mathbb{E}\|\xi_1(t) - \xi_2(t)\|^p,
\end{aligned}$$

so

$$I_1 \leqslant \frac{M^p}{\theta^p}L_F \sup_{t\in\mathbb{R}^+}\mathbb{E}\|\xi_1(s) - \xi_2(s)\|^p. \tag{7}$$

On the other side, relying on Lemma 4.2, Holder's inequality, and the Lipschitz condition, we obtain

$$\begin{aligned}
I_2 &= \mathbb{E}\|\int_0^t S_g(t-s)[G(s,\xi_1(s)) - G(s,\xi_2(s))]dB(s)\|^p \\
&\leqslant C_p\mathbb{E}\left(\int_0^t \|S_g(t-s)\|^2\|G(s,\xi_1(s)) - G(s,\xi_2(s))\|^2 ds\right)^{\frac{p}{2}} \\
&\leqslant C_p\left(\int_0^t \|S_g(t-s)\|^2 ds\right)^{\frac{p-2}{2}} \times \int_0^t \|S_g(t-s)\|^2\mathbb{E}\|G(s,\xi_1(s)) - G(s,\xi_2(s))\|^p ds \\
&\leqslant C_p M^{p-2}\left(\int_0^t e^{-2\theta(t-s)}ds\right)^{\frac{p-2}{2}} \times M^2\int_0^t e^{-2\theta(t-s)}\mathbb{E}\|G(s,\xi_1(s)) - G(s,\xi_2(s))\|^p ds \\
&\leqslant C_p\frac{M^p}{(2\theta)^{\frac{p}{2}}}.L_G \sup_{t\in\mathbb{R}^+}\mathbb{E}\|\xi_1(t) - \xi_2(t)\|^p,
\end{aligned}$$

so

$$I_2 \leqslant C_p\frac{M^p}{(2\theta)^{\frac{p}{2}}}.L_G \sup_{t\in\mathbb{R}^+}\mathbb{E}\|\xi_1(t) - \xi_2(t)\|^p. \tag{8}$$

Grounded on (7), (8), we get

$$\mathbb{E}\|(\mathcal{F}\xi_1)(t) - (\mathcal{F}\xi_2)(t)\|^p \leqslant 2^{p-1}M^p\left(\frac{L_F}{\theta^p} + \frac{C_pL_G}{(2\theta)^{\frac{p}{2}}}\right)\|\xi_1 - \xi_2\|_\infty^p.$$

As a result, $\mathcal{F}$ has a unique fixed point in $PSAP_\omega(\mathbb{R}^+, \mathbb{L}(\Omega,\mathbb{H}), m_1, m_2)$. Therefore, based on the Banach fixed point theorem, equation (2) has a unique solution in $PSAP_\omega(\mathbb{R}^+, \mathbb{L}(\Omega,\mathbb{H}), m_1, m_2)$.

■

5. Application

Let's consider the following equation

$$\begin{cases} d\xi(t,x) = \frac{\partial^2}{\partial^2 x}\xi(t,x)dt + F(t,\xi(t,x))dt + G(t,\xi(t,x))dB(t), \\ (t,x) \in \mathbb{R}^+ \times [0,1], \\ \xi(t,0) = \xi(t,1) = 0 \text{ for } t \in \mathbb{R}^+. \end{cases} \tag{9}$$

Suppose that m_1 is the Lebesgue measure and m_2 is a positive measure, where its Radon-Nkodym derivative is

$$\varrho(t) = \begin{cases} e^t & \text{if } t \leqslant 0 \\ 1 & \text{if } t > 0. \end{cases}$$

Hence, with reference to [3], m_1 and m_2 satisfy (H_1) and (H_2). In order to write (9) in the same way as (2), the following linear operator is considered

$$A : D(A) \subset \mathbb{L}^2(0,1) \to \mathbb{L}^2(0,1).$$

It is provided by

$$D(A) = \Big\{\xi \text{ continuous } / \xi' \text{ absolutely continuous on } [0,1], \xi'' \in \mathbb{L}^2(0,1) \text{ and } \xi(0) = \xi(1) = 0\Big\},$$

$$A\xi = \xi'' \text{ for all } \xi \in D(A).$$

It is well known that A produces a C_0 semi-group $(S_g(t))_{t\geqslant 0}$ such that $\|S_g(t)\| \leqslant e^{-\theta t}$ for $t, \theta \geqslant 0$. Let

$$F(t,\xi) = (\sin t + \sin 2\pi\sqrt{2}t)\xi,$$

$$G(t,\xi) = (\sin 2t + \sin t)\xi.$$

We have $F, G \in P\mathcal{S}AP_\omega(\mathbb{R}^+ \times \mathbb{L}^p(\Omega, \mathbb{L}^2(0,1)), \mathbb{L}^p(\Omega, \mathbb{L}^2(0,1)), m_1, m_2)$. It is simple to check that F and G satisfy the Lipschitz conditions in Theorem 4.5, where $M = 1$, $L_F = 2^p$, and $L_G = 2^p$. Departing from Theorem 4.5, we infer that equation (9) has a unique (m_1, m_2)-pseudo $\mathcal{S}$-asymptotically ω-periodic mild solution. ∎

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