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Design and connection of parametric surfaces through given regular curves

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Abstract. This study focused on the connections between two parametric surfaces through the given regular curves. Accordingly, we analyzed a C^0-continuous connected surface in terms of the marching-scale functions of these surfaces. It should be noted that, in general, differentiability along a common curve for a C^0-continuously connected surface is not guaranteed. To solve this problem, we introduced a C^1-continuous connection and proved its existence for such continuous connections. These connections are improved versions of the G^1-connection of developable surfaces introduced by Paluszny. Moreover, we suggested applications to illustrate the C^1-continuous connection using Bézier curves and some marching-scale functions of the parametric surfaces.

1. Introduction

Surface construction from a given curve is of significant interest in computational and applied mathematics. In recent years, surface construction from one or more curves has garnered attention from several researchers [1, 3, 7, 9, 10, 14, 15, 18]. Wang et al. [17] proposed the surface interpolation of a curve as an isoparametric geodesic for surface construction. In particular, they presented the surface family in terms of marching-scale functions and the Frenet frames of a given curve, and studied the construction of surfaces that possess this curve as a geodesic. Subsequently, many mathematicians have studied parametric surface families through a given curve. Zhao and Wang [19] used the Frenet formulas to characterise developable surfaces through a given space curve (specifically in special cases where the curve is a geodesic on the surface) and presented the results to design developable surfaces. Kasap et al. [8] analyzed the problem of constructing parametric surface from a given geodesic curve, as demonstrated in [17], and considered more general marching-scale functions represented by factor decomposition. In addition, Sánchez-Reyes [16] introduced methods for constructing minimal surfaces from a prescribed geodesic to solve Plateau's problem. Moreover, Bayram et al. [1] studied the problem of finding a surface family from a given spatial asymptotic curve; Li, Wang, and Zhu [12] derived the sufficient condition for a given curve as a line of curvature of the surface under general expressions of the marching-scale functions.

A developable surface can be formed by bending or rolling a planar surface without stretching or tearing it. In other words, it can be isometrically developed or unrolled onto a plane. Developable surfaces are widely used in materials that are not amenable to stretching; their applications include the

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formation of ship hulls, ducts, shoes, clothing, and automobile parts, such as upholstery, body panels, and windshields [4]. Various studies have analysed designs with developable surfaces [11, 14, 19]. Hu, Wu, and Wang [6] presented a class of methods for constructing locally controlled developable H-Bézier surfaces through a given characteristic curve, and further proposed a novel method for constructing developable surfaces using generalised C-Bézier bases with shape parameters [5]. Moreover, the problem of designing developable surfaces with Bézier patches was addressed in [2]. Zhao and Wang [19] proposed a new method for designing developable surfaces by constructing a surface pencil through a given curve as a geodesic. However, only one developable surface is not sufficient for practical applications. Therefore, a connection of two developable surfaces given abutting two curves must be considered. Paluszny [14] studied the G^1 joining of surface patches along a common pregeodesic and discussed the G^1 connection between developable surfaces along common rulings. The G^1 connection between two developable surfaces containing G^1 abutting pregeodesics may be useful in practical applications because it offers a wider family of G^1 patches containing G^1 two curves as a piecewise pregeodesic. Moreover, Li et al. [11] proposed a G^1 connection for developable surfaces through cubic Bézier curves as a geodesic. It should be noted that Paluszny and Li et al. only dealt with the connection between two developable surfaces abutting the curves and rulings; however, there may exist other surface connections through the curves on two parametric surfaces. Accordingly, we introduce the following problem: *Which method can be used to connect any two parametric surfaces through given regular curves, and can such surfaces be constructed and designed using this method?*

In this paper, we discuss the aforementioned problem and provide a partial solution. The remainder of this paper is organised as follows. In Section 2, we briefly review parametric surfaces through given curves. In Section 3, we introduce two surface connections, namely, C^0 and C^1-continuous connections; moreover, we define the essential and degenerate functions to describe these connections. In the last section, we present a method for designing the C^1-connection of surfaces through the given Bézier curves.

2. Parametric surface through a given curve

Let α be a spatially regular curve with parametric r. It is well known that the Frenet frames of the curve, $\alpha(r)$, are defined by:

$$\begin{aligned} \mathbf{T}(r) &= \frac{\dot\alpha(r)}{\|\dot\alpha(r)\|}, \\ \mathbf{B}(r) &= \frac{\dot\alpha(r)\times\ddot\alpha(r)}{\|\dot\alpha(r)\times\ddot\alpha(r)\|}, \\ \mathbf{N}(r) &= \mathbf{B}(r)\times\mathbf{T}(r). \end{aligned} \tag{1}$$

Additionally, the curvature, κ, and torsion, τ, are given by:

$$\kappa(r) = \frac{\|\dot\alpha(r)\times\ddot\alpha(r)\|}{\|\dot\alpha(r)\|^3}, \quad \tau(r) = \frac{<\dot\alpha(r)\times\ddot\alpha(r), \dddot\alpha(r)>}{\|\dot\alpha(r)\times\ddot\alpha(r)\|^2}. \tag{2}$$

In contrast, the Frenet formula of a frame, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, for a regular curve, $\alpha(r)$, satisfies the following relationship:

$$\frac{d}{dr}\begin{pmatrix} \mathbf{T}(r) \\ \mathbf{N}(r) \\ \mathbf{B}(r) \end{pmatrix} = \begin{pmatrix} 0 & v(r)\kappa(r) & 0 \\ -v(r)\kappa(r) & 0 & v(r)\tau(r) \\ 0 & -v(r)\tau(r) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{T}(r) \\ \mathbf{N}(r) \\ \mathbf{B}(r) \end{pmatrix},$$

where $v(r) = \|\dot\alpha(r)\|$.

In this study, we considered a surface generated from a regular curve: $\alpha(r)$ (see [17]).

$$\begin{aligned} \mathbf{X}(r,t) = \alpha(r) + (F(r,t)\ G(r,t)\ H(r,t)) &\begin{pmatrix} \mathbf{T}(r) \\ \mathbf{N}(r) \\ \mathbf{B}(r) \end{pmatrix} \\ r_0 \le r \le r_1, \quad t_0 \le t \le t_1, & \end{aligned} \tag{3}$$

where $F(r,t), G(r,t)$, and $H(r,t)$ denote the smooth functions.
If parameter t denotes the time, functions $F(r,t), G(r,t)$, and $H(r,t)$ can be viewed as the directed marching distances of a unit point in time t in direction $\mathbf{T}(r)$, $\mathbf{N}(r)$, and $\mathbf{B}(r)$, respectively. In particular, the values of functions $F(r,t), G(r,t)$, and $H(r,t)$ indicate extension-, flexion-, and retortion-like effects of the unit point over time t, respectively, starting from $\alpha(r)$. Moreover, functions $F(r,t), G(r,t)$, and $H(r,t)$ are considered marching-scale functions in directions $\mathbf{T}(r)$, $\mathbf{N}(r)$, and $\mathbf{B}(r)$, respectively.

Definition 2.1. *A curve, $\alpha(r)$, on a parametric surface, $\mathbf{X}(r,t)$, defined in* (3) *is said to be an isoparametric curve if it is a parametric curve, i.e., there exists a parameter, t_0, such that $\alpha(r) = \mathbf{X}(r,t_0)$.*

Owing to Definition 2.1 and (3), we obtain the following result, which is useful late:

Lemma 2.2. *Curve $\alpha(r)$ on parametric surface $\mathbf{X}(r,t)$, defined by* (3)*, is an isoparametric curve if and only if the marching-scale functions satisfy $F(r,t_0) = G(r,t_0) = H(r,t_0) = 0$.*

3. Connections of two parametric surfaces

In the study of surface connections, $IS(\alpha, F, G, H)$ denotes the parametric surface, $\mathbf{X}(r,t)$, defined by (3), through an isoparametric curve, $\alpha(r) = \mathbf{X}(r,t_0)$.

3.1. C^0-continuous connection

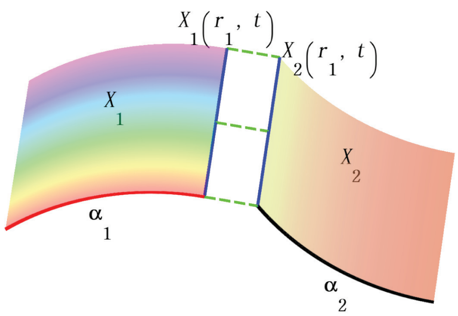
Let $\mathbf{X}_i$ denote parametric surfaces defined by

$$\mathbf{X}_i(r,t) = \alpha_i(r) + (F_i(r,t)\ G_i(r,t)\ H_i(r,t)) \begin{pmatrix} \mathbf{T}_i(r) \\ \mathbf{N}_i(r) \\ \mathbf{B}_i(r) \end{pmatrix} \qquad (4)$$
$$r_{i-1} \le r \le r_i, \quad t_0 \le t \le t_1,$$

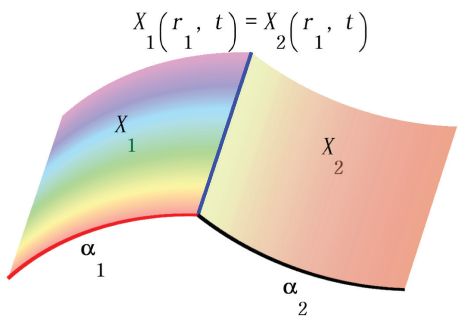
where $F_i(r,t), G_i(r,t)$, and $H_i(r,t)$ denote the smooth functions for $i = 1, 2$.

First, we introduce a method for surface connection by gluing the t-parameter curves of two parametric surfaces, $\mathbf{X}_1$ and $\mathbf{X}_2$, along $r = r_1$.

Definition 3.1. *If $\mathbf{X}_1(r_1,t) = \mathbf{X}_2(r_1,t)$, then the surface connected by $\mathbf{X}_1$ and $\mathbf{X}_2$ along $r = r_1$ is said to be a C^0-continuously connected surface, which can be denoted by $SC^0(\mathbf{X}_1,\ \mathbf{X}_2)$ (see Figure 1).*



(a) Two parametric surfaces, $\mathbf{X}_1$ and $\mathbf{X}_2$



(b) is shows the continuous connection of $\mathbf{X}_1$ and $\mathbf{X}_2$ along $r = r_1$

Figure 1 : Continuously connected surface, $SC^0(\mathbf{X}_1, \mathbf{X}_2)$

Definition 3.2. *Let α_i denote the regular curves for $r_{i-1} \le r \le r_i$ and $i = 1, 2$. If $\alpha_1^{(j)}(r_1) = \alpha_2^{(j)}(r_1)$ for $j = 0, 1, \cdots, m$, where $\alpha_i^{(j)}$ is the j-th derivative of α_i. Then, the two curves α_1 and α_2 are said to be j-th continuous at $r = r_1$, which can be denoted by $C^j(\alpha_1,\ \alpha_2)$.*

While studying parametric surfaces through a given curve, Wang et al. [17] used marching-scale functions expressed as two factors, whereas Kasap et al. [8] selected more general marching-scale functions. It should be noted that marching-scale functions play an important role in the construction of parametric surfaces. Thus, it is important to explain the construction of marching-scale functions for surface connection. We now suggest sets of two variable functions for the design and construction of parametric surface connections through a given space curve.

Let $\mathcal{F}$ be a set of two variable functions: $\mathcal{F} = \{f \mid f : \mathbf{D} \subset \mathbb{R}^2 \to \mathbb{R}\}$, where $(r,t) \in \mathbf{D} = [r_0, r_1] \times [t_0, t_1]$.

Definition 3.3. *For a given function $\hat{f} \in \mathcal{F}$, function $\hat{f}$ is called an essential C^0 function along $r = r_1$ if $f(r_1, t) = \hat{f}(r_1, t)$ for any $f \in \mathcal{F}$. The set of essential C^0 functions is denoted by $C^0_{\hat{f}(r_1,t)}$.*

Definition 3.4. *A function $f \in \mathcal{F}$ is called a degenerate C^0 function along $r = r_1$ if $f(r_1, t) = 0$. The set of degenerate C^0 functions is denoted by $\mathcal{D}^0_{r=r_1}$.*

From an essential C^0 function and degenerate C^0 function along $r = r_1$, we can obtain the following properties.

Proposition 3.5. *Let $\hat{f} \in C^0_{\hat{f}(r_1,t)}$ and $\tilde{f} \in \mathcal{D}^0_{r=r_1}$. Then, for any $f \in \mathcal{F}$, we have*
(1) $\hat{f} + \tilde{f} \in C^0_{\hat{f}(r_1,t)}$.
(2) $f\tilde{f} \in \mathcal{D}^0_{r=r_1}$.
(3) $\hat{f} + f\tilde{f} \in C^0_{\hat{f}(r_1,t)}$.

Proof. The sum of functions $\hat{f}$ and $\tilde{f}$ along $r = r_1$ is

$$(\hat{f} + \tilde{f})(r_1, t) = \hat{f}(r_1, t) + \tilde{f}(r_1, t).$$

Because $\tilde{f} \in \mathcal{D}^0_{r=r_1}$, i.e., $\tilde{f}(r_1, t) = 0$, we have

$$(\hat{f} + \tilde{f})(r_1, t) = \hat{f}(r_1, t),$$

which follows $\hat{f} + \tilde{f} \in C^0_{\hat{f}(r_1,t)}$.

Moreover, the product of functions f and $\tilde{f}$ along $r = r_1$ becomes

$$(f\tilde{f})(r_1, t) = f(r_1, t)\tilde{f}(r_1, t)$$

for any $f \in \mathcal{F}$. Note that $\tilde{f}(r_1, t) = 0$; thus, we have:

$$(f\tilde{f})(r_1, t) = 0,$$

which implies that $f\tilde{f} \in \mathcal{D}^0_{r=r_1}$.
Finally, (1) and (2) lead to $\hat{f} + f\tilde{f} \in C^0_{\hat{f}(r_1,t)}$. □

Using Proposition 3.5 (3), we can construct marching-scale functions that share the essential C^0 functions, i.e., $\hat{f}$, $\hat{g}$, and $\hat{h}$; accordingly, we obtain the following:

Corollary 3.6. *Let $\tilde{f}, \tilde{g}, \tilde{h} \in \mathcal{D}^0_{r=r_1}$ and $\hat{f} \in C^0_{\hat{f}(r_1,t)}$, $\hat{g} \in C^0_{\hat{g}(r_1,t)}$, $\hat{h} \in C^0_{\hat{h}(r_1,t)}$ for $i = 1, 2$. Marching-scale functions F_i, G_i and H_i can be defined as follows:*

$$F_i = \hat{f} + f_i\tilde{f}, \quad G_i = \hat{g} + g_i\tilde{g}, \quad H_i = \hat{h} + h_i\tilde{h}$$

for some $f_i, g_i, h_i \in \mathcal{F}$. Then, $F_i \in C^0_{\hat{f}(r_1,t)}$, $G_i \in C^0_{\hat{g}(r_1,t)}$, and $H_i \in C^0_{\hat{h}(r_1,t)}$.

Remark 3.7. *The degenerate C^0 function plays an important role in continuous surface connection. We can easily construct degenerate C^0 functions along $r = r_1$ in various forms, such as $r - r_1$, $\sin(r - r_1)$, and $\sinh(r - r_1)$.*

Theorem 3.8. *Let $\mathbf{X}_i$ denote parametric surfaces defined by $\mathbf{X}_i(r,t) = IS(\alpha_i, F_i, G_i, H_i)$, $r_{i-1} \le r \le r_i$, $t_0 \le t \le t_1$ with $F_i \in C^0_{\hat{f}(r_1,t)}$, $G_i \in C^0_{\hat{g}(r_1,t)}$, $H_i \in C^0_{\hat{h}(r_1,t)}$ for $i = 1, 2$. If the isoparametric curves are $C^2(\alpha_1, \alpha_2)$ at $r = r_1$, then we have a C^0-continuously connected surface, $SC^0(\mathbf{X}_1,\ \mathbf{X}_2)$.*

Proof. Let $\mathbf{X}_i$ denote parametric surfaces defined by $\mathbf{X}_i(r,t) = IS(\alpha_i, F_i, G_i, H_i)$, $r_{i-1} \le r \le r_i$, $t_0 \le t \le t_1$ for $i = 1, 2$. Then, inserting $r = r_1$ yields:

$$\begin{aligned} \mathbf{X}_1(r_1,t) &= \alpha_1(r_1) + F_1(r_1,t)\mathbf{T}_1(r_1) + G_1(r_1,t)\mathbf{N}_1(r_1) + H_1(r_1,t)\mathbf{B}_1(r_1), \\ \mathbf{X}_2(r_1,t) &= \alpha_2(r_1) + F_2(r_1,t)\mathbf{T}_2(r_1) + G_2(r_1,t)\mathbf{N}_2(r_1) + H_2(r_1,t)\mathbf{B}_2(r_1). \end{aligned} \tag{5}$$

If the isoparametric curves are $C^2(\alpha_1, \alpha_2)$ at $r = r_1$, we obtain the following:

$$\begin{gathered} \alpha_1(r_1) = \alpha_2(r_1), \\ \mathbf{T}_1(r_1) = \mathbf{T}_2(r_1), \quad \mathbf{N}_1(r_1) = \mathbf{N}_2(r_1), \quad \mathbf{B}_1(r_1) = \mathbf{B}_2(r_1). \end{gathered} \tag{6}$$

We assume that $F_i \in C^0_{\hat{f}(r_1,t)}$, $G_i \in C^0_{\hat{g}(r_1,t)}$, and $H_i \in C^0_{\hat{h}(r_1,t)}$ for $i = 1, 2$. Then, we also have

$$F_1(r_1,t) - F_2(r_1,t) = G_1(r_1,t) - G_2(r_1,t) = H_1(r_1,t) - H_2(r_1,t) = 0. \tag{7}$$

To demonstrate that $\mathbf{X}_1(r,t) = \mathbf{X}_2(r,t)$ along $r = r_1$, we confirm $\mathbf{X}_1(r_1,t) - \mathbf{X}_2(r_1,t) = 0$. Equation (5) implies that

$$\mathbf{X}_1(r_1,t) - \mathbf{X}_2(r_1,t) = \sum_{i=1}^{2} (-1)^{i-1}\{\alpha_i(r_1) + F_i(r_1,t)\mathbf{T}_i(r_1) + G_i(r_1,t)\mathbf{N}_i(r_1) + H_i(r_1,t)\mathbf{B}_i(r_1)\}.$$

Moreover, (6) results in:

$$\begin{aligned} \mathbf{X}_1(r_1,t) - \mathbf{X}_2(r_1,t) = &\{F_1(r_1,t) - F_2(r_1,t)\}\mathbf{T}_1(r_1) \\ &+ \{G_1(r_1,t) - G_2(r_1,t)\}\mathbf{N}_1(r_1) + \{H_1(r_1,t) - H_2(r_1,t)\}\mathbf{B}_1(r_1). \end{aligned}$$

Thus, (7) leads to $\mathbf{X}_1(r_1,t) - \mathbf{X}_2(r_1,t) = 0$. □

3.2. C^1-continuous connection

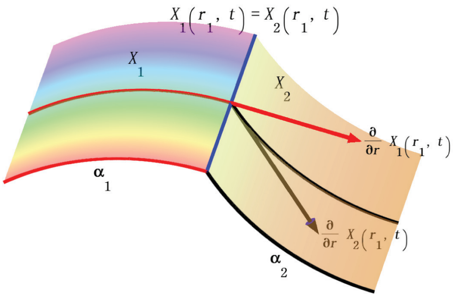
Although $SC^0(\mathbf{X}_1, \mathbf{X}_2)$ is continuous along the common curve, $\mathbf{X}_1(r_1,t) = \mathbf{X}_2(r_1,t)$, of surfaces $\mathbf{X}_1$ and $\mathbf{X}_2$ (see Figure 1(b)), differentiability is not guaranteed along the common curve of the connected surface. In general, surfaces $SC^0(\mathbf{X}_1, \mathbf{X}_2)$ are not connected smoothly. This problem can be solved using degenerate C^0 functions.

Definition 3.9. *If $\mathbf{X}_1(r_1,t) = \mathbf{X}_2(r_1,t)$, $\mathbf{X}_{1t}(r_1,t) = \mathbf{X}_{2t}(r_1,t)$, and $\mathbf{X}_{1r}(r_1,t) = \mathbf{X}_{2r}(r_1,t)$, then the surfaces are called C^1-continuously connected by $\mathbf{X}_1$ and $\mathbf{X}_2$ along $r = r_1$, denoted by $SC^1(\mathbf{X}_1,\ \mathbf{X}_2)$ (see Figure 2 (b)).*

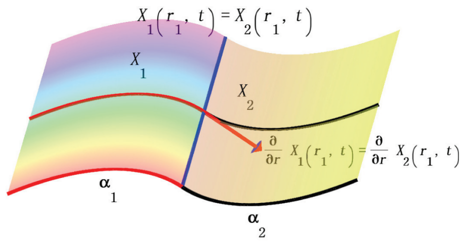
To provide smoothness to the surface connection, $SC^0(\mathbf{X}_1,\ \mathbf{X}_2)$, we now consider a set of smooth functions that share an essential C^0 function.

Definition 3.10. *For a given function $\hat{f} \in C^0_{\hat{f}(r_1,t)}$, function $\hat{f}$ is called an essential C^1 function along $r = r_1$ if $f_r(r_1,t) = \hat{f}_r(r_1,t)$ for any $f \in C^0_{\hat{f}(r_1,t)}$. The set of essential C^1 functions is denoted by $C^1_{\hat{f}(r_1,t)}$.*

Definition 3.11. *A function $f \in \mathcal{D}^0_{r=r_1}$ is called a degenerate C^1 function along $r = r_1$ if $f_r(r_1,t) = 0$. The set of degenerate C^1 functions is denoted by $\mathcal{D}^1_{r=r_1}$.*



(a) Tangent vectors along $r = r_1$ of surfaces $\mathbf{X}_1$ and $\mathbf{X}_2$ on $SC^0(\mathbf{X}_1, \mathbf{X}_2)$, respectively.



(b) Smoothness of connected surface $SC^1(\mathbf{X}_1, \mathbf{X}_2)$ which coincide with two tangent vectors along $r = r_1$.

Figure 2 : Continuously connected surface $SC^1(\mathbf{X}_1, \mathbf{X}_2)$

Proposition 3.12. *Let $\hat{f} \in C^1_{\hat{f}(r_1,t)}$ and $\tilde{f} \in \mathcal{D}^1_{r=r_1}$. Then, for any $f \in \mathcal{F}$, we have:*
(1) $\hat{f} + \tilde{f} \in C^1_{\hat{f}(r_1,t)}$.
(2) $f\tilde{f} \in \mathcal{D}^1_{r=r_1}$.
(3) $\hat{f} + f\tilde{f} \in C^1_{\hat{f}(r_1,t)}$.

Proof. The sum of functions $\hat{f}$ and $\tilde{f}$ along $r = r_1$ is

$$(\hat{f} + \tilde{f})(r_1, t) = \hat{f}(r_1, t) + \tilde{f}(r_1, t) = \hat{f}(r_1, t).$$

Because $\tilde{f} \in \mathcal{D}^1_{r=r_1}$, i.e., $\tilde{f}(r_1, t) = 0$ and $\tilde{f}_r(r_1, t) = 0$, we have

$$(\hat{f} + \tilde{f})(r_1, t) = \hat{f}(r_1, t).$$

In addition, the partial derivative of $\hat{f} + \tilde{f}$ along $r = r_1$ becomes

$$(\hat{f} + \tilde{f})_r(r_1, t) = \hat{f}_r(r_1, t) + \tilde{f}_r(r_1, t).$$

Moreover, it is known that $\tilde{f}_r(r_1, t) = 0$; accordingly,

$$(\hat{f} + \tilde{f})_r(r_1, t) = \hat{f}_r(r_1, t).$$

Thus, we obtain $\hat{f} + \tilde{f} \in C^1_{\hat{f}(r_1,t)}$.

For any $f \in \mathcal{F}$, $\tilde{f} \in \mathcal{D}^1_{r=r_1}$; because $\tilde{f}(r_1, t) = 0$, we have

$$(f\tilde{f})(r_1, t) = f(r_1, t)\tilde{f}(r_1, t) = 0.$$

In addition, the partial derivative of $(f\tilde{f})(r_1, t)$ along $r = r_1$ is:

$$(f\tilde{f})_r(r_1, t) = f(r_1, t)\tilde{f}_r(r_1, t) + f_r(r_1, t)\tilde{f}(r_1, t).$$

Thus, we have $(f\tilde{f})_r(r_1, t) = 0$ because $\tilde{f}(r_1, t) = \tilde{f}_r(r_1, t) = 0$. Hence, $(f\tilde{f}) \in \mathcal{D}^1_{r=r_1}$.
Using Propositions 3.12 (1) and (2), we finally obtain $\hat{f} + f\tilde{f} \in C^1_{\hat{f}(r_1,t)}$. □

Thus, from Proposition 3.12 (3), we construct marching-scale functions that share the essential C^1 functions, $\hat{f}$, $\hat{g}$, and $\hat{h}$, for the connection of surfaces.

Corollary 3.13. *Let $\tilde{f}, \tilde{g}, \tilde{h} \in \mathcal{D}^1_{r=r_1}$ and $\hat{f} \in C^1_{\hat{f}(r_1,t)}$, $\hat{g} \in C^1_{\hat{g}(r_1,t)}$, $\hat{h} \in C^1_{\hat{h}(r_1,t)}$ for $i = 1, 2$. Marching-scale functions, F_i, G_i and H_i, can be defined as follows:*

$$F_i = \hat{f} + f_i\tilde{f}, \quad G_i = \hat{g} + g_i\tilde{g}, \quad H_i = \hat{h} + h_i\tilde{h}$$

for some $f_i, g_i, h_i \in \mathcal{F}$. Then, $F_i \in C^1_{\hat{f}(r_1,t)}$, $G_i \in C^1_{\hat{g}(r_1,t)}$, and $H_i \in C^1_{\hat{h}(r_1,t)}$.

Proposition 3.14. *If $f, g \in C^1_{\hat{f}(r_1,t)}$, then $f - g \in \mathcal{D}^1_{r=r_1}$.*

Proof. Assume that $f, g \in C^1_{\hat{f}(r_1,t)}$. Then, we have

$$f(r_1, t) = g(r_1, t) = \hat{f}(r_1, t),$$
$$f_r(r_1, t) = g_r(r_1, t) = \hat{f}_r(r_1, t).$$

Accordingly,

$$(f - g)(r_1, t) = f(r_1, t) - g(r_1, t) = 0,$$
$$(f - g)_r(r_1, t) = f_r(r_1, t) - g_r(r_1, t) = 0.$$

Thus, we obtain $f - g \in \mathcal{D}^1_{r=r_1}$. □

Proposition 3.15. *If $\tilde{f}, \tilde{g} \in \mathcal{D}^0_{r=r_1}$, $\tilde{f}\tilde{g} \in \mathcal{D}^1_{r=r_1}$.*

Proof. Suppose that $\tilde{f}, \tilde{g} \in \mathcal{D}^0_{r=r_1}$. Then, we have

$$\tilde{f}(r_1, t) = \tilde{g}(r_1, t) = 0 \quad \text{and} \quad \tilde{f}(r_1, t)\tilde{g}(r_1, t) = 0.$$

The partial derivative of $\tilde{f}\tilde{g}$ with respect to r is:

$$(\tilde{f}\tilde{g})_r = \tilde{f}_r\tilde{g} + \tilde{f}\tilde{g}_r.$$

By taking $r = r_1$, we obtain

$$(\tilde{f}\tilde{g})_r(r_1, t) = \tilde{f}_r(r_1, t)\tilde{g}(r_1, t) + \tilde{f}(r_1, t)\tilde{g}_r(r_1, t).$$

Note that $\tilde{f}(r_1, t) = 0$ and $\tilde{g}(r_1, t) = 0$; thus, we have

$$(\tilde{f}\tilde{g})_r(r_1, t) = 0.$$

This leads to $\tilde{f}\tilde{g} \in \mathcal{D}^1_{r=r_1}$. □

Remark 3.16. *From Proposition 3.15, we can easily determine degenerate C^1 functions along $r = r_1$ as $(r - r_1)^2$, and* $\sin^2(r - r_1)$, $\sinh^2(r - r_1)$.

Using the essential C^1 functions, we obtain the following result for a C^1-surface continuously connected by $\mathbf{X}_1$ and $\mathbf{X}_2$ along $r = r_1$.

Theorem 3.17. *Let $\mathbf{X}_i$ denote parametric surfaces defined by $\mathbf{X}_i(r, t) = IS(\alpha_i, F_i, G_i, H_i)$, $r_{i-1} \le r \le r_i$, and $t_0 \le t \le t_1$ for $i = 1, 2$. If $F_i \in C^1_{\hat{f}(r_1,t)}$, $G_i \in C^1_{\hat{g}(r_1,t)}$, $H_i \in C^1_{\hat{h}(r_1,t)}$ for $i = 1, 2$ and the isoparametric curves are $C^3(\alpha_1, \alpha_2)$ at $r = r_1$, we have a C^1-continuously connected surface, $SC^1(\mathbf{X}_1,\ \mathbf{X}_2)$.*

Proof. Let $\mathbf{X}_i$ denote parametric surfaces defined by $\mathbf{X}_i(r,t) = IS(\alpha_i, F_i, G_i, H_i)$, $r_{i-1} \leq r \leq r_i$, and $t_0 \leq t \leq t_1$ for $i = 1, 2$. If $F_i \in C^1_{\hat{f}(r_1,t)}$, $G_i \in C^1_{\hat{g}(r_1,t)}$, and $H_i \in C^1_{\hat{h}(r_1,t)}$, then $F_i \in C^0_{\hat{f}(r_1,t)}$, $G_i \in C^0_{\hat{g}(r_1,t)}$, and $H_i \in C^0_{\hat{h}(r_1,t)}$ for $i = 1, 2$, respectively. In addition, if the isoparametric curves are $C^3(\alpha_1, \alpha_2)$ at $r = r_1$, the curves imply $C^2(\alpha_1, \alpha_2)$ at $r = r_1$. Owing to Theorem 3.8, we have $\mathbf{X}_1(r_1,t) = \mathbf{X}_2(r_1,t)$. This immediately provides $\mathbf{X}_{1t}(r_1,t) = \mathbf{X}_{2t}(r_1,t)$ for $t_0 \leq t \leq t_1$.
However, the partial derivative of $\mathbf{X}_i(r,t)$ with respect to r leads to:

$$\mathbf{X}_{ir} = (v_i + F_{ir} - v_i\kappa_i G_i)\,\mathbf{T}_i(r) + (G_{ir} + v_i\kappa_i F_i - v_i\tau_i H_i)\mathbf{N}_i(r) + (v_i\tau_i G_i + H_{ir})\mathbf{B}_i(r), \tag{8}$$

where κ_i and τ_i denote the curvatures and torsions of α_i, respectively, and $v_i = \|\dot{\alpha}_i(r)\|$ for $i = 1, 2$.
To demonstrate that $\mathbf{X}_{1r}(r_1,t) = \mathbf{X}_{2r}(r_1,t)$ along $r = r_1$, we can confirm $\mathbf{X}_{1r}(r_1,t) - \mathbf{X}_{2r}(r_1,t) = 0$ for $t_0 \leq t \leq t_1$. From (8), we have:

$$\begin{aligned}\mathbf{X}_1(r_1,t) - \mathbf{X}_2(r_1,t) = \sum_{i=1}^{2}(-1)^{i+1}[\{(v_i(r_1) + F_{ir}(r_1,t) - v_i(r_1)\kappa_i(r_1)G_i(r_1,t)\}\mathbf{T}_i(r_1)\\ + \{G_{ir}(r_1,t) + v_i(r_1)\kappa_i(r_1)F_i(r_1,t) - v_i(r_1)\tau_i(r_1)H_i(r_1,t)\}\mathbf{N}_i(r_1)\\ + \{v_i(r_1)\tau_i(r_1)G_i(r_1,t) + H_{ir}(r_1,t)\}\mathbf{B}_i(r_1)].\end{aligned}$$

If curves α_1 and α_2 are $C^3(\alpha_1, \alpha_2)$ at $r = r_1$, from (1) and (2), we have

$$\begin{aligned}&\mathbf{T}_1(r_1) = \mathbf{T}_2(r_1),\ \mathbf{N}_1(r_1) = \mathbf{N}_2(r_1),\ \mathbf{B}_1(r_1) = \mathbf{B}_2(r_1),\\ &v_1(r_1) = v_2(r_1),\ \kappa_1(r_1) = \kappa_2(r_1), \tau_1(r_1) = \tau_2(r_1).\end{aligned} \tag{9}$$

Accordingly, we obtain:

$$\begin{aligned}\mathbf{X}_1(r_1,t) - \mathbf{X}_2(r_1,t) = \sum_{i=1}^{2}(-1)^{i+1}[\{F_{ir}(r_1,t) - v_1(r_1)\kappa_1(r_1)G_i(r_1,t)\}\mathbf{T}_1(r_1)\\ + \{G_{ir}(r_1,t) + v_1(r_1)\kappa_1(r_1)F_i(r_1,t) - v_1(r_1)\tau_1(r_1)H_i(r_1,t)\}\mathbf{N}_1(r_1)\\ + \{v_1(r_1)\tau_1(r_1)G_i(r_1,t) + H_{ir}(r_1,t)\}\mathbf{B}_1(r_1)].\end{aligned}$$

Since $F_i \in C^1_{\hat{f}(r_1,t)}$, $G_i \in C^1_{\hat{g}(r_1,t)}$, and $H_i \in C^1_{\hat{h}(r_1,t)}$ for $i = 1, 2$, from Proposition 3.14, we obtain $F_1 - F_2$, $G_1 - G_2$, and $H_1 - H_2$ as the degenerate C^1 functions along $r = r_1$, i.e.,

$$\sum_{i=1}^{2}(-1)^{i+1}F_i(r_1,t) = \sum_{i=1}^{2}(-1)^{i+1}G_i(r_1,t) = \sum_{i=1}^{2}(-1)^{i+1}H_i(r_1,t) = 0 \tag{10}$$

and

$$\sum_{i=1}^{2}(-1)^{i+1}F_{ir}(r_1,t) = \sum_{i=1}^{2}(-1)^{i+1}G_{ir}(r_1,t) = \sum_{i=1}^{2}(-1)^{i+1}H_{ir}(r_1,t) = 0. \tag{11}$$

Thus, (10) and (11) result in $\mathbf{X}_{1r}(r_1,t) - \mathbf{X}_{2r}(r_1,t) = 0$. This indicates that we can construct C^1-continuously connected surfaces using $\mathbf{X}_1$ and $\mathbf{X}_2$ along $r = r_1$. □

4. Applications

In this section, we describe a method for designing surface connections using the given Bézier curves.

A Bézier curve of degree n is specified by a sequence of $n + 1$ points called control points. Polygons obtained by joining the control points with line segments in the prescribed order are called a control polygon. Controlling the shape of a Bézier curve is facilitated by the fact that the control polygon reflects the basic shape of the curve.

Definition 4.1. *[13] An arbitrary interval Bézier curve,* $\mathbf{B}(r)$*, of degree* n *with control points* $\mathbf{b_0}$*,* $\mathbf{b_1}$*,* $\cdots$*, and* $\mathbf{b_n}$*, defined on interval* $[a, b]$ *is given by*

$$\mathbf{B}(r) = \sum_{i=0}^{n} \mathbf{b_i} B_{i,n}\left(\frac{r-a}{b-a}\right),$$

where

$$B_{i,n}(r) = \begin{cases} \frac{n!}{(n-i)!\,i!}\left(1 - \frac{r-a}{b-a}\right)^{n-i}\left(\frac{r-a}{b-a}\right)^{i}, & \text{if } 0 \le i \le n \\ 0, & \text{otherwise} \end{cases}$$

are called the Bernstein polynomials.

4.1. C^1-continuous connection of surfaces through Bézier curves

In this subsection, we explain the C^1-continuous connections between two parametric surfaces with C^3-continuous Bézier curves; we consider two Bézier curves, $\alpha_1(r)$ ($r \in [0, 1]$) and $\alpha_2(r)$ ($r \in [1, 2]$), with the following control points.

$$\begin{aligned} &\alpha_1(r): \\ &\mathbf{b_0}(0,0,0), \mathbf{b_1}(0,2,-0.1), \mathbf{b_2}(1,3,-0.2), \mathbf{b_3}(2,3,-0.2), \mathbf{b_4}(4,2,-0.3), \\ &\alpha_2(r): \\ &\mathbf{c_0}(4,2,-0.3), \mathbf{c_1}(6,1,-0.4), \mathbf{c_2}(9,-1,-0.6), \mathbf{c_3}(14,-4,-1.1), \mathbf{c_4}(7,-7,-1.2). \end{aligned} \tag{12}$$

Because $\alpha_1^{(j)}(1) = \alpha_2^{(j)}(1)$ for $j = 0, 1, 2, 3$, Bézier curves $\alpha_1(r)$ and $\alpha_2(r)$ are C^3-continuous at $r = 1$.

Example 4.2. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ denote parametric surfaces given by

$$\begin{aligned} \mathbf{X}_1(r,t) =& IS(\alpha_1, \hat{f} + f_1\tilde{f}, \hat{g} + g_1\tilde{g}, \hat{h} + h_1\tilde{h}) \\ & 0 \le r \le 1, \quad 0 \le t \le 1, \end{aligned}$$

and

$$\begin{aligned} \mathbf{X}_2(r,t) =& IS(\alpha_2, \hat{f} + f_2\tilde{f}, \hat{g} + g_2\tilde{g}, \hat{h} + h_2\tilde{h}) \\ & 1 \le r \le 2, \quad 0 \le t \le 1, \end{aligned}$$

where

$$\begin{array}{llll} \hat{f} = 2t, & \tilde{f} = (r-1)^2, & f_1 = 2t, & f_2 = t^2, \\ \hat{g} = t^2, & \tilde{g} = (r-1)^2, & g_1 = t^2, & g_2 = t^2, \\ \hat{h} = t(t+1), & \tilde{h} = (r-1)^2, & h_1 = 2t, & h_2 = 3t. \end{array}$$

The C^1-continuously connected surface, $SC^1(\mathbf{X}_1, \mathbf{X}_2)$, of parametric surfaces $\mathbf{X}_1$ and $\mathbf{X}_2$ through Bézier curves α_1 and α_2, respectively, is depicted in Figure 3(a).

Example 4.3. Consider the parametric surfaces defined by

$$\begin{aligned} \mathbf{X}_1(r,t) =& IS(\alpha_1, \hat{f} + f_1\tilde{f}, \hat{g} + g_1\tilde{g}, \hat{h} + h_1\tilde{h}) \\ & 0 \le r \le 1, \quad 0 \le t \le 1 \end{aligned}$$

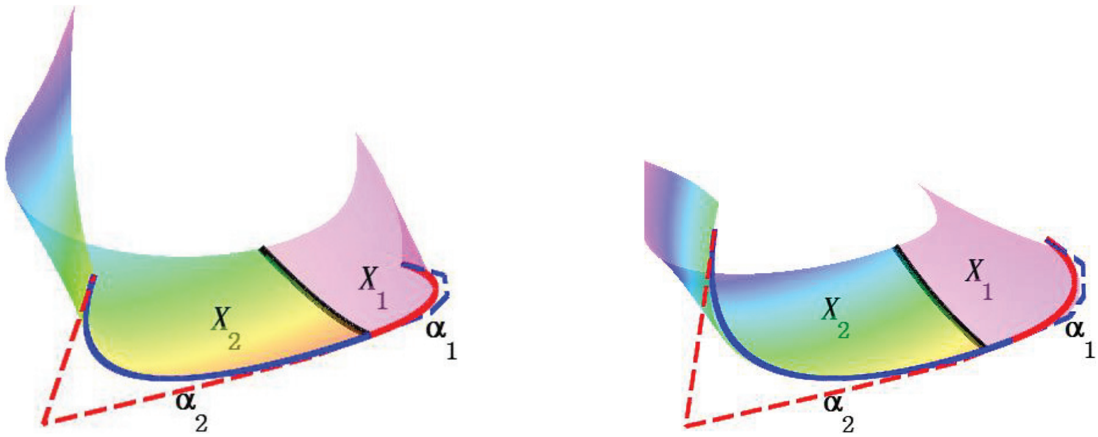
and

$$\begin{aligned} \mathbf{X}_2(r,t) =& IS(\alpha_2, \hat{f} + f_2\tilde{f}, \hat{g} + g_2\tilde{g}, \hat{h} + h_2\tilde{h}) \\ & 1 \le r \le 2, \quad 0 \le t \le 1 \end{aligned}$$

where

$$\hat{f} = \cos\left(\frac{\pi}{2}t\right), \quad \tilde{f} = \sin^2\left(\frac{\pi}{2}(r-1)\right), \ f_1 = 0.3t, \qquad f_2 = 0.1t^2,$$
$$\hat{g} = \sin\left(\frac{\pi}{2}t\right), \quad \tilde{g} = \sin^2\left(\frac{\pi}{2}(r-1)\right), \ g_1 = 0.1t^2, \qquad g_2 = 0.2t,$$
$$\hat{h} = \frac{\pi}{2}t, \qquad \tilde{h} = \sin^2\left(\frac{\pi}{2}(r-1)\right), \ h_1 = 0.2t, \qquad h_2 = 0.3t.$$

The C^1-continuously connected surface, $SC^1(\mathbf{X}_1, \mathbf{X}_2)$, of parametric surfaces $\mathbf{X}_1$ and $\mathbf{X}_2$ through Bézier curves α_1 and α_2, respectively, is shown in Figure 3(b).



(a) (b)

Figure 3 : C^1-continuous connections of two parametric surfaces with C^3-continuous Bézier curves.

4.2. C^1-continuous connection of developable surfaces through Bézier curves

In this subsection, we explain the C^1-continuous connections between two developable surfaces with C^3-continuous Bézier curves.

Li, Wang, and Zhu [11] discussed that an isoparametric surface, $\mathbf{X}(r, t)$, is developable if and only if it can be represented by:

$$\mathbf{X}(r, t) = \alpha(r) + tg(r)\{\tau(r)\mathbf{T}(r) + \kappa(r)\mathbf{B}(r)\}, \tag{13}$$

where

$$g(r) = \frac{p(r)\|\alpha'(r)\|}{\tau(r)} = \frac{\tilde{p}(r)\|\alpha'(r) \times \alpha''(t)\|}{\kappa(r)}$$

for any function $p(t)$ and $\tilde{p}(t)$. Specifically, if α is a planar curve, then the developable surface is

$$\mathbf{X}(r, t) = \alpha(r) + tq(r)\mathbf{B}(r) \tag{14}$$

for any function, $q(t)$. Using (13) and (14), we obtain examples of C^1-continuous connections of developable surfaces.

Example 4.4. *Let α_1 and α_2 be planar Bézier curves satisfying C^3-continuous at $r = 1$ with the control points:*

$$\alpha_1(r)(r \in [0, 1]) : \mathbf{b_0}(0, 0, 0), \mathbf{b_1}(0, 4, 0), \mathbf{b_2}(2, 6, 0), \mathbf{b_3}(4, 4, 0)$$

and

$$\alpha_2(r)(r \in [1,2]) : \mathbf{c_0}(4,4,0), \mathbf{c_1}(6,2,0), \mathbf{c_2}(8,-4,0), \mathbf{c_3}(8,-16,0).$$

Consider the parametric surfaces defined by:

$$\begin{aligned}\mathbf{X}_1(r,t) =& \alpha_1(r) + tq_1(r)\mathbf{B}_1(r)\\ & 0 \le r \le 1, \quad 0 \le t \le 1\end{aligned}$$

and

$$\begin{aligned}\mathbf{X}_2(r,t) =& \alpha_2(r) + tq_2(r)\mathbf{B}_2(r)\\ & 1 \le r \le 2, \quad 0 \le t \le 1,\end{aligned}$$

where

$$q_1(r) = 4 + 0.2(r-1)^2, \quad q_2(r) = 4 + 0.1(r-1)^2.$$

Then, we can design a C^1-continuous connection between two developable surfaces through planar Bézier curves, α_1 and α_2, as shown in Figure 4(a).

Example 4.5. *Let α_1 and α_2 be spatial Bézier curves satisfying C^3-continuous at $r = 1$ with the control points:*

$$\alpha_1(r)(r \in [0,1]) : \mathbf{b_0}(0,0,0), \mathbf{b_1}(0,4,\frac{1}{2}), \mathbf{b_2}(2,6,1), \mathbf{b_3}(4,4,\frac{3}{2})$$

and

$$\alpha_2(r)(r \in [1,2]) : \mathbf{c_0}(4,4,\frac{3}{2}), \mathbf{c_1}(\frac{11}{2},\frac{5}{2},\frac{15}{8}), \mathbf{c_2}(7,-1,\frac{9}{4}), \mathbf{c_3}(8,-7,\frac{21}{8}), \mathbf{c_4}(7,-10,3).$$

Then, the curvature, κ_i, and torsion, τ_i, of the spatial Bézier curve, α_i, for $i = 1,2$ are obtained by

$$\begin{aligned}\kappa_1(r) &= \frac{16}{3}\,\frac{\sqrt{64r^4 - 128r^3 + 194r^2 - 128r + 66}}{(32r^4 + 64r^2 - 128r + 65)^{\frac{3}{2}}}, \quad \tau_1(r) = -\frac{2}{3(32r^4 - 64r^3 + 97r^2 - 64r + 33)},\\ \kappa_2(r) &= \frac{48\sqrt{784r^8 - 4480r^7 + 8416r^6 - 1280r^5 - 13411r^4 + 10738r^3 + 5590r^2 - 7570r + 1825}}{(2368r^6 - 15168r^5 + 36000r^4 - 39872r^3 + 5590r^2 - 7570r + 1825)^{\frac{3}{2}}},\\ \tau_2(r) &= -\frac{6(7r^2 - 10r + 5)}{784r^8 - 4480r^7 + 8416r^6 - 1280r^5 - 13411r^4 + 10738r^3 + 5590r^2 - 7570r + 1825}.\end{aligned}$$

To construct a C^1-continuous connection between two developable surfaces, we consider parametric surfaces as follows:

$$\begin{aligned}\mathbf{X}_1(r,t) =& \alpha_1(r) + tg_1(r)\{\tau_1(r)\mathbf{T}_1(r) + \kappa_1(r)\mathbf{B}_1(r)\}\\ & 0 \le r \le 1, \quad 0 \le t \le 1,\end{aligned}$$

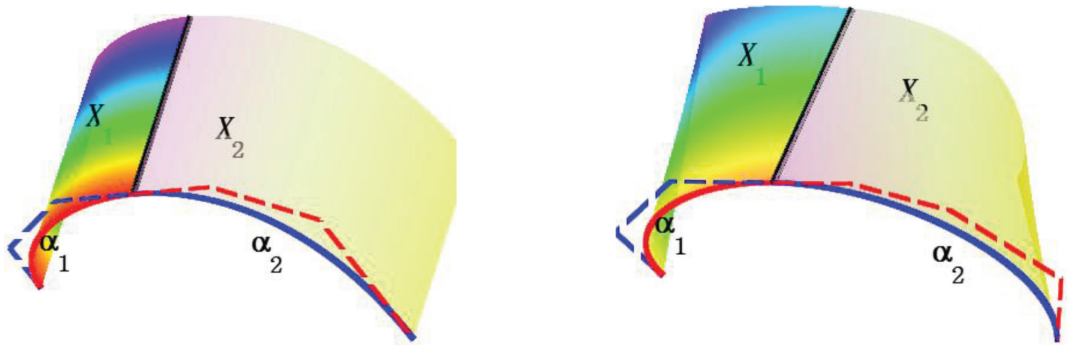
and

$$\begin{aligned}\mathbf{X}_2(r,t) =& \alpha_2(r) + tg_2(r)\{\tau_2(r)\mathbf{T}_2(r) + \kappa_2(r)\mathbf{B}_2(r)\}\\ & 1 \le r \le 2, \quad 0 \le t \le 1,\end{aligned}$$

where

$$g_1(r) = \frac{8 + 4(r-1)^2}{\kappa_1(r)}, \quad g_2(r) = \frac{8 + 3(r-1)^2}{\kappa_2(r)}.$$

Accordingly, we can construct a surface connection between developable surfaces $\mathbf{X}_1$ and $\mathbf{X}_2$ with $C^3(\alpha_1, \alpha_2)$, as shown in Figure 4(b).



(a) (b)

Figure 4 : C^1-continuous connection between developable surfaces $\mathbf{X}_1$ and $\mathbf{X}_2$ through Bézier curves α_1 and α_2.

5. Conclusions and Further Study

In this study, we presented the following problem: 'Which method can be used to connect any two parametric surfaces through given regular curves, and can such surfaces be constructed and designed using this method?' To solve this problem, we introduced the method of C^0- and C^1-continuously connected surfaces through the given curves on parametric surfaces. To illustrate the construction and design of surface connections, we defined essential and degenerate functions for the given marching-scale functions of parametric surfaces. Moreover, we used Bézier curves as the curves given on parametric surfaces. Essential and degenerate functions are important for surface connection. Essential functions influence the construction of surfaces and degenerate functions cause the deformation of partial surfaces, although the curve connected by the surfaces is maintained. Finally, further studies using the connection of parametric surfaces can be conducted. Another question to consider would be as follows: 'How can algebraic or geometric characteristics of essential and degenerate functions affect marching-scale functions?'

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