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Subspace-super recurrence of operators

Otmane Benchiheb[a,*], Mohamed Amouch[a]

[a]*Chouaib Doukkali University. Department of Mathematics, Faculty of science El Jadida, Morocco*

Abstract. This paper is a continuation of our recent work on super-recurrence of operators [3]. We introduce and study subspace-super recurrence of operators. We give the relationship between this new class of operators, super recurrent operators, and other well known class of operators of linear topological dynamics. Several examples and proprieties are given. In particular, we give the relationship between subspace-super recurrence of an operator T and the set of its eigenvectors. Also we present some sufficient conditions of subspace-super recurrence and we demonstrate a subspace-super recurrence criterion. As application, we study the case of shifts operators.

1. Introduction and preliminaries

In the following, X denotes a complex Banach space while $\mathcal{B}(X)$ will stand for the algebra of all operators (linear continuous maps) acting on X.

For a given operator T and a given vector $x \in X$, the orbit of x under T is represented by

$$\mathrm{Orb}(x,T) := \{T^n x : n \in \mathbb{N}\} = \{x, Tx, T^2x, \dots\}.$$

If there exists a vector x such that it's orbit under T is dense in the space X, then the operator T is said to be *hypercyclic*. In this case, the vector x is called a hypercyclic vector for T and the set of all hypercyclic vector for T is denoted by $\mathrm{HC}(T)$.

A useful tool for proving hypercyclicity, as introduced by Birkhoff [12], is that of *topological transitivity*. This means that for any two nonempty and open subsets U and V of X, there exists an $n \in \mathbb{N}$ such that

$$T^{-n}(U) \cap V \neq \emptyset.$$

In fact, he demonstrated that in a complex Banach separable space X, T is hypercyclic if and only if it is topologically transitive.

In a similar manner, T is said to be *supercyclic* if we assure the existence of a vector x whose projective orbit

$$\mathbb{C} \cdot \mathrm{Orb}(x,T) := \{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$$

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* Corresponding author: Otmane Benchiheb
Email addresses: otmane.benchiheb@gmail.com, benchiheb.o@ucd.ac.ma (Otmane Benchiheb), amouch.m@ucd.ac.ma (Mohamed Amouch)

is dense in X. In this case, the vector x is called a supercyclic vector for T. The set of all supercyclic vector for T is denoted by $\mathrm{SC}(T)$. As in the hypercyclicity case, in a complex separable Banach space X the operator T is supercyclic if and only if for each couple (U, V) of nonempty and open subsets of X, there are some $n \in \mathbb{N}$ and some $\lambda \in \mathbb{C}$ such that

$$\lambda T^{-n}(U) \cap V \neq \emptyset.$$

For more information about hypercyclic operators, supercyclic operators and related notions see [2, 5, 8, 20, 21].

Another important notion in the dynamical system is that of recurrence: the operator T is said to be recurrent if for each U a nonempty and open subset of X there is some $n \in \mathbb{N}$ such that

$$T^{-n}(U) \cap U \neq \emptyset.$$

A vector $x \in X$ is called a recurrent vector for T if there exists a strictly increasing sequence (n_k) of positive integers such that

$$T^{n_k}x \longrightarrow x \quad \text{as} \quad k \longrightarrow \infty.$$

We denote by $\mathrm{Rec}(T)$ the set of all recurrent vectors for T. We have that T is recurrent if and only if $\mathrm{Rec}(T)$ is dense in X. For more information about this class of operator see for example [13, 15, 17, 22]. The concept of recurrence was also examined for C_0-semigroups in the work conducted by Moosapoor in [26].

Similarly, an operator T acting on a Banach space X is said to be super-recurrent if for each nonempty and open subset U of X, there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that

$$\lambda T^{-n}(U) \cap U \neq \emptyset.$$

A vector $x \in X$ is called a super-recurrent vector for T if there exists a strictly increasing sequence (n_k) of positive integers and a sequence (λ_k) of complex numbers such that

$$\lambda_k T^{n_k}x \longrightarrow x \quad \text{as} \quad k \longrightarrow \infty.$$

The set of all super-recurrent vectors is denoted by $\mathrm{SRec}(T)$. Again, an operator T is super-recurrent if and only if $\mathrm{SRec}(T)$ is dense in X. For more information about super-recurrent operators see [3, 9, 10]. The notion of super-recurrence was studied also for C_0-semigroups in [4].

In 2011, Madore and Martínez-Avendaño [24] introduced the notion of subspace hypercyclicity: an operator T is said to be subspace hypercyclic for subspace $\mathcal{M}$ or simply $\mathcal{M}$-hypercyclic, provided there is a vector x vector whose orbit under T intersects the subspace in a relatively dense set. i.e, $\mathrm{Orb}(x, T) \cap \mathcal{M}$ is relatively dense in $\mathcal{M}$. The vector x is called a subspace-hypercyclic vector. The operator T is said to be subspace-transitive with respect to $\mathcal{M}$ or $\mathcal{M}$-transitive if for each couple (U, V) of nonempty relatively open subsets of $\mathcal{M}$, there is some $n \in \mathbb{N}$ such that

$$T^{-n}(U) \cap V$$

contains a relatively open nonempty subset of $\mathcal{M}$. It has shown in [24, Theorem 3.5] that the subspace-transitivity implies the subspace hypercyclicity. However, the converse is not true, see [24, Example 3.8].

In 2022, Moosapoor [25] introduced the notion of subspace-recurrent operators: an operator T acting on X is called subspace-recurrent with respect to a nonzero subspace $\mathcal{M}$ or simply $\mathcal{M}$-recurrent, if for each U a nonempty and open subset of X, there is some n such that

$$T^{-n}(U) \cap U$$

is a nonempty and open in $\mathcal{M}$. A vector x is called a $\mathcal{M}$-recurrent vector if there exists an increasing sequence (n_k) of positive integers such that $T^{n_k}x \in \mathcal{M}$ and

$$T^{n_k}x \longrightarrow x, \quad \text{as} \quad k \longrightarrow \infty.$$

The set of all $\mathcal{M}$-recurrent vectors for T is denoted by $\mathrm{Rec}_{\mathcal{M}}(T)$. As in the recurrence and super-recurrence cases, we have that T is $\mathcal{M}$-recurrent if and only if $\mathrm{Rec}_{\mathcal{M}}(T)$ is dense in $\mathcal{M}$.

Note that the concepts of subspace supercyclicity and subspace diskcyclicity have also been investigated. On one hand, Xian-Feng, Yong-Lu, and Yun-Hua explored subspace supercyclicity in their work [32]. On the other hand, Bamerni and Kılıçman delved into the subspace diskcyclicity in [7]. Moosapoor has made significant contributions to these concepts. In her work [27], she presented noteworthy criteria concerning subspace supercyclicity. Additionally, his study in [28] investigated the subspace diskcyclic behavior within the context of C_0-semigroups.

This paper is devoted to introduce and study the notion of $\mathcal{M}$-super recurrent operators. The paper is organized as follows:

In Section 2, we introduce the notions of $\mathcal{M}$-super recurrent operators and $\mathcal{M}$-super recurrent vectors for a given operator and we give the relationship between the class of super-recurrent operators and all other class mentioned above. Moreover, we prove some proprieties of $\mathcal{M}$-super recurrent operators. In particular, we show that an operator T is $\mathcal{M}$-super recurrent with respect to a closed subspace $\mathcal{M}$ if and only of it admits a dense set of $\mathcal{M}$-recurrent vectors. In addition to that we prove that some spectral proprieties satisfied by hypercyclic, supercyclic, recurrent, and super-recurrent operators fails to hold in the case of $\mathcal{M}$-super recurrent operators.

In section 3, we give some sufficient conditions of $\mathcal{M}$-super recurrent operators. In this manner, we characterize the subspace-super recurrence of an operator in term its eigenvectors. Finally, we state a subspace-super recurrence criterion.

In section 5, we study the case of shifts operators. In fact, we prove that any shift operator is $\mathcal{M}$-super recurrent with respect to some closed and non trivial subspace.

2. $\mathcal{M}$-super recurrent operators

In the following, $\mathcal{M}$ will denote a nonzero subspace of X and T will be an operator acting on X.

We begin this section by our main definition.

Definition 2.1. *An operator T is subspace-super recurrent or $\mathcal{M}$-super recurrent with respect to $\mathcal{M}$ ($\mathcal{M}$-super recurrent for short) provided for each nonempty and open subset of $\mathcal{M}$ there are some $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that*

$$\lambda T^{-n}(U) \cap U$$

contains a relatively open nonempty subset of $\mathcal{M}$.

In the following, we give examples of subspace-super recurrent operators that are not super-recurrent. Also, we give an example of a subspace-super recurrent operator that is not subspace-recurrent.

Example 2.2. *We provide the three following examples.*

1. *Let X be a Banach space and $A, B \in \mathcal{B}(X)$. Suppose that A is super-recurrent and B is not. Let $\mathcal{M} = X \oplus \{0\}$ and $T = A \oplus B$. Then it is not difficult to verify that T is $\mathcal{M}$-super recurrent. However, T cannot be super-recurrent by [3, Proposition 3.7].*
2. *Assume that $X = \ell^p(\mathbb{N})$; $1 < p < \infty$. Let T be a diagonal operator defined on $\ell^p(\mathbb{N})$ by*

$$T(x_1, x_2, x_3, x_4 \dots) = (x_1, 2x_2, x_3, 2x_4, \dots), \quad \textit{for all } (x_1, x_2, x_3, x_4 \dots) \in \ell^p(\mathbb{N}).$$

   Clearly T cannot be super-recurrent. However, if $\mathcal{M} = \{(x_k) \in \ell^p(\mathbb{N}) : x_{2k} = 0\}$, then it is not difficult to see that T is $\mathcal{M}$-super-recurrent.
3. *Let X be a complex Banach space and T an operator acting on X. By definitions, if T is subspace-recurrent, then it is $\mathcal{M}$-super recurrent. However, the converse of this statement does not hold in general. For instance, let $T = 2I$, then T is $\mathcal{M}$-super recurrent with respect to each nontrivial and closed subspace of X. On other hand, T is not subspace-recurrent with respect to any closed nontrivial subspace of X.*

Indeed, let's assume that T is subspace-recurrent with respect to some closed and nontrivial subspace $\mathcal{M}$ of X. Take $x \in \mathcal{M}$ such that $\|x\| = 2$. Define $U = B\left(x, \frac{1}{2}\right) \cap \mathcal{M}$. This sets up U as a nonempty, open subset of $\mathcal{M}$. As a result of T being $\mathcal{M}$-super recurrent, there exist some $n \geq 1$ and $y \in B\left(x, \frac{1}{2}\right)$ such that $2^n y = T^n y \in B\left(x, \frac{1}{2}\right)$. Firstly, we have

$$\begin{aligned}
\|y\| &= \|y - x + x\| \\
&\geq \left|\|y - x| - \|x\|\right| \\
&\geq \|x\| - \|y - x\| \\
&\geq 2 - \frac{1}{2} = \frac{3}{2}.
\end{aligned}$$

Secondly, we consider

$$\begin{aligned}
\|x - T^n y\| &\geq \left|\|2^n y\| - \|x\|\right| \\
&\geq 2^n \|y\| - \|x\| \\
&\geq 3 \times 2^{n-1} - 2 > 1.
\end{aligned}$$

However, this is clearly not feasible given that $\|x - T^n y\| < \dfrac{1}{2}$. Hence, it follows that T cannot be subspace-recurrent with respect to any closed nontrivial subspace of X.

Remark 2.3. *We can always assume that the scalar λ in Definition 2.1 is non-zero. Indeed, since X is a Banach space and $\mathcal{M}$ is closed in X, it follows that $\mathcal{M}$ itself is a Banach space. Therefore, it lacks isolated points.*

Now, let U be any nonempty and open subset of $\mathcal{M}$. Consequently, $V = U \setminus \{0\}$ is also a nonempty and open subset of $\mathcal{M}$ since $\mathcal{M}$ doesn't have isolated points. As T is $\mathcal{M}$-super recurrent, there exist $n \in \mathbb{N}$ and a non-zero complex number λ such that the set

$$\lambda T^{-n}(V) \cap V$$

contains a relatively open non-empty subset of $\mathcal{M}$. Since V is a subset of U, it follows that the set

$$\lambda T^{-n}(U) \cap U$$

contains a relatively open non-empty subset of $\mathcal{M}$, and we also have $\lambda \neq 0$.

Let U be a nonempty and open subset of $\mathcal{M}$. In the following lemma, we show that $\alpha T^{-n}(U)$ hit U for infinitely many couples (α, n).

Lemma 2.4. *Assume that T is $\mathcal{M}$-super recurrent operator and let U be a nonempty and open subset of $\mathcal{M}$. Then the set*

$$\{(n, \alpha) \in \mathbb{N} \times \mathbb{C} : \alpha T^{-n}(U) \cap U \textit{ is nonempty and open in } \mathcal{M}\}$$

is infinite.

Proof. Alternatively, suppose that the set

$$\{(n, \alpha) \in \mathbb{N} \times \mathbb{C} : \alpha T^{-n}(U) \cap U \text{ is nonempty and open in } \mathcal{M}\}$$

is finite. We may suppose, without loose of generality, that

$$\{(n, \alpha) \in \mathbb{N} \times \mathbb{C} : \alpha T^{-n}(U) \cap U \text{ is nonempty and open in } \mathcal{M}\} = \{(1, \alpha_1), (2, \alpha_2), \ldots, (k, \alpha_k)\}.$$

Let $U_k = \alpha_k T^{-k}(U) \cap U$. Then U_k is nonempty and open in $\mathcal{M}$. Since T is $\mathcal{M}$-super recurrent, it follows that there exist some $\alpha \in \mathbb{C}$ and some $m \in \mathbb{N}$ such that $\alpha T^{-m}(U_k) \cap U_k \neq \emptyset$. This implies that $\alpha\alpha_k T^{-(m+k)}(U) \cap U$ is a nonempty open subset of $\mathcal{M}$, which is impossible since $m + k$ is greater than k. □

In the following definition, we define the notion of subspace-recurrent vectors for T.

Definition 2.5. *We say that a vector $x \in \mathcal{M}$ is an $\mathcal{M}$-super recurrent vector if there are an increasing sequence (n_k) of positive integers and a sequence (λ_k) of complex scalars such that $T^{n_k}x \in \mathcal{M}$ and*

$$\lambda_k T^{n_k} x \longrightarrow x.$$

The set of all $\mathcal{M}$-super recurrent is denoted by $\mathrm{SRec}_{\mathcal{M}}(T)$.

Remark 2.6. *A vector $x \in \mathcal{M}$ is an $\mathcal{M}$-super recurrent vector if there exists a sequence $(\lambda_k) \subset \mathbb{C}$ such that for each $\varepsilon > 0$ the set*

$$\{n \in \mathbb{N} : T^n x \in \mathcal{M} \ \ \text{and} \ \ \|\lambda_n T^n x - x\| < \varepsilon\}$$

is infinite.

Theorem 2.7. *The following are equivalent*:

1. *T is $\mathcal{M}$-super recurrent;*
2. $\overline{\mathrm{SRec}_{\mathcal{M}}(T)} = \mathcal{M}$.

Proof. First assume that $\overline{\mathrm{SRec}_{\mathcal{M}}(T)} = \mathcal{M}$ and let $U_{\mathcal{M}}$ be an open and nonempty subset of $\mathcal{M}$. There exists a $\mathcal{M}$-super recurrent vector x such that $x \in U_{\mathcal{M}}$. Using this, one can find a $\lambda \in \mathbb{C}$ and a $n \in \mathcal{N}$ such that $\lambda T^{-n}(U_{\mathcal{M}}) \cap U_{\mathcal{M}} \neq \emptyset$. Since $U_{\mathcal{M}}$ is arbitrary taken, it follows that T is $\mathcal{M}$-super recurrent.

Reciprocally, assume that T is $\mathcal{M}$-super recurrent and let $U = B(x_0, \varepsilon) \cap \mathcal{M}$ with $x_0 \in \mathcal{M}$ and $\varepsilon < 1$. Then there are $n_1 \in \mathbb{N}$ and $\lambda_1 \in \mathbb{C}$ such that $\lambda_1 T^{-n_1}(U) \cap U$ is a nonempty open subset of $\mathcal{M}$. Hence one can find some $x_1 \in \mathcal{M}$ and $\varepsilon_1 < \frac{1}{2}$ such that

$$U_2 = B(x_1, \varepsilon_1) \cap \mathcal{M} \subset \lambda_1 T^{-n_1}(U) \cap U \subset U.$$

Using the fact that T is $\mathcal{M}$-super recurrent and U_2 is open in $\mathcal{M}$, we can assure the existence of some $n_2 > n_1$ and some $\lambda_2 \in \mathbb{C}$ such that $\lambda_2 T^{-n_2}(U_2) \cap U_2$ is nonempty and open in $\mathcal{M}$. Again, there are x_2 and ε_2 with $\varepsilon_2 < \frac{1}{2^2}$ such that

$$U_3 = B(x_2, \varepsilon_2) \cap \mathcal{M} \subset U_2 = B(x_1, \varepsilon_1) \cap \mathcal{M}.$$

By induction, we can construct a increasing sequences (n_k) of positive integers, a sequence (λ_k) of complex numbers, a sequence (x_k) of elements of $\mathcal{M}$, and a sequence $\varepsilon_k < \frac{1}{2^k}$ such that

$$B(x_k, \varepsilon_k) \cap \mathcal{M} \subset B(x_{k-1}, \varepsilon_{k-1}) \cap \mathcal{M}$$

and

$$\begin{aligned}
\frac{1}{\lambda_k} T^{n_k}(B(x_k, \varepsilon_k) \cap \mathcal{M}) &\subset \frac{1}{\lambda_k} T^{n_k}(\lambda_k T^{-n_k}(B(x_{k-1}, \varepsilon_{k-1}) \cap \mathcal{M}) \cap (B(x_{k-1}, \varepsilon_{k-1}) \cap \mathcal{M})) \\
&\subset \frac{1}{\lambda_k} T^{n_k}(\lambda_k T^{-n_k}(B(x_{k-1}, \varepsilon_{k-1}) \cap \mathcal{M}) \\
&\subset B(x_{k-1}, \varepsilon_{k-1}) \cap \mathcal{M}.
\end{aligned}$$

Since X is complete, it follows by Cantor's theorem that there exists a vector $y \in X$ such that

$$\bigcap_{k \in \mathbb{N}} B(x_k, \varepsilon_k) \cap \mathcal{M} = \{y\}.$$

Now to complete the proof, we need to show that y is an $\mathcal{M}$-super recurrent vector since $y \in U$. For all k, we have that $y \in B(x_{k+1}, \varepsilon_{k+1}) \cap \mathcal{M} \subset \lambda_k T^{-n_k}(B(x_k, \varepsilon_k) \cap \mathcal{M})$. This implies that $\dfrac{1}{\lambda_k} T^{n_k} y \in \mathcal{M}$ and

$$\left\| \frac{1}{\lambda_k} T^{n_k} y - x_k \right\| < \varepsilon_k, \tag{1}$$

In addition, since $y \in \bigcap_k B(x_k, \varepsilon_k) \cap \mathcal{M}$, it follows that

$$\|x_k - y\| < \varepsilon_k. \tag{2}$$

Now let $k \in \mathbb{N}$. Then by using (1) and (2) we have that

$$\left\| \frac{1}{\lambda_k} T^{n_k} y - y \right\| \leq \|x_k - y\| + \left\| \frac{1}{\lambda_{k\,k}} T^{n_k} y - x_k \right\| < \frac{1}{2^{k-1}}.$$

This means that y is $\mathcal{M}$-super recurrent vector for T. □

Let X and Y be two Banach spaces. If T and S are operators acting on X and Y respectively, then T and S are called conjugate or similar if there exists some homeomorphism $\phi : X \longrightarrow Y$ such that $S \circ \phi = \phi \circ T$, see [20, Definition 1.5].

Proposition 2.8. *Assume that $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$ are similar. Then T is $\mathcal{M}$-super recurrent in X if and only if S is $\mathcal{M}$-super recurrent in Y.*

Proof. Assume that T is $\mathcal{M}$-super recurrent with respect to some nontrivial and closed subspace $\mathcal{M}$. Let $\mathcal{N} := \phi(\mathcal{M})$. Let x be a $\mathcal{M}$-super recurrent vector for T. Then there exists a strictly increasing sequence (n_k) of positive integers such that $T^{n_k}x \in \mathcal{M}$ and $T^{n_k}x \longrightarrow x$. Let $y = \phi(x)$. Then,

$$S^{n_k} y = S^{n_k}(\phi(x)) = \phi(T^{n_k} x) \in \mathcal{N} \quad \text{and} \quad S^{n_k} y \longrightarrow y.$$

This means that y is a $\mathcal{N}$-super-recurrent vector for S. Hence the result holds by Theorem 2.7. □

Proposition 2.9. *Let T and S be operators acting on Banach spaces X and Y respectively. If $T \oplus S$ is $\mathcal{M}$-super recurrent on $X \oplus Y$, then T and S are $\mathcal{M}$-super recurrent on X and Y respectively.*

Proof. Assume that $T \oplus S$ is subspace super-recurrent on $X \oplus Y$ and let $\mathcal{M} \oplus \mathcal{N}$ be nonempty and closed subspace of $X \oplus Y$ such that $T \oplus S$ is $\mathcal{M} \oplus \mathcal{N}$-super-recurrent. Let $x \oplus y \in \mathrm{Rec}_{\mathcal{M}\oplus\mathcal{N}}(T \oplus S$. Then there exist a strictly increasing sequence $(n_k) \subset \mathbb{N}$ and a sequence $(\lambda_k) \subset \mathbb{C}$ such that

$$\lambda_k(T^{n_k} x \oplus S^{n_k} y) = \lambda_k(T \oplus S)^{n_k}(x \oplus y) \longrightarrow x \oplus y.$$

The last statement immediately implies that

$$\lambda_k T^{n_k} x \longrightarrow x \quad \text{and} \quad \lambda_k S^{n_k} y \longrightarrow y.$$

Hence $x \in \mathrm{Rec}_{\mathcal{M}}(T)$ and $y \in \mathrm{Rec}_{\mathcal{N}}(S)$. Now, one can use Theorem 2.7 to deduce that T is $\mathcal{M}$-super recurrent and S is $\mathcal{N}$-super-recurrent. □

Let T be an operator acting on a Banach space X and let $p > 1$. Ansari in [1] showed that T is hypercyclic (resp. supercyclic) if and only if T^p hypercyclic (resp. supercyclic). Later, Costakis, Manoussos, and Parissis in [15](resp. Amouch and Benchiheb in [3]) proved that T is recurrent (resp. super-recurrent) if and only if T^p recurrent (resp. super-recurrent). Recently, Moosapoor in [25] demonstrated that subspace-recurrence of T^p implies subspace-recurrence of T.

The following proposition establish that the $\mathcal{M}$-super recurrence of T^p, with $p > 1$, implies the $\mathcal{M}$-super recurrence of T. The proof is evident so we omit it.

Proposition 2.10. *Let T be an operator acting on a Banach space X and let $p > 1$. If T^p is $\mathcal{M}$-super recurrent, then T is $\mathcal{M}$-super recurrent. Moreover,*

$$\mathrm{SRec}_{\mathcal{M}}(T^p) \subset \mathrm{SRec}_{\mathcal{M}}(T).$$

In whats follows, $\sigma(T)$ and $\sigma_p(T)$ will denote the spectrum and the point spectrum respectively of an operator T while $r(T)$ will denote the spectral radius of T.

In the hypercyclicity and recurrence cases, it has proven in [15, 23] that each component of the spectrum of a hypercyclic or recurrent operator T intersects the unit circle.

In same manner, in [3, 16] it has proven the existence of a circle $\{z \in \mathbb{C} : |z| = R\}$, called a supercyclicity or the super-recurrence circle for T, such that each component of the spectrum of T meets this circle.

For $\sigma_p(T^*)$: if T is hypercyclic, then $\sigma_p(T^*) = \emptyset$, see [8, Proposition 1.7]. If T is supercyclic, then $\sigma_p(T^*) \subset \{\lambda\}$ for some $\lambda \in \mathbb{C}$, see [8, Theorem 1.24] or [16].

In the case of recurrent and super-recurrent operators the results are a slightly different. If T is recurrent (resp. super-recurrent) then $\sigma_p(T^*) \subset \mathbb{T}$, see [15, Proposition 2.14] (resp. there exists $R > 0$ such that $\sigma_p(T^*) \subset \{z \in \mathbb{C} : |z| = R\}$, see [3, Theorem 4.2]).

Surprisingly, all those results fails to hold in the case of subspace-super recurrence as shows the following example.

Example 2.11. *Consider the space $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ and let T be an operator defined on $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ by:*

$$\begin{array}{rcccc} T & : & \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} & \longrightarrow & \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ & & x \oplus y \oplus z & \longmapsto & x \oplus 2y \oplus 3z. \end{array}$$

Then T is $\mathcal{M}$-super recurrent with respect to a closed and nontrivial subspace $\mathcal{M}$ (take for example $\mathcal{M} = \mathbb{C} \oplus \{0\} \oplus \{0\}$). On the other hand, one can easily check that

$$\sigma(T) = \sigma_p(T^*) = \{1, 2, 3\}.$$

Hence, the spectral proprieties mentioned cannot hold for the operator T.

3. Some Sufficient Conditions for Subspace-super recurrence

The next theorems give some equivalent assertions to subspace-super recurrence.

Theorem 3.1. *Let T be an operator acting on a Banach space X. Let $\mathcal{M}$ be closed and nontrivial subspace of X. The following asserting are equivalent*:

1. *T is $\mathcal{M}$-super recurrent;*
2. *for every nonempty and relatively open set U in $\mathcal{M}$, there exists $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that*

$$\lambda T^{-n}(U) \cap U \neq \emptyset \quad \text{and} \quad T^n(\mathcal{M}) \subset \mathcal{M};$$

3. *for every nonempty and relatively open set U in $\mathcal{M}$, there exists $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that*

$$\lambda T^{-n}(U) \cap U$$

is nonempty and relatively open in $\mathcal{M}$.

Proof. (1) $\Rightarrow$ (2) : Assume that T is $\mathcal{M}$-super recurrent and let U be a nonempty and relatively open set in $\mathcal{M}$. Then there exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ and a nonempty relatively open set V in $\mathcal{M}$ such that

$$V \subset \lambda T^{-n}(U) \cap U.$$

On one hand $\lambda T^{-n}(U) \cap U$ is nonempty since V is nonempty. On the other hand, since $V \subset \lambda T^{-n}(U)$, we get that

$$\frac{1}{\lambda} T^n(V) \subset U \subset \mathcal{M}.$$

Now let $x \in \mathcal{M}$, $x_0 \in V$, and $r > 0$ small enough such that $(x_0 + rx) \in V$. Then we have that

$$\frac{1}{\lambda} T^n x_0 + \frac{1}{\lambda} T^n(rx) = \frac{1}{\lambda} T^n(x_0 + rx) \in \mathcal{M}.$$

Using the previous equation one can easily deduce that $T^n x \in \mathcal{M}$. Since x is arbitrary in $\mathcal{M}$, it follows that $T^n(\mathcal{M}) \subset \mathcal{M}$.
(2) $\Rightarrow$ (3) : Let U be a nonempty relatively open in $\mathcal{M}$. Then there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $\lambda T^{-n}(U) \cap U$ is nonempty and $T^n(\mathcal{M}) \subset \mathcal{M}$. The last fact implies that $T^n(\mathcal{M})_{|\mathcal{M}} \in \mathcal{B}(\mathcal{M})$. Hence $\lambda T^{-n}(U)$ is a relatively open in $\mathcal{M}$. Thus, $\lambda T^{-n}(U) \cap U$ is a relatively open in $\mathcal{M}$.
(3) $\Rightarrow$ (1) : is trivial. □

Theorem 3.2. *Let T be an operator on X and $\mathcal{M}$ be a subspace of X. The following asserting are equivalent*:

1. *T is $\mathcal{M}$-super recurrent;*
2. *for any $x \in \mathcal{M}$, there exist a sequences $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathcal{M}$, and a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ such that*
$$x_k \longrightarrow x \quad \text{and} \quad \mathcal{M} \ni \lambda_k T^{n_k} x_k \longrightarrow x;$$
3. *for all $x \in \mathcal{M}$ and for any W a neighborhood of zero in $\mathcal{M}$, there are $y \in \mathcal{M}$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{C}$ such that $\lambda T^n y \in \mathcal{M}$ and*
$$\lambda T^n y - x \in W \quad \text{and} \quad x - y \in W.$$

Proof. (1) $\Rightarrow$ (2) : Let $x \in \mathcal{M}$ and $k \in \mathbb{N} \setminus \{0\}$. Then $U_k := B\left(x, \frac{1}{k}\right) \cap \mathcal{M}$ is nonempty and open in $\mathcal{M}$. Hence, there exist $n_k \in \mathbb{N}$, $\lambda_k \in \mathbb{C}$, and $x_k \in U_k$ such that $\lambda_k T^{n_k} x_k \in U_k$. Thus,
$$\|x_k - x\| < \frac{1}{k} \quad \text{and} \quad \|\lambda_k T^{n_k} x_k - x\| < \frac{1}{k}.$$
This implies that $x_k \longrightarrow x$ and $\lambda_k T^{n_k} x_k \longrightarrow x$.
(2) $\Rightarrow$ (3) : is trivial.
(3) $\Rightarrow$ (1) : Let U be a nonempty relatively open set in $\mathcal{M}$ and let $x \in U$. Then there exits a sequence $(y_k)_{k\in\mathbb{N}} \subset \mathcal{M}$, a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$, and a sequence $(n_k)_{k\in\mathbb{N}}$ such that y_k and $\lambda_k T^{n_k} y_k$ both converging into x in $\mathcal{M}$, which implies the desired result. □

In the following, we talk about operators that have a large set of eigenvectors and their relation with $\mathcal{M}$-super recurrent operator. We begin our study by the following useful and simple lemma.

Lemma 3.3. *Let T be an operator acting on a Banach space X. Let $\mathcal{M}$ be closed and nontrivial subspace of X. Assume that $x \in \mathcal{M}$ is eigenvectors for T, then x is a $\mathcal{M}$-super recurrent vector for T.*

Proof. There exists $\mu \in \mathbb{C}$ such that $Tx = \mu x$. It follows that $T^k x = \mu^k x \in \mathcal{M}$ for all k since $\mathcal{M}$ is a subspace of X. By taking $\lambda_k = \mu^{-k}$ for all k, the result hold. □

In following theorem, we prove that if T has a large set of eigenvectors, then it is subspace-super recurrence with respect to a closed and nontrivial subspace $\mathcal{M}$.

Theorem 3.4. *Let T be an operator acting on a Banach space X. Assume that the set of all eigenvectors for T is dense in X. Then, T is subspace-super recurrence with respect to a closed and nontrivial subspace $\mathcal{M}$.*

Proof. Let $\mathcal{A}$ be the set of all eigenvectors for T and suppose that $\mathcal{A}$ is dense in X. By [6, Theorem 2.1], there exists a closed and nontrivial subspace $\mathcal{M}$ such that $\mathcal{A} \cap \mathcal{M}$ is dense in $\mathcal{M}$, and the result hold then by Lemma 3.3 and Theorem 2.7. □

In the following proposition, we prove that under some additional assumption, the converse of Proposition 2.10 holds.

Proposition 3.5. *Let T be an operator acting on X. Assume that the set of all eigenvectors for T is dense in $\mathcal{M}$. Then the following hold*:

1. *T^n is $\mathcal{M}$-super recurrent for all $n \in \mathbb{N}$.*
2. *if T is invertible, then T^{-n} is $\mathcal{M}$-super recurrent for all $n \in \mathbb{N}$.*

Proof. For (1), let n be a positive integer such that $n > 1$. Let x be a eigenvectors for T, then one can find $\mu \in \mathbb{C}$ such that $Tx = \mu x$. This implies that $T^n x = \mu^n x$. Hence x is a eigenvectors for T^n. Then the result hold by using Lemma 3.3 and Theorem 2.7.
For (2), let x be a nonzero a eigenvectors for T, then there exists a nonzero complex number μ such that $T^n x = \mu x$. This implies that $T^{-n}x = \mu^{-1}x$. Hence x is a a eigenvectors for T^{-n}. As in (1), we deduce that T^{-n} is $\mathcal{M}$-super recurrent. □

Before establishing a Subspace-Super Recurrence Criterion, we prove the following lemma.

Lemma 3.6. *Assume that there are a dense subset $\mathcal{M}_0$ of $\mathcal{M}$, a sequence (λ_k) of complex numbers, and a sequence (n_k) of strictly increasing positive integers such that*

1. $\lambda_k T^{n_k} x \longrightarrow x$ *for all* $x \in \mathcal{M}_0$;
2. $T^{n_k}(\mathcal{M}) \subseteq \mathcal{M}$ *for all* $k \in \mathbb{N}$,

then T is $\mathcal{M}$-super recurrent.

Proof. Let U be an open and nonempty subset of M. There exists $x \in \mathcal{M}_0$ such that $x \in U$ since $\mathcal{M}_0$ is dense in $\mathcal{M}$. By condition (1), we have that $\lambda_k T^{n_k} x \longrightarrow x$. Since U is an open, it follows that there exists some positive integers k such that $x \in U \cap \lambda_k T^{-n_k}(U)$. Now, by using condition (2), one can easily deduce that $U \cap \lambda_k T^{-n_k}(U)$ is a nonempty and open subset of $\mathcal{M}$. □

One of the best tools to ensure the hypercyclicity is what's so called *Hypercyclicity Criterion*. This criterion was introduced by several formulation in [11, 18, 23]. Later, this criterion was introduced in [30] for the supercyclicity and in [25] for the subspace-recurrence. In the following theorem we give this criterion for the subspace-super recurrence case.

Theorem 3.7. *(Subspace-super Recurrence Criterion) Assume that there exist a dense set $\mathcal{M}_0$ of $\mathcal{M}$, a sequence (λ_k) of complex numbers, and strictly increasing sequence (n_k) of positive integers such that*:

1. $\lambda_n T^{n_k} x \longrightarrow 0$ *for every* $x \in \mathcal{M}_0$;
2. *for all $x \in \mathcal{M}_0$, there exists a sequence (x_k) of $\mathcal{M}$ and a sequence (λ_k) of $\mathbb{C}$ that satisfy*
$$x_k \longrightarrow 0 \quad \textit{and} \quad \lambda_k T^{n_k} x_k \longrightarrow x_k,$$
3. $T^{n_k}(\mathcal{M}) \subset \mathcal{M}$, *for all* $k \in \mathbb{N}$.

Proof. Let U be a nonempty relatively open set in $\mathcal{M}$. Since $\mathcal{M}_0$ is dense in $\mathcal{M}$, it follows that there is a vector $x \in \mathcal{M}_0$ such that $x \in U$. Hence, there is some $\varepsilon > 0$ such that $B(x,\varepsilon) \cap \mathcal{M} \subset U$. Since $x \in \mathcal{M}_0$, it follows by condition (1) that $\lambda_n T^{n_k} x \longrightarrow 0$. Now using condition (2), one can find a sequence (x_k) of $\mathcal{M}$ such that $x_k \longrightarrow 0$ and $\lambda_k T^{n_k} x_k \longrightarrow x_k$. Hence there is some positive integer k such that
$$\|\lambda_k T^{n_k} x\| < \frac{\varepsilon}{2}, \quad \|x_k\| < \varepsilon, \quad \text{and} \quad \|\lambda_k T^{n_k} x_k - x\| < \frac{\varepsilon}{2}.$$
On one hand, we have that $x + x_k \in U$ since $\|(x + x_k) - x\| = \|x_k\| < \varepsilon$. On the other hand, we have that
$$\|\lambda_k T^{n_k}(x + x_k) - x\| \leq \|\lambda_k T^{n_k} x\| + \|\lambda_k T^{n_k} x_k - x\| < \varepsilon.$$
Hence $\lambda_k T^{-n_k}(U) \cap U \neq \emptyset$. Finally, by using condition (3), we conclude that $\lambda_k T^{-n_k}(U) \cap U \neq \emptyset$ is a nonempty relatively open set in $\mathcal{M}$. □

4. Weighted shifts operators

Let $X = \ell^p(\mathbb{N})$ with $1 < p < \infty$, or $c_0(\mathbb{N})$ the set consists of every finite sequence of $\ell^p(\mathbb{N})$. We define the unilateral weighted shift operator B_a on $\ell^p(\mathbb{N})$ by

$$B_a(x_0, x_1, x_2, \dots) = (a_0x_1, a_1x_2, \dots),, \quad \text{for all } (x_0, x_1, x_2, \dots) \in \ell^p(\mathbb{N}),$$

where $a = (a_0, a_1, a_2, \dots)$ is a nonzero bounded sequence (called weight of B_a) of complex numbers. Similarly, we can define the bilateral weighted shifts operators acting on $X = \ell^p(\mathbb{Z})$ with $1 < p < \infty$, or $c_0(\mathbb{Z})$ the set consists of every finite sequence of $\ell^p(\mathbb{Z})$.

Operator-theorists frequently test their theories on the class of shifts operators, see [31] for more information about shifts operators.

The first who dealt with the dynamic of shifts operators is Salas. In fact, he gave a fundamental characterization of the hypercyclicity and the supercyclicity of shifts operators, see [29, 30].

For the recurrence case, the recurrent shifts operators were characterized by Costakis, Manoussos, and Parissis in [15].

In this section, we study the subspace-super recurrence of shifts. The next theorem affirm that a shift operator is always $\mathcal{M}$-super recurrent.

Theorem 4.1. *Let B_a be the unilateral or the bilateral weighted shift with weight sequence $a = (a_0, a_1, a_2, \dots)$. Then there is a non-trivial closed subspace $\mathcal{M}$ such that T is $\mathcal{M}$-super recurrent operator.*

Proof. We will prove only the case of $\ell^p(\mathbb{N})$ with $1 < p < \infty$ since the proofs of the other cases is identical. Without loss of generality, we may suppose that $a_{3k} > 1$ for all k. Consider the subspace $\mathcal{M}$ of $\ell^p(\mathbb{N})$ defined by

$$\mathcal{M} := \{(x_k) : x_{3k} = 0 \text{ for all } k \in \mathbb{N}\}.$$

Firstly, let $\mathcal{M}_0 = c_0(\mathbb{N})$. Then $\mathcal{M}_0$ is dense in $\ell^p(\mathbb{N})$. and it is clear that for all $x \in \mathcal{M}_0$ we have that $T^{3k}x \longrightarrow 0$.

Secondly, let S_a be the forward weighted shift with weight sequence $a = (a_0, a_1, a_3, \dots)$; that is the operator defined on $\ell^p(\mathbb{N})$ by

$$S_a(x_0, x_1, x_2, \dots) = (0, a_0^{-1}x_0, a_1^{-1}x_1, \dots),, \quad \text{for all } (x_0, x_1, x_2, \dots) \in \ell^p(\mathbb{N}).$$

For all k, let $x_k = \dfrac{1}{\prod_{i=0}^{k} a_{3i}} x$, where x is any arbitrary fixed element of $\mathcal{M}$. Since $x \in \mathcal{M}$, it follows that $x_k \in \mathcal{M}$ for all k. Moreover, we have that

$$\|x_k\| = \left\| \frac{1}{\prod_{i=0}^{k} a_{3i}} x \right\| = \left| \frac{1}{\prod_{i=0}^{k} a_{3i}} \right| \|x\| \longrightarrow 0$$

since $a_{3i} > 1$ for all i. In addition to that, it is clear that $\lambda_k B_a^{3k} x_k \longrightarrow x$, where $\lambda_k = \prod_{i=0}^{k} a_{3i}$ for all k.

Finally, it is clear that $B_a^{3k}(\mathcal{M}) \subset \mathcal{M}$.

Hence, the conditions of Theorem 3.7 are satisfied. Thus, B_a is $\mathcal{M}$-super recurrent. □

Remark 4.2. *Since the spaces $\ell^\infty(\mathbb{N})$ and $\ell^\infty(\mathbb{Z})$ are non separable, it follows that B_a are never hypercyclic nor supercyclic on $\ell^\infty(\mathbb{N})$ or $\ell^\infty(\mathbb{Z})$. Even the recurrence can exist in some spaces which are not separable, there is no unilateral or bilateral weighted backward shifts on $\ell^\infty(\mathbb{N})$ or $\ell^\infty(\mathbb{Z})$, see [15, Theorem 5.1]. However, in the subspace-recurrence or the subspace-super recurrence, one can follow the same ideas as in the proof of Theorem 4.1 to prove that unilateral or bilateral weighted backward shifts on $\ell^\infty(\mathbb{N})$ or $\ell^\infty(\mathbb{Z})$ are always subspace-recurrent or the $\mathcal{M}$-super recurrent.*

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References

[1] S. I. Ansari, *Hypercyclic and cyclic vectors*, Journal of Functional Analysis. **128** (1995), 374–383.
[2] M. Amouch, O. Benchihe, *Some versions of supercyclicity for a set of operators*, Filomat. **35** (2021), 1619–1627.
[3] M. Amouch, O. Benchiheb, *On a Class of Super-Recurrent Operators*, Filomat. **36** (2022), 3701–3708.
[4] M. Amouch, H. Hassani, *On super-recurrence of strongly continuous semigroup*, Advances in Operator Theory. **7** (2022).
[5] M. Ansari, B. K. Robati, K. Hedayatian, *On the density and transitivity of sets of operators*, Turkish Journal of Mathematics. **42** (2018), 181–189.
[6] N. Bamerni, V. Kadets, A Kılıçman, *Hypercyclic operators are subspace hypercyclic*, Journal of Mathematical Analysis and Applications. **435** (2016), 1812–1815.
[7] N. Bamerni, A. Kılıçman, *On subspace-diskcyclicity*, Arab Journal of Mathematical Sciences. **23**(2017), 133–140.
[8] F. Bayart, É. Matheron, *Dynamics of linear operators*, No. 179. Cambridge university press, 2009.
[9] O. Benchiheb, N. Karim, M Amouch, *On Compositional dynamics on Hardy space*, Boletim da Sociedade Paranaense de Matemática. **41** (2023), 1–9.
[10] O. Benchiheb, F. Sadek, M. Amouch, *On super-rigid and uniformly super-rigid operators*, Afrika Matematika. **34** (2023).
[11] J. P. Bés, *Three problems on hypercyclic operators*, PhD, Bowling Green State University, Bowling Green, OH, USA, 1998.
[12] G. D. Birkhoff, *Surface transformations and their dynamical applications*, 1–119, 1922.
[13] A. Bonilla, K. Grosse-Erdmann, .López-Martínez, A. Peris, *Frequently recurrent operators*, Journal of Functional Analysis. **283**(2020), 109713.
[14] G. Costakis, I. Parissis, *Szemerédi's theorem, frequent hypercyclicity and multiple recurrence*, Mathematica Scandinavica. **110** (2012), 251–272.
[15] G. Costakis, A. Manoussos, I. Parissis, *Recurrent linear operators*, Complex Analysis and Operator Theory. **8** (2014), 1601–1643.
[16] N. S. Feldman, T. L. Miller, V. G. Miller, *Hypercyclic and supercyclic cohyponormal operators* Acta Sci. Math.(Szeged). **68** (2002), 303–328.
[17] J. Galán, et al, *Product recurrence for weighted backward shifts*, Applied Mathematics and Information Sciences. **9** (2015), 2361–2365.
[18] R. Gethner, J. Shapiro, *Universal vectors for operators on spaces of holomorphic functions* Proceedings of the American Mathematical Society. **100** (1987), 281–288.
[19] G. Godefro, and J. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, Journal of Functional Analysis, **98** (1991), 229–269.
[20] K. Grosse-Erdmann, A. Peris, *Linear chaos*, Springer Science and Business Media, 2011.
[21] Z. Jamil, *On Hereditarily Codiskcyclic Operators*, Baghdad Sci. J. **19** (2020), 309–312.
[22] N. Karim, O. Benchiheb, M. Amouch, *Recurrence of multiples of composition operators on weighted Dirichlet spaces*, Advances in Operator Theory. **7** (2022), 1–15.
[23] C. Kitai, *Invariant closed sets for linear operators*, PhD, University of Toronto, Toronto, Canada, 1982.
[24] B. Madore, R. Martínez-Avendaño, *Subspace hypercyclicity*, Journal of mathematical analysis and applications. **373** (2011), 502–511.
[25] M. Moosapoor, *On subspace-recurrent operators*, Tamkang Journal of Mathematics. **53** (2022), 363–371.
[26] M. Moosapoor, *On the Recurrent C_0-Semigroups, Their Existence, and Some Criteria*, Journal of Mathematics, (2021), 1–7.
[27] M. Moosapoor, *On subspace-supercyclic operators*, Aust. J. Math. anal. Appl. **17** (2020), 1–8.
[28] M. Moosapoor, *On the existence of subspace-diskcyclic C_0-semigroups and some criteria*, Journal of Mahani Mathematical Research. (2023), 513–521.
[29] . Salas, *Hypercyclic weighted shifts* Transactions of the American Mathematical Society. **347** (1995), 993–1004.
[30] H. Salas, *Supercyclicity and weighted shifts*, Studia Math. **135** (1999), 55–74.
[31] A. Shields, *Weighted shifts operators and analytic function theory, Topics of Operator Theory*, Math. Surveys Monographs, vol. 13, Amer. Math. Soc, Providence, RI, 1974, pp. 49–128.
[32] Z. Xian-Feng, S. Yong-Lu, Z. Yun-Hua, *Subspace-supercyclicity and common subspace-supercyclic vectors*, J. East China Norm. Univ. **1** (2012), 106–112