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# Mean $p$-angular distance orthogonality in normed linear spaces

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**Abstract.** In this paper, we introduce and study a new concept of orthogonality, namely, mean $p$-angular distance orthogonality in normed linear spaces. We investigate main properties of this type of orthogonality and its relation to some other previously defined generalized orthogonalities. Then, we study $\alpha$-existence, $\alpha$-diagonal existence and S-existence theorems for this type of orthogonality. Moreover, a new characterization of inner product spaces is given in terms of property (H), properly formulated for the mean $p$-angular distance orthogonality.

## 1. Introduction

One of the most important concepts which is used to describe and understand the geometric properties of abstract Euclidean spaces, is the concept of orthogonality. Recall that an abstract Euclidean (or an inner product) space is a real normed linear space $X$ in which the norm $\|\cdot\|$ comes from an inner product $\langle\cdot,\cdot\rangle$; i.e., $\|x\| = \langle x, x\rangle^{1/2}$ for all $x \in X$. In an abstract Euclidean space $(X, \langle\cdot,\cdot\rangle)$, a vector $x$ is said to be orthogonal to a vector $y$ (denoted by $x \perp y$) if $\langle x, y\rangle = 0$. Let $x, y, z \in X$. It is easy to see that orthogonality in $X$ satisfies the following properties:

(1) *Non-degeneracy*: $x \perp x$ if and only if $x = 0$;

(2) *Symmetry*: if $x \perp y$, then $y \perp x$;

(3) *Simplification*: if $x \perp y$, then $\lambda x \perp \lambda y$ for all $\lambda \in \mathbb{R}$;

(4) *Homogeneity*: if $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(5) *Continuity*: let $\{x_n\}$, $\{y_n\}$ be two sequences in $X$ such that $x_n \to x$ and $y_n \to y$. If $x_n \perp y_n$ for each $n \in \mathbb{N}$, then $x \perp y$;

(6) *Left additivity*: if $y \perp x$ and $z \perp x$, then $(y + z) \perp x$;

(7) *Right additivity*: if $x \perp y$ and $x \perp z$, then $x \perp (y + z)$;

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(8) *Left (Right) $\alpha$-existence*: for any two linearly independent vectors $x, y \in X$, there exists $\alpha \in \mathbb{R}$ such that $(\alpha x + y) \perp x$ ($x \perp (\alpha x + y)$);

(9) *$\alpha$-diagonal existence*: for any $x, y \in X \setminus \{0\}$ there exists a real number $\alpha > 0$ such that $x + \alpha y \perp x - \alpha y$.

(10) *S-existence*: if $Y$ is any two-dimensional subspace of $X$, then for every $x \in Y$ and any positive real number $\gamma$ there exists $y \in Y$ such that $\|y\| = \gamma$ and $y \perp x$.

Let $(X, \|\cdot\|)$ be a real normed linear space. An orthogonality in $X$ is a binary relation which coincides with the above (usual) notion of orthogonality when the norm on $X$ is induced by an inner product; that is, when $X$ is an abstract Euclidean space. Although the study of generalized orthogonalities in $X$ provides us with a lot of information about the geometric structure of $X$, there is not a unique way to define a generalized notion of orthogonality in $X$. For any possible such definition, it should be investigated to see which of the above-mentioned properties of the usual orthogonality in inner product spaces, continues to hold. Birkhoff-James orthogonality is one of the most important orthogonality types, suggested by Birkhoff [9] and developed by James [18, 19]. A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ in the sense of Birkhoff-James, denoted by $x \perp_{BJ} y$, if

$$\|x + \lambda y\| \geq \|x\| \quad (\forall \lambda \in \mathbb{R}).$$

In 1945, James introduced isosceles orthogonality and Pythagorean orthogonality [17]. In 1957, Singer [28] restricted the definition of isosceles orthogonality to the unit sphere and defined unitary version of isosceles orthogonality which is known as Singer orthogonality. In 1962, Carlsson [10] extended the concepts of isosceles and Pythagorean orthogonalities: a vector $x \in X$ is called Carlsson orthogonal to a vector $y \in X$ if $\sum_{i=1}^{m} \alpha_i \|\beta_i x + \gamma_i y\|^2 = 0$, where $m \in \mathbb{N}$, and $\alpha_i$, $\beta_i$, and $\gamma_i$ ($i = 1, \ldots, m$) are given real numbers satisfying

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1. \tag{1}$$

Furthermore, the family of unitary Carlsson orthogonality has been investigated by Alonso and Benítez in [1, 5]. Utilizing the $2-HH$-norm, Kikianty and Dragomir [15, 20] introduced Hermite-Hadamard type of Pythagorean (HH-P), isosceles (HH-I), and more generally, Carlsson (HH-C) orthogonality. A vector $x \in X$ is called HH-C-orthogonal to a vector $y \in X$ if

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 dt = 0,$$

where $m \in \mathbb{N}$, and $\alpha_i$, $\beta_i$ and $\gamma_i$ ($i = 1, \ldots, m$) are given real numbers satisfying (1). Recently, unitary version of HH-C-orthogonality has been also studied in [13].

It is known that all the above definitions are equivalent to the usual orthogonality in an inner product space. Also, note that all of these generalized orthogonalities satisfy the properties of nondegeneracy, continuity and existence. However, some other properties of orthogonality such as symmetry, homogeneity and additvity need not always carry over to generalized orthogonalities. For example, Birkhoff-James orthogonality is homogeneous and not symmetric, while isosceles and Singer orthogonalities are symmetric but not homogeneous. To study more about different types of orthogonalities in real normed linear spaces, their properties and relations between them, we refer to [1, 4, 6, 7, 29] and the references therein.

Many interesting characterizations of inner product spaces related to various notions of generalized orthogonality in normed linear spaces have been presented during 20th century; see e.g., [1, 5, 6, 8, 12]. It has been proved in [17] that isosceles and Pythagorean orthogonalities are homogeneous in a normed linear space if and only if the norm is induced from an inner product. Some more characterizations of inner product spaces involving isosceles and Singer orthogonalities have been presented in [2, 3, 22, 30]. More generally, Carlsson [10] proved that Carlsson orthogonality in a real normed linear space is homogeneous (or left additive) if and only if the underlying space is an abstract Euclidean space. Analogously, it was proved in [20] that HH-C-orthogonality is homogeneous (or left additive) in a normed linear space $X$ if and only if the norm of $X$ comes from an inner product. Moreover, some characterizations of the real inner product spaces using the notion of HH-C-orthogonality and its relation with Birkhoff–James orthogonality, have been provided in [13, 14]. Specifically, we remind the following strong characterization of inner product spaces based on the Birkhoff-James orthogonality.

**Theorem 1.1 ([19]).** *A real normed linear space of dimension at least 3 is an abstract Euclidean space if and only if Birkhoff–James orthogonality is symmetric.*

In [24], Maligranda considered the *p-angular distance* ($p \in \mathbb{R}$) between any two nonzero vectors $x$ and $y$ in a real normed linear space $(X, \|\cdot\|)$ as a generalization of the concept of angular distance (to which it reduces when $p = 0$):

$$\alpha_p[x, y] = \left\| \frac{x}{\|x\|^{1-p}} - \frac{y}{\|y\|^{1-p}} \right\|.$$

Based on this concept, a vector $x \in X$ is said to be *p-angular distance orthogonal* to a vector $y \in X$, written as $x \perp_A^p y$, whenever $\|x\|\,\|y\| = 0$ or

$$\alpha_p[x, y] = \left\| \frac{x}{\|x\|^{1-p}} - \frac{y}{\|y\|^{1-p}} \right\| = \left\| \frac{x}{\|x\|^{1-p}} + \frac{y}{\|y\|^{1-p}} \right\| = \alpha_p[x, -y].$$

Clearly, the $p$-angular distance orthogonality is reduced to Singer orthogonality if $p = 0$ and to isosceles orthogonality if $p = 1$. The main properties of $p$-angular distance orthogonality have been provided in [27]. In particular, there was shown that $X$ is an abstract Euclidean space if and only if the $p$-angular distance orthogonality is either homogeneous or additive in $X$.

In this paper, we define (and obtain the main properties of) a new notion of orthogonality called *mean p-angular distance orthogonality*. We illustrate that the mean $p$-angular distance orthogonality is independent from $p$-angular distance orthogonality and from the Birkhoff-James orthogonality. Next, we prove that the mean $p$-angular distance orthogonality has $\alpha$-existence and $\alpha$-diagonal existence properties. Some interesting examples are given to show that this orthogonality is neither symmetric nor homogeneous. Then, we prove that the mean $p$-angular distance orthogonality has $S$-existence property. Moreover, following the method of Carlsson [10], a new characterization of inner product spaces is obtained in terms of property (H), properly formulated for the mean $p$-angular distance orthogonality.

It should be noted that this new notion of orthogonality is defined and studied in the context of real normed linear spaces and the proofs of the main existence results appeared in this paper, depend heavily on some essential tools from real functions theory such as the Intermediate Value Theorem. Accordingly, there is no direct application of our methods to complex normed linear spaces. However, for more study about the definition and properties of numerous types of generalized orthogonalities in complex normed linear spaces, we refer to the comprehensive survey article [25].

## 2. Mean $p$-angular distance orthogonality

In this section, inspiring from the concepts of $p$-angular distance orthogonality [27], HH-I-orthogonality [15] and its unitary version, UHH-I-orthogonality [13], we are going to introduce and study a new notion of orthogonality in normed linear spaces. Let $(X, \langle\, .\, , .\, \rangle)$ be an abstract Euclidean space and let $p \neq 1$ be a positive real number. Note that if

$$\frac{1+t}{2} \frac{x}{\|x\|^{1-p}} \perp \frac{1-t}{2} \frac{y}{\|y\|^{1-p}},$$

for almost every $t \in [0, 1]$, then

$$\left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} + \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 = \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} - \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2,$$

for almost every $t \in [0, 1]$, and therefore,

$$\int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} + \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 dt = \int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} - \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 dt.$$

This motivates us to define the mean $p$-angular distance orthogonality as follows:

**Definition 2.1.** *Let $(X, \|\cdot\|)$ be a real normed linear space, $x, y \in X$ and let $p \neq 1$ be a positive real number. We say that a vector $x \in X$ is mean p-angular distance orthogonal to a vector $y \in X$, written as $x \perp^p_{mA} y$, if and only if $\|x\|\,\|y\| = 0$ or*

$$\int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} + \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 dt = \int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} - \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 dt. \tag{2}$$

It is clear that for any two vectors $x, y \in X$, $x \perp^p_{mA} y$ if and only if $(-x) \perp^p_{mA} y$ if and only if $x \perp^p_{mA} (-y)$ if and only if $(-x) \perp^p_{mA} (-y)$. Moreover, one can easily show that the mean $p$-angular distance orthogonality reduces to the usual one when $X$ is an inner product space. The following proposition gives us some of the main properties of the mean $p$-angular distance orthogonality.

**Proposition 2.2.** *The mean p-angular distance orthogonality satisfies nondegeneracy, simplification and continuity properties.*

*Proof.* To see that mean $p$-angular distance orthogonality satisfies the continuity property, let $x, y \in X$ be two nonzero vectors and $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $x_n \perp^p_{mA} y_n$ for any $n \in \mathbb{N}$. We may assume without loss of generality that $x_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$. Then, by the continuity of the norm functional $x \mapsto \|x\|$ and the Bounded Convergence Theorem, we obtain that

$$\begin{aligned}
\int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} + \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 dt &= \lim_{n\to\infty} \int_0^1 \left\| \frac{1+t}{2} \frac{x_n}{\|x_n\|^{1-p}} + \frac{1-t}{2} \frac{y_n}{\|y_n\|^{1-p}} \right\|^2 dt \\
&= \lim_{n\to\infty} \int_0^1 \left\| \frac{1+t}{2} \frac{x_n}{\|x_n\|^{1-p}} - \frac{1-t}{2} \frac{y_n}{\|y_n\|^{1-p}} \right\|^2 dt \\
&= \int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} - \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 dt.
\end{aligned}$$

Hence, $x \perp^p_{mA} y$. □

Next, we prove that the existence of two nonzero vectors in a normed linear space which are orthogonal in the sense of mean $p$-angular distance, implies that the dimension of the space is at least 2.

**Proposition 2.3.** *Let $(X, \|.\|)$ be a real normed linear space. For any two nonzero vectors $x, y \in X$, if $x$ is mean p-angular distance orthogonal to $y$, then $x$ and $y$ are linearly independent.*

*Proof.* Assume to the contrary that $x \perp^p_{mA} y$, but $x$ and $y$ are linearly dependent. Then there exists a real number $\beta \neq 0$ such that $y = \beta x$. If $\beta > 0$, then

$$\int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} + \frac{1-t}{2} \frac{\beta x}{\|\beta x\|^{1-p}} \right\|^2 dt = \frac{\|x\|^{2p}}{4} \int_0^1 |\, 1 + t + \beta^p(1-t)\,|^2\, dt$$

and

$$\int_0^1 \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} - \frac{1-t}{2} \frac{\beta x}{\|\beta x\|^{1-p}} \right\|^2 dt = \frac{\|x\|^{2p}}{4} \int_0^1 |\, 1 + t - \beta^p(1-t)\,|^2\, dt.$$

Thus, from $x \perp^p_{mA} \beta x$ we obtain that

$$0 = \frac{1}{4}\|x\|^{2p} \left( \int_0^1 |\, 1 + t + \beta^p(1-t)\,|^2\, dt - \int_0^1 |\, 1 + t - \beta^p(1-t)\,|^2\, dt \right) = \|x\|^{2p}\beta^p \int_0^1 (1-t^2) dt = \frac{2}{3}\|x\|^{2p}\beta^p,$$

which is true only when $x = 0$. Now, if $\beta < 0$, then $-\beta > 0$ and since $x \perp^p_{mA} y$ if and only if $x \perp^p_{mA} (-y)$, we can infer, by a similar argument, from $x \perp^p (-\beta)x$ that $x = 0$ which is absurd. □

Of the properties (listed at the beginning of the paper) that one may hope to hold for a generalized orthogonality relation in a real normed linear space, symmetry and homogeneity are not satisfied by mean $p$-angular distance orthogonality:

**Example 2.4.** *Consider the real linear space $X = \mathbb{R}^2$ equipped with the norm $\|(\alpha,\beta)\| = |\alpha| + |\beta|$. Let $x = (2,3)$ and $y = (1,-1)$. Then $\|x\| = 5, \|y\| = 2$. We show that for all $p > 1$, $x \perp^p_{mA} y$ but $y \not\perp^p_{mA} x$. Indeed, for $t \in [0,1]$ we have*

$$\Big\|(1+t)\frac{x}{\|x\|^{1-p}} + (1-t)\frac{y}{\|y\|^{1-p}}\Big\| = \Big\|(1+t)\frac{x}{\|x\|^{1-p}} - (1-t)\frac{y}{\|y\|^{1-p}}\Big\| = 5^p(1+t).$$

*Thus $x \perp^p_{mA} y$. On the other hand, by simple computations based on the definition of mean p-angular distance orthogonality, we see that $y \perp^p_{mA} x$ if and only if*

$$\int_0^{c_p} 5^{2p}(1-t)^2dt + \int_{c_p}^1 \Big(2^p(1+t) - 5^{p-1}(1-t)\Big)^2 dt = \int_0^{d_p} 5^{2p}(1-t)^2dt + \int_{d_p}^1 \Big(2^p(1+t) + 5^{p-1}(1-t)\Big)^2 dt,$$

*in which*

$$c_p = \frac{3\times 5^{p-1} - 2^{p-1}}{3\times 5^{p-1} + 2^{p-1}} \quad \text{and} \quad d_p = \frac{2\times 5^{p-1} - 2^{p-1}}{2\times 5^{p-1} + 2^{p-1}}.$$

*Since $c_p > d_p$, our orthogonality condition amounts to the equation*

$$\int_{c_p}^1 4(1-t^2)2^p\times 5^{p-1}dt + \int_{d_p}^{c_p}\Big[\Big(2^p(1+t) + 5^{p-1}(1-t)\Big)^2 - 5^{2p}(1-t)^2\Big]dt = 0.$$

*But, the first integral on the left hand side of this equation is positive. Moreover, since $\frac{1-d_p}{1+d_p} = \frac{1}{2}(\frac{2}{5})^{p-1} = \frac{5}{4}(\frac{2}{5})^p$, we have for all $t \in (d_p, c_p]$ that*

$$2^p(1+t) + 5^{p-1}(1-t) > 5^p(1-t),$$

*and therefore, the second integral in also positive. From this, we conclude that $y \not\perp^p_{mA} x$.*

**Example 2.5.** *Consider the real linear space $X = \mathbb{R}^2$ equipped with the norm $\|(\alpha,\beta)\| = \max\{|\alpha|,|\beta|\}$. Let $n \in \mathbb{N}$, $x = (1,n)$ and $y = (1,0)$. Then $\|y\| = 1, \|x\| = n$. First, we show that $x \perp^p_{mA} y$ for large enough $n$. We have*

$$\int_0^1 \Big\|\frac{1+t}{2}\frac{x}{\|x\|^{1-p}} \pm \frac{1-t}{2}\frac{y}{\|y\|^{1-p}}\Big\|^2 dt = \int_0^1 \Big\|\Big(\frac{1+t}{2n^{1-p}} \pm \frac{1-t}{2}, \frac{(1+t)n^p}{2}\Big)\Big\|^2 dt = \frac{1}{4}n^{2p}\int_0^1 \Big\|\Big(\frac{1+t}{n} \pm \frac{1-t}{n^p}, 1+t\Big)\Big\|^2 dt.$$

*Since,*

$$0 \le \Big|\frac{1+t}{n} \pm \frac{1-t}{n^p}\Big| \le \frac{2}{n} + \frac{1}{n^p} \longrightarrow 0 \qquad as \qquad n \longrightarrow \infty,$$

*we conclude that for sufficiently large $n \in \mathbb{N}$ and all $t \in [0,1]$,*

$$\max\Big\{\Big|\frac{1+t}{n} \pm \frac{1-t}{n^p}\Big|, 1+t\Big\} = 1+t,$$

*and therefore,*

$$\int_0^1 \Big\|\frac{1+t}{2}\frac{x}{\|x\|^{1-p}} \pm \frac{1-t}{2}\frac{y}{\|y\|^{1-p}}\Big\|^2 dt = \frac{1}{4}n^{2p}\int_0^1 (1+t)^2dt = \frac{7}{12}n^{2p}.$$

*Thus $x = (1,n) \perp^p_{mA} y = (1,0)$ for any large enough $n \in \mathbb{N}$. Fix such a natural number $n$; we show that $(\lambda x) \not\perp^p_{mA} y$ for $\lambda = n^{-1}$. In this case, it is easy to compute the right hand side of (2):*

$$\int_0^1 \Big\|\frac{1+t}{2}\frac{\lambda x}{\|\lambda x\|^{1-p}} - \frac{1-t}{2}\frac{y}{\|y\|^{1-p}}\Big\|^2 dt = \int_0^1 \Big\|\frac{1+t}{2}(n^{-1},1) - \frac{1-t}{2}(1,0)\Big\|^2 dt = \int_0^1 \Big(\frac{1+t}{2}\Big)^2 dt = \frac{7}{12}.$$

*Regarding the left hand side of (2), however, we have to dissect the defining integral into two integrals according as t is less than or equal to the constant $c_n = \frac{1}{2n-1}$ ($n \in \mathbb{N}$) or greater than it:*

$$\begin{aligned}\int_0^1 \Big\| \frac{1+t}{2} \frac{\lambda x}{\|\lambda x\|^{1-p}} + \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \Big\|^2 dt &= \int_0^1 \Big\| \Big(\frac{1+t}{2n} + \frac{1-t}{2}, \frac{1+t}{2}\Big) \Big\|^2 dt = \int_0^{c_n} \Big(\frac{1+t}{2n} + \frac{1-t}{2}\Big)^2 dt + \int_{c_n}^1 \Big(\frac{1+t}{2}\Big)^2 dt \\ &= \frac{7}{12} + \frac{1}{4n^2(2n-1)} + \frac{1}{4n^2(2n-1)^2} + \frac{1}{12n^2(2n-1)^3} - \frac{1}{2(2n-1)^2} \\ &+ \frac{1}{2n(2n-1)} - \frac{1}{6n(2n-1)^3}.\end{aligned}$$

*Now, note that in this case, (2) amounts to the equation* $24n^3 - 24n^2 + 4n + 1 = 0$ *which does not have any natural root n. Therefore,* $(\lambda x) \not\perp_{mA}^p y$.

The following example reveals that our mean $p$-angular distance orthogonality is independent from $p$-angular distance orthogonality introduced and studied in [27]. Indeed, we give two nonzero vectors in an appropriate real normed linear space which are $p$-angular distance orthogonal but are not orthogonal in the sense of our new mean $p$-angular distance.

**Example 2.6.** *Consider the real linear space* $X = \mathbb{R}^2$ *equipped with the norm* $\|(\alpha, \beta)\| = |\alpha| + |\beta|$. *If* $x = (\frac{1}{2}, 1)$ *and* $y = (1, -\frac{1}{2})$, *then we have* $\|x\| = \|y\| = \frac{3}{2}$, *and so it is easy to check that*

$$\Big\| \frac{x}{\|x\|^{1-p}} + \frac{y}{\|y\|^{1-p}} \Big\| = \Big\| \frac{x}{\|x\|^{1-p}} - \frac{y}{\|y\|^{1-p}} \Big\| = 2\Big(\frac{3}{2}\Big)^{p-1}.$$

*Therefore,* $x \perp_A^p y$. *But by simple computations, we see that* $x \not\perp_{mA}^p y$. *In fact,*

$$\int_0^1 \Big\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} + \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \Big\|^2 dt = \Big(\frac{3}{2}\Big)^{2p-2} \int_0^1 \Big(\frac{2+t}{2}\Big)^2 dt = \frac{19}{12}\Big(\frac{3}{2}\Big)^{2p-2},$$

*but*

$$\int_0^1 \Big\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} - \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \Big\|^2 dt = \frac{1}{4}\Big(\frac{3}{2}\Big)^{2p-2} \left( \int_0^{\frac{1}{3}} (2-t)^2 dt + \int_{\frac{1}{3}}^1 (2t+1)^2 dt \right) = \frac{393}{324}\Big(\frac{3}{2}\Big)^{2p-2}.$$

## 3. Existence theorems for mean $p$-angular distance orthogonality

Our main goal in this section is to prove existence results for the mean $p$-angular distance orthogonality. We prove that this type of generalized orthogonality has $\alpha$-existence, $\alpha$-diagonal existence and $S$-existence properties in any real normed linear space.

**Theorem 3.1.** *Let* $(X, \|\cdot\|)$ *be a real normed linear space. Then the mean p-angular distance orthogonality has left* $\alpha$*-existence property; i.e., for any two linearly independent vectors* $x, y \in X$, *there exists* $\alpha \in \mathbb{R}$ *such that* $(\alpha x + y) \perp_{mA}^p x$.

*Proof.* Assume that $x$ and $y$ are linearly independent vectors in $X$. Define the function $\Psi_{x,y} : \mathbb{R} \longrightarrow \mathbb{R}$, abbreviated by $\Psi$, as follows:

$$\Psi(\alpha) = \int_0^1 \psi(x, y, \alpha, t) dt,$$

where

$$\psi(x, y, \alpha, t) = \Big\| \frac{1+t}{2} \frac{\alpha x + y}{\|\alpha x + y\|^{1-p}} + \frac{1-t}{2} \frac{x}{\|x\|^{1-p}} \Big\|^2 - \Big\| \frac{1+t}{2} \frac{\alpha x + y}{\|\alpha x + y\|^{1-p}} - \frac{1-t}{2} \frac{x}{\|x\|^{1-p}} \Big\|^2.$$

In order to estimate the values of $\Psi$, we write the integrand as the product of two functions:

$$\varphi_1(x,y,\alpha,t) = \left\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} + \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\right\| - \left\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} - \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\right\|,$$

and

$$\varphi_2(x,y,\alpha,t) = \left\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} + \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\right\| + \left\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} - \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\right\|.$$

Then we have $\psi(x,y,\alpha,t) = \varphi_1(x,y,\alpha,t)\varphi_2(x,y,\alpha,t)$. Let

$$c_1 := c_1(x,y,\alpha,t) = \frac{1+t}{2}\alpha\|\alpha x+y\|^{p-1} + \frac{1-t}{2}\|x\|^{p-1},$$

and

$$d_1 := d_1(x,y,\alpha,t) = \frac{1+t}{2}\alpha\|\alpha x+y\|^{p-1} - \frac{1-t}{2}\|x\|^{p-1}.$$

Since $p>0$, we have $\lim_{\alpha\to\infty}\alpha\|\alpha x+y\|^{p-1} = \infty$, and so we can select $\alpha>0$ so large that

$$(1+t)\alpha\|\alpha x+y\|^{p-1} > (1-t)\|x\|^{p-1}$$

holds for all $t\in[0,1]$. Therefore, $c_1(x,y,\alpha,t)>0$ and $d_1(x,y,\alpha,t)>0$ for sufficiently large $\alpha>0$ and all $t\in[0,1]$. Regarding $\varphi_1$, we may write

$$\begin{aligned}\varphi_1(x,y,\alpha,t) =& \left\|\Big(\frac{1+t}{2}\alpha\|\alpha x+y\|^{p-1} + \frac{1-t}{2}\|x\|^{p-1}\Big)x + \frac{1+t}{2}\|\alpha x+y\|^{p-1}y\right\| \\ &- \left\|\Big(\frac{1+t}{2}\alpha\|\alpha x+y\|^{p-1} - \frac{1-t}{2}\|x\|^{p-1}\Big)x + \frac{1+t}{2}\|\alpha x+y\|^{p-1}y\right\|.\end{aligned}$$

Taking away the multipliers of $x$ inside the norms in the first and the second part of the above equation, yields for large enough values of $\alpha>0$ that

$$\begin{aligned}\varphi_1(x,y,\alpha,t) &= c_1\left\|x + \frac{(1+t)\|\alpha x+y\|^{p-1}}{(1+t)\alpha\|\alpha x+y\|^{p-1}+(1-t)\|x\|^{p-1}}y\right\| - d_1\left\|x + \frac{(1+t)\|\alpha x+y\|^{p-1}}{(1+t)\alpha\|\alpha x+y\|^{p-1}-(1-t)\|x\|^{p-1}}y\right\| \\ &= c_1\|x+(2c_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}y\| - d_1\|x+(2d_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}y\|.\end{aligned}$$

Since $c_1 = d_1 + (1-t)\|x\|^{p-1}$, we have

$$\begin{aligned}\varphi_1(x,y,t,\alpha) &= (1-t)\|x\|^{p-1}\|x+(2c_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}y\| \\ &\quad - d_1\Big(\|x+(2d_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}y\| - \|x+(2c_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}y\|\Big) \\ &\geq (1-t)\|x\|^{p-1}\|x+(2c_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}y\| - (2c_1)^{-1}(1-t^2)\|\alpha x+y\|^{p-1}\|x\|^{p-1}\|y\| \\ &= (1-t)\|x\|^{p-1}\Big(\|x+(2c_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}y\| - (2c_1)^{-1}(1+t)\|\alpha x+y\|^{p-1}\|y\|\Big) \\ &\geq (1-t)\|x\|^{p-1}\Big(\|x\| - c_1^{-1}(1+t)\|\alpha x+y\|^{p-1}\|y\|\Big).\end{aligned}$$

Moreover, by triangle inequality, we find out for $\alpha\in\mathbb{R}$ and all $t\in[0,1]$ that

$$\begin{aligned}\varphi_2(x,y,\alpha,t) &= \left\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} + \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\right\| + \left\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} - \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\right\| \\ &\geq \left\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} + \frac{1-t}{2}\frac{x}{\|x\|^{1-p}} + \frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} - \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\right\| \\ &= (1+t)\|\alpha x+y\|^p \geq \|\alpha x+y\|^p > 0,\end{aligned}$$

since $x$ and $y$ have been assumed to be linearly independent. Now, since for every $\alpha>0$ and all $t\in[0,1]$,

$$c_1^{-1}(1+t)\|\alpha x+y\|^{p-1} = \frac{2(1+t)\|\alpha x+y\|^{p-1}}{(1+t)\alpha\|\alpha x+y\|^{p-1}+(1-t)\|x\|^{p-1}} \leq \frac{2}{\alpha},$$

we conclude that

$$\Psi(\alpha) = \int_0^1 \psi(x, y, \alpha, t)dt = \int_0^1 \varphi_1(x, y, \alpha, t)\varphi_2(x, y, \alpha, t)dt \geq \frac{2}{3}\|x\|^{p-1}\Big(\|x\| - 2\alpha^{-1}\|y\|\Big)\|\alpha x + y\|^p.$$

Therefore, $\Psi(\alpha) > 0$ for sufficiently large $\alpha > 0$. Since $\psi(x, y, \alpha, t) = -\psi(x, -y, -\alpha, t)$, we get by the same reasoning that $\Psi(\alpha) < 0$ for sufficiently small $\alpha < 0$. So, by the Intermediate Value Theorem for real continuous functions, there exists $\alpha \in \mathbb{R}$ such that $\Psi(\alpha) = 0$; i.e., $(\alpha x + y) \perp^p_{mA} x$. $\square$

**Remark 3.2.** *In the case that $0 < p < 1$, we can also obtain for any linearly independent vectors $x$ and $y$ in real normed linear space $(X, \|\cdot\|)$ a bound for the norm of $z = \alpha x + y$ (which according to Theorem 3.1 is mean p-angular distance orthogonal to x) in terms of $\|x\|$ and $\|y\|$ which in turn, gives us a bound for $|\alpha|$. First note that with a simple application of the triangle inequality, we have*

$$\begin{aligned}\Big\|\frac{1+t}{2}\frac{z}{\|z\|^{1-p}} + \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\Big\|^2 &= \frac{1}{\|z\|^{2-2p}}\Big\|\Big[\frac{1+t}{2}\alpha + \frac{1-t}{2}\Big(\frac{\|z\|}{\|x\|}\Big)^{1-p}\Big]x + \frac{1+t}{2}y\Big\|^2 \\ &\geq \frac{1}{\|z\|^{2-2p}}\Big(\Big[\frac{1+t}{2}\alpha + \frac{1-t}{2}\Big(\frac{\|z\|}{\|x\|}\Big)^{1-p}\Big]\|x\| - \frac{1+t}{2}\|y\|\Big)^2,\end{aligned}$$

*and*

$$\begin{aligned}\Big\|\frac{1+t}{2}\frac{z}{\|z\|^{1-p}} - \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\Big\|^2 &= \frac{1}{\|z\|^{2-2p}}\Big\|\Big[\frac{1+t}{2}\alpha - \frac{1-t}{2}\Big(\frac{\|z\|}{\|x\|}\Big)^{1-p}\Big]x + \frac{1+t}{2}y\Big\|^2 \\ &\leq \frac{1}{\|z\|^{2-2p}}\Big(\Big|\frac{1+t}{2}\alpha - \frac{1-t}{2}\Big(\frac{\|z\|}{\|x\|}\Big)^{1-p}\Big|\|x\| + \frac{1+t}{2}\|y\|\Big)^2.\end{aligned}$$

*Then, taking the power on the right sides of the above two inequalities and then computing the integral of both sides on $[0, 1]$, yield*

$$\begin{aligned}\int_0^1 \Big\|\frac{1+t}{2}\frac{z}{\|z\|^{1-p}} + \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\Big\|^2 dt \geq \frac{1}{\|z\|^{2-2p}}\Big(&\frac{7}{12}\alpha^2\|x\|^2 + \frac{1}{3}\alpha\|z\|^{1-p}\|x\|^{1+p} + \frac{1}{12}\|z\|^{2-2p}\|x\|^{2p} - \frac{7}{6}\alpha\|x\|\,\|y\| \\ &- \frac{1}{3}\|z\|^{1-p}\|x\|^p\|y\| + \frac{7}{12}\|y\|^2\Big),\end{aligned}$$

*and*

$$\begin{aligned}\int_0^1 \Big\|\frac{1+t}{2}\frac{z}{\|z\|^{1-p}} - \frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\Big\|^2 dt \leq \frac{1}{\|z\|^{2-2p}}\Big(&\frac{7}{12}\alpha^2\|x\|^2 - \frac{1}{3}\alpha\|z\|^{1-p}\|x\|^{1+p} + \frac{1}{12}\|z\|^{2-2p}\|x\|^{2p} + \frac{7}{6}\alpha\|x\|\,\|y\| \\ &- \frac{1}{3}\|z\|^{1-p}\|x\|^p\|y\| + \frac{7}{12}\|y\|^2\Big).\end{aligned}$$

*Therefore,*

$$\|z\|^{1-p}\|x\|^p \leq \frac{7}{2}\|y\|. \tag{3}$$

*We conclude from (3) and $|\alpha|\,\|x\| - \|y\| \leq \|z\|$ that for any linearly independent $x, y \in X$, if $0 < p < 1$, then the set of all $\alpha \in \mathbb{R}$ satisfying $(\alpha x + y) \perp^p_{mA} x$ is bounded. In fact,*

$$|\alpha| \leq \Big(\frac{7}{2}\frac{\|y\|}{\|x\|}\Big)^{\frac{1}{1-p}} + \frac{\|y\|}{\|x\|}.$$

The following examples reveal that our mean $p$-angular distance orthogonality is independent from Birkhoff-James orthogonality. In the first example, by means of Theorem 3.1 (left-$\alpha$-existence for mean $p$-angular distance orthogonality), we give two nonzero vectors in an appropriate real normed linear space which are mean $p$-angular distance orthogonal but are not orthogonal in the sense of Birkhoff-James.

**Example 3.3.** *Consider the real linear space $X = \mathbb{R}^2$ equipped with the norm $\|(\alpha,\beta)\| = |\alpha| + |\beta|$. If $x = (1,2)$ and $y = (2,-1)$, then $x$ and $y$ are linearly independent and by the left-$\alpha$-existence result (Theorem 3.1), there exists a real number $\alpha$ such that $(\alpha x + y) \perp_{mA}^{p} x$; i.e., $(2+\alpha, 2\alpha - 1) \perp_{mA}^{p} (1,2)$. It is easy to see that $\alpha \neq 2^{-1}$. Indeed, letting*

$$p_0 = \frac{\log 3}{\log 6 - \log 5},$$

*we have for $0 < p < p_0$ that $(\frac{5}{2}, 0) \perp_{mA}^{p} (1,2)$ if and only if*

$$\int_0^1 \Big((1+t)(\frac{5}{2})^p + (1-t)3^p\Big)^2 dt = \int_0^1 \Big((1+t)(\frac{5}{2})^p + (1-t)3^{p-1}\Big)^2 dt,$$

*which is equivalent to $\frac{2}{3}9^p + (\frac{15}{2})^p = 0$ and this is absurd. For $p \geq p_0$, we have $(\frac{5}{2}, 0) \perp_{mA}^{p} (1,2)$ if and only if*

$$\int_0^1 \Big((1+t)(\frac{5}{2})^p + (1-t)3^p\Big)^2 dt = \int_0^{c_p} \Big((1-t)3^p - (1+t)(\frac{5}{2})^p\Big)^2 dt + \int_{c_p}^1 \Big((1-t)3^{p-1} + (1+t)(\frac{5}{2})^p\Big)^2 dt, \tag{4}$$

*in which $c_p = \frac{3^{p-1} - (\frac{5}{2})^p}{3^{p-1} + (\frac{5}{2})^p}$. The equation (4) has only one negative solution:*

$$p = \frac{\log 3 - \log(3 + 2\sqrt{3})}{\log 6 - \log 5}.$$

*Therefore, $(\frac{5}{2}, 0) \not\perp_{mA}^{p} (1,2)$ for all $p \geq p_0$. But corresponding to every $\alpha \neq 2^{-1}$ we can find $\lambda \in \mathbb{R}$ such that $\|(\alpha x + y) + \lambda x\| < \|\alpha x + y\|$. That is to say,*

$$|2 + \alpha + \lambda| + |2\alpha - 1 + 2\lambda| < |2 + \alpha| + |2\alpha - 1|.$$

*Thus $(\alpha x + y) \not\perp_{BJ} x$.*

In the second example, we give two nonzero vectors in an appropriate real normed linear space which are Birkhoff-James orthogonal but are not orthogonal in the sense of mean $p$-angular distance.

**Example 3.4.** *Consider the real linear space $X = \mathbb{R}^2$ equipped with the norm $\|(\alpha,\beta)\| = |\alpha| + |\beta|$. If $x = (1,0)$ and $y = (1,-1)$, then we have $\|x\| = 1$ and $\|y\| = 2$. It can easily be seen that*

$$\|x + \lambda y\| = |1 + \lambda| + |\lambda| \geq 1 = \|x\| \quad (\forall \lambda \in \mathbb{R}).$$

*Hence $x \perp_{BJ} y$. But, we show that $x \not\perp_{mA}^{p} y$. To this end, first we compute the left hand side of (2) for the above two vectors:*

$$\begin{aligned}\int_0^1 \Big\|\frac{1+t}{2}\frac{x}{\|x\|^{1-p}} + \frac{1-t}{2}\frac{y}{\|y\|^{1-p}}\Big\|^2 dt &= \int_0^1 \Big\|\frac{1+t}{2}(1,0) + \frac{1-t}{2}(2^{p-1}, -2^{p-1})\Big\|^2 dt \\ &= \frac{1}{4}\int_0^1 (1 + t + 2^p(1-t))^2 dt \\ &= \frac{1}{12}(7 + 2^{p+2} + 2^{2p}).\end{aligned}$$

*Next, letting the real constant $c_p = \frac{2^p - 2}{2^p + 2}$, the right hand side of (2) for the above two special vectors is*

$$\begin{aligned}\int_0^1 \Big\|\frac{1+t}{2}\frac{x}{\|x\|^{1-p}} - \frac{1-t}{2}\frac{y}{\|y\|^{1-p}}\Big\|^2 dt &= \int_0^1 \Big\|\frac{1+t}{2}(1,0) - \frac{1-t}{2}(2^{p-1}, -2^{p-1})\Big\|^2 dt \\ &= \frac{1}{4}\int_0^{c_p} (-(1+t) + 2^p(1-t))^2 dt + \frac{1}{4}\int_{c_p}^1 (1+t)^2 dt \\ &= \frac{7}{12} + \frac{4 \times 2^{3p} - 24 \times 2^{2p} + 2^{5p} + 6 \times 2^{4p}}{12(2^p + 2)^3}.\end{aligned}$$

*In this case, (2) amounts to the equation*

$$(2^p+2)^3(2^p+2^2)=4\times 2^{2p}-24\times 2^p+2^{4p}+6\times 2^{3p}.$$

*However, one can verify with the help of a mathematical software such as Maple 2015 that this equation does not have any real solution. Thus $x \not\perp_{mA}^{p} y$.*

**Theorem 3.5.** *Let $(X, \|\cdot\|)$ be a real normed linear space. Then the mean p-angular distance orthogonality has $\alpha$-diagonal existence property; i.e., for any two linearly independent $x, y \in X$, there exists a real number $\alpha$ such that $x+\alpha y \perp_{mA}^{p} x-\alpha y$.*

*Proof.* Define the continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ by $f(\lambda)=\int_0^1 \varphi(\lambda,t)dt$, where

$$\varphi(\lambda,t)=\Big\|\frac{1+t}{2}\frac{x+\lambda y}{\|x+\lambda y\|^{1-p}}+\frac{1-t}{2}\frac{x-\lambda y}{\|x-\lambda y\|^{1-p}}\Big\|^2-\Big\|\frac{1+t}{2}\frac{x+\lambda y}{\|x+\lambda y\|^{1-p}}-\frac{1-t}{2}\frac{x-\lambda y}{\|x-\lambda y\|^{1-p}}\Big\|^2.$$

We are to show the existence of a real number $\alpha$ such that $f(\alpha)=0$. For each $\lambda>0$, we have

$$\begin{aligned}\frac{\varphi(\lambda,t)}{\lambda^{2p}}&=\frac{(1+t)^2}{4}\left[\Big\|\Big(\frac{1}{\lambda^p\|x+\lambda y\|^{1-p}}+\frac{1-t}{1+t}\frac{1}{\lambda^p\|x-\lambda y\|^{1-p}}\Big)x+\Big(\frac{1}{\lambda^p\|x+\lambda y\|^{1-p}}-\frac{1-t}{1+t}\frac{1}{\lambda^p\|x-\lambda y\|^{1-p}}\Big)\lambda y\Big\|^2\right]\\&\quad-\frac{(1+t)^2}{4}\left[\Big\|\Big(\frac{1}{\lambda^p\|x+\lambda y\|^{1-p}}-\frac{1-t}{1+t}\frac{1}{\lambda^p\|x-\lambda y\|^{1-p}}\Big)x+\Big(\frac{1}{\lambda^p\|x+\lambda y\|^{1-p}}+\frac{1-t}{1+t}\frac{1}{\lambda^p\|x-\lambda y\|^{1-p}}\Big)\lambda y\Big\|^2\right].\end{aligned}$$

Hence,

$$\lim_{\lambda\to\infty}\frac{\varphi(\lambda,t)}{\lambda^{2p}}=\frac{1}{4}(1+t)^2\Big|1-\frac{1-t}{1+t}\Big|^2\|y\|^{2p}-\frac{1}{4}(1+t)^2\Big|1+\frac{1-t}{1+t}\Big|^2\|y\|^{2p}=-(1-t^2)\|y\|^{2p}.$$

Now, by the Lebesgue Dominated Convergence Theorem, we obtain that

$$\lim_{\lambda\to\infty}\frac{f(\lambda)}{\lambda^{2p}}=\|y\|^{2p}\int_0^1(t^2-1)dt=-\frac{2}{3}\|y\|^{2p}<0.$$

On the other hand,

$$f(0)=\|x\|^{2p}\int_0^1(1-t^2)dt=\frac{2}{3}\|x\|^{2p}>0.$$

Therefore, by the Intermediate Value Theorem for real continuous functions, there exists $\alpha\in\mathbb{R}$ such that $f(\alpha)=0$. □

**Remark 3.6.** *With the same notations and by the same argument as in the proof of Theorem 3.5, we have*

$$\lim_{\lambda\to\infty}\frac{\varphi(-\lambda,t)}{\lambda^{2p}}=-(1-t^2)\|y\|^{2p}.$$

*Again by the Lebesgue Dominated Convergence Theorem, we obtain that*

$$\lim_{\lambda\to-\infty}\frac{f(\lambda)}{\lambda^{2p}}=\|y\|^{2p}\int_0^1(t^2-1)dt=-\frac{2}{3}\|y\|^{2p}<0.$$

*From this and the proof of Theorem 3.5, one can assert the existence of at least two different real numbers $\alpha$ (one is positive and the other negative), satisfying the diagonal orthogonality. Accordingly, diagonal uniqueness does not hold for the mean p-angular distance orthogonality. The next example makes this fact more clear.*

**Example 3.7.** *In the real linear space $X=\mathbb{R}^2$ equipped with the norm $\|(\alpha,\beta)\|=|\alpha|+|\beta|$, let $x=(1,0)$ and $y=(0,1)$. Then $x+\alpha y\perp_{mA}^{p} x-\alpha y$ if and only if*

$$\int_0^1(1+|\alpha|t)^2dt=\int_0^1(t+|\alpha|)^2dt.$$

*It follows that $|\alpha|=1$, and so $\alpha=\pm 1$.*

As we said at the beginning of the paper, another important existence result that one usually pursue to see whether or not it is true for a newly defined generalized orthogonality in real normed linear spaces, is $S$-existence which we are to prove it for our mean $p$-angular distance orthogonality. Note that since the mean $p$-angular distance orthogonality is not homogeneous, $S$-existence and $\alpha$-existence are not equivalent.

**Theorem 3.8.** *Let $(X, \|\cdot\|)$ be a real normed linear space and let $Y$ be a two-dimensional subspace of $X$. Then for every $x \in Y$ and any positive real number $\gamma$, there exists $y \in Y$ with $\|y\| = \gamma$ such that $y \perp_{mA}^{p} x$.*

*Proof.* Let $x \in Y$. Because the dimension of $Y$ is two, we can find a nonzero vector $z \in Y$ such that $x$ and $z$ are linearly independent. Hence for each $\lambda \in \mathbb{R}$, $z + \lambda x \neq 0$. Define the continuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\varphi(\lambda) = \int_0^1 \left( \Big\| \frac{1+t}{2} \gamma^p \frac{z+\lambda x}{\|z+\lambda x\|} + \frac{1-t}{2} \frac{x}{\|x\|^{1-p}} \Big\|^2 - \Big\| \frac{1+t}{2} \gamma^p \frac{z+\lambda x}{\|z+\lambda x\|} - \frac{1-t}{2} \frac{x}{\|x\|^{1-p}} \Big\|^2 \right) dt.$$

With the same argument as in the proof of Theorem 3.5, we write this as

$$\varphi(\lambda) = \int_0^1 \left( \frac{(1+t)^2}{4} \Big\| \frac{\gamma^p z}{\|z+\lambda x\|} + \Big( \frac{1-t}{1+t} \frac{1}{\|x\|^{1-p}} + \frac{\gamma^p \lambda}{\|z+\lambda x\|} \Big) x \Big\|^2 - \frac{(1+t)^2}{4} \Big\| \frac{\gamma^p z}{\|z+\lambda x\|} + \Big( -\frac{1-t}{1+t} \frac{1}{\|x\|^{1-p}} + \frac{\gamma^p \lambda}{\|z+\lambda x\|} \Big) x \Big\|^2 \right) dt.$$

Since

$$\frac{\gamma^p z}{\|z+\lambda x\|} \longrightarrow 0 \quad \text{and} \quad \frac{\gamma^p \lambda}{\|z+\lambda x\|} \longrightarrow \frac{\gamma^p}{\|x\|} \qquad \text{as} \quad \lambda \longrightarrow \infty,$$

we conclude that

$$\begin{aligned}
\lim_{\lambda\to\infty} \varphi(\lambda) &= \int_0^1 \left( \frac{(1+t)^2}{4} \Big\| \Big( \gamma^p \frac{1}{\|x\|} + \frac{1-t}{1+t} \frac{1}{\|x\|^{1-p}} \Big) x \Big\|^2 - \frac{(1-t)^2}{4} \Big\| \Big( \gamma^p \frac{1}{\|x\|} - \frac{1-t}{1+t} \frac{1}{\|x\|^{1-p}} \Big) x \Big\|^2 \right) dt \\
&= \frac{1}{4} \int_0^1 \Big( ((1+t)\gamma^p + (1-t)\|x\|^p)^2 - ((1+t)\gamma^p - (1-t)\|x\|^p)^2 \Big) dt \\
&= \int_0^1 (1-t^2)\gamma^p \|x\|^p dt = \frac{2}{3} \gamma^p \|x\|^p > 0.
\end{aligned}$$

On the other hand, we have

$$\frac{\gamma^p z}{\|z+\lambda x\|} \longrightarrow 0 \quad \text{and} \quad \frac{\gamma^p \lambda}{\|z+\lambda x\|} \longrightarrow \frac{-\gamma^p}{\|x\|} \quad \text{as} \quad \lambda \longrightarrow -\infty.$$

Therefore,

$$\begin{aligned}
\lim_{\lambda\to-\infty} \varphi(\lambda) &= \int_0^1 \left( \frac{(1+t)^2}{4} \Big\| \Big( -\gamma^p \frac{1}{\|x\|} + \frac{1-t}{1+t} \frac{1}{\|x\|^{1-p}} \Big) x \Big\|^2 - \frac{(1+t)^2}{4} \Big\| \Big( -\gamma^p \frac{1}{\|x\|} - \frac{1-t}{1+t} \frac{1}{\|x\|^{1-p}} \Big) x \Big\|^2 \right) dt \\
&= \frac{1}{4} \int_0^1 \Big( (-(1+t)\gamma^p + (1-t)\|x\|^p)^2 - ((1+t)\gamma^p + (1-t)\|x\|^p)^2 \Big) dt \\
&= -\int_0^1 (1-t^2)\gamma^p \|x\|^p dt = -\frac{2}{3} \gamma^p \|x\|^p < 0.
\end{aligned}$$

Since $\varphi$ is continuous as a function of $\lambda$, it follows from the Intermediate Value Theorem for real continuous functions that there exists a $\lambda_0 \in \mathbb{R}$ such that $\varphi(\lambda_0) = 0$. Thus, taking $y = \gamma \frac{z+\lambda_0 x}{\|z+\lambda_0 x\|}$ we have $\|y\| = \gamma$ and $y \perp_{mA}^{p} x$. □

## 4. Characterization of inner product spaces

In this section, we give some new characterizations of inner product spaces in terms of homogeneity (or additivity) of the mean $p$-angular distance orthogonality. Let $(X, \|\cdot\|)$ be a real normed linear space. It is obvious that if the norm is induced by an inner product, then the mean $p$-angular distance orthogonality is homogeneous. Following Carlsson [10], and Kikianty and Dragomir [20], we impose a condition on the mean $p$-angular orthogonality, called (H)-type property, which is weaker than homogeneity. We prove that this condition is sufficient for the norm of $X$ to be Gâteaux differentiable, and along with symmetry, entails that $X$ is an abstract Euclidean space.

In order to prove the main result of this section, we use some properties of Gâteaux differential of the norm. Let $(X, \|\cdot\|)$ be a real normed linear space and let $x, y \in X$. The functionals

$$\tau_-(x,y) = \lim_{t\to 0^-} \frac{\|x+ty\| - \|x\|}{t} \quad , \quad \tau_+(x,y) = \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

are called Gâteaux left and Gâteaux right derivative of the norm at $x$ in direction $y$, respectively. The norm $\|\cdot\|$ is said to be Gâteaux differentiable at $x$ in direction $y$ if $\tau_-(x,y) = \tau_+(x,y) =: \tau(x,y)$. If the norm $\|\cdot\|$ is Gâteaux differentiable at $x$ in all directions $y$, then we say that the norm $\|\cdot\|$ is Gâteaux differentiable at $x$. One can prove by the very definition that for all $\alpha, \beta \in \mathbb{R}$, if $\alpha\beta > 0$, then we have

$$\tau_+(\alpha x, \beta y) = \mid \beta \mid \tau_+(x,y), \quad \tau_-(\alpha x, \beta y) = \mid \beta \mid \tau_-(x,y),$$

and if $\alpha\beta < 0$, then we have

$$\tau_+(\alpha x, \beta y) = - \mid \beta \mid \tau_-(x,y), \quad \tau_-(\alpha x, \beta y) = - \mid \beta \mid \tau_+(x,y).$$

**Proposition 4.1 ([10]).** *Let $(X, \|\cdot\|)$ be a real normed linear space. If there exist two real numbers $\alpha$ and $\beta$ with $\alpha + \beta \neq 0$ such that $\alpha\tau_+(x,y) + \beta\tau_-(x,y)$ is a continuous function of $x, y \in X$ with $x \neq 0$, then the norm of $X$ is Gâteaux differentiable.*

For more information about norm derivatives the reader is referred to [16] and the references therein in which this recently published article considered some new functional equations containing norm derivatives on finite dimensional real normed linear spaces.

In addition to this, we apply the following lemma as an important tool during the proof of our main result in this section.

**Lemma 4.2 ([10]).** *In a real normed linear space $(X, \|\cdot\|)$, for any two vectors $u, v \in X$ and for all $a \in \mathbb{R}$ we have*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \left( \|(\lambda + a)u + v\|^2 - \|\lambda u + v\|^2 \right) = 2a\|u\|^2.$$

**Definition 4.3.** *Let $(X, \|\cdot\|)$ be a real normed linear space. We say that mean $p$-angular distance orthogonality satisfies property (H) in $X$, if $x \perp_{mA}^p y$ implies*

$$\lim_{n\to\infty} \frac{1}{n^p} \int_0^1 \left[\varphi(n,t,x,y) - \psi(n,t,x,y)\right] dt = 0,$$

*where*

$$\varphi(n,t,x,y) := \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} + n^p \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2 \quad \text{and} \quad \psi(n,t,x,y) := \left\| \frac{1+t}{2} \frac{x}{\|x\|^{1-p}} - n^p \frac{1-t}{2} \frac{y}{\|y\|^{1-p}} \right\|^2.$$

It is obvious that if the mean $p$-angular distance orthogonality is positively homogeneous (or additive) to the right in $X$, then it has property (H).

**Theorem 4.4.** *Let $(X, \|\cdot\|)$ be a real normed linear space. If the mean $p$-angular distance orthogonality satisfies property $(H)$, then for any two vectors $x, y \in X$, $x \neq 0$ there exists a unique $\alpha \in \mathbb{R}$, as a specific linear combination of $\tau_+(x,y)$ and $\tau_-(x,y)$, such that $(\alpha x + y) \perp_{mA}^p x$.*

*Proof.* Assume first that $x, y \in X$ are linearly independent. By Theorem 3.1, there exists a real number $\alpha$ such that $(\alpha x + y) \perp_{mA}^p x$. Applying the same notations as in Definition 4.3, we may write

$$\begin{aligned}
\frac{1}{n^p}\varphi(n,t,\alpha x + y, x) &= \frac{1}{n^p} \left\| \frac{1+t}{2} \frac{\alpha x + y}{\|\alpha x + y\|^{1-p}} + n^p \frac{1-t}{2} \frac{x}{\|x\|^{1-p}} \right\|^2 \\
&= \frac{1}{n^p} \left\| \Big( \frac{1+t}{2} \frac{\alpha}{\|\alpha x + y\|^{1-p}} + n^p \frac{1-t}{2} \frac{1}{\|x\|^{1-p}} \Big) x + \frac{1+t}{2} \frac{y}{\|\alpha x + y\|^{1-p}} \right\|^2,
\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n^p}\psi(n,t,\alpha x+y,x) &= \frac{1}{n^p}\Big\|\frac{1+t}{2}\frac{\alpha x+y}{\|\alpha x+y\|^{1-p}} - n^p\frac{1-t}{2}\frac{x}{\|x\|^{1-p}}\Big\|^2\\ &= \frac{1}{n^p}\Big\|\Big(\frac{1+t}{2}\frac{\alpha}{\|\alpha x+y\|^{1-p}} - n^p\frac{1-t}{2}\frac{1}{\|x\|^{1-p}}\Big)x + \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|^2.\end{aligned}$$

Therefore, by Lemma 4.2, we obtain

$$\frac{1}{n^p}\varphi(n,t,\alpha x+y,x) = \frac{1}{n^p}\Big\|n^p\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} + \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|^2 + \frac{1-t^2}{2}\frac{\alpha}{\|x\|^{1-p}\|\alpha x+y\|^{1-p}}\|x\|^2 + \theta_1(n,t,x,y),$$

and

$$\frac{1}{n^p}\psi(n,t,\alpha x+y,x) = \frac{1}{n^p}\Big\|n^p\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} - \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|^2 - \frac{1-t^2}{2}\frac{\alpha}{\|x\|^{1-p}\|\alpha x+y\|^{1-p}}\|x\|^2 + \theta_2(n,t,x,y),$$

in which for fixed $x, y \in X$, $\theta_1$ and $\theta_2$ are functions of $t, n$ such that

$$\theta_1(n,t,x,y) \longrightarrow 0, \qquad \theta_2(n,t,x,y) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Thus we have

$$\begin{aligned}&\frac{1}{n^p}\varphi(n,t,\alpha x+y,x) - \frac{1}{n^p}\psi(n,t,\alpha x+y,x)\\ &= \frac{1}{n^p}\left(\Big\|n^p\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} + \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|^2 - \Big\|n^p\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} - \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|^2\right)\\ &+ (1-t^2)\frac{\alpha}{\|x\|^{1-p}\|\alpha x+y\|^{1-p}}\|x\|^2 + \theta_1(n,t,x,y) - \theta_2(n,t,x,y).\end{aligned}$$

We call $A$ the first term on the right hand side of the above equation and we write it as the product of

$$B = \left(\Big\|\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} + \frac{1+t}{2n^p}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\| + \Big\|\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} - \frac{1+t}{2n^p}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|\right),$$

and

$$C = \left(\Big\|n^p\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} + \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\| - \Big\|n^p\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} - \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|\right).$$

Now, if $n \to \infty$, then it is easy to see that $B$ tends to $(1-t)\|x\|^p$ and if we write

$$C = \frac{\Big\|\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} + \frac{1+t}{2n^p}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\| - \Big\|\frac{1-t}{2}\frac{x}{\|x\|^{1-p}} - \frac{1+t}{2n^p}\frac{y}{\|\alpha x+y\|^{1-p}}\Big\|}{n^{-p}},$$

then we can see that $C$ tends to

$$\tau_+\left(\frac{1-t}{2}\frac{x}{\|x\|^{1-p}}, \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\right) + \tau_-\left(\frac{1-t}{2}\frac{x}{\|x\|^{1-p}}, \frac{1+t}{2}\frac{y}{\|\alpha x+y\|^{1-p}}\right) = \frac{1+t}{2\|\alpha x+y\|^{1-p}}[\tau_+(x,y) + \tau_-(x,y)].$$

Consequently, the limit of $A$ as $n \to \infty$ is

$$\frac{(1-t^2)\|x\|^p}{2\|\alpha x+y\|^{1-p}}[\tau_+(x,y) + \tau_-(x,y)].$$

Since the mean $p$-angular orthogonality in $X$ satisfies property (H), we conclude that

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{1}{n^p}\int_0^1 (\varphi(n,t,\alpha x+y,x) - \psi(n,t,\alpha x+y,x))\,dt\\ &= \int_0^1 \frac{(1-t^2)\|x\|^p}{2\|\alpha x+y\|^{1-p}}[\tau_+(x,y)+\tau_-(x,y)]dt + \int_0^1 \frac{(1-t^2)\alpha}{\|x\|^{1-p}\|\alpha x+y\|^{1-p}}\|x\|^2 dt\\ &= \frac{1}{3}\frac{\|x\|^p}{\|\alpha x+y\|^{1-p}}[\tau_+(x,y)+\tau_-(x,y)] + \frac{2}{3}\frac{\alpha\|x\|^{p+1}}{\|\alpha x+y\|^{1-p}}.\end{aligned}$$

Therefore, $\tau_+(x,y) + \tau_-(x,y) = -2\alpha\|x\|$ and so

$$\alpha(x,y) = -\frac{1}{2\|x\|}[\tau_+(x,y) + \tau_-(x,y)]. \tag{5}$$

If $x \neq 0$ and $y$ are not linearly independent, then $y = \beta x$ for some $\beta \in \mathbb{R}$, and both sides of (5) are equal to $-\beta$. This shows that (5) holds for any two vectors $x \neq 0$ and $y$ in $X$. □

**Lemma 4.5.** *The function $\alpha = \alpha(x,y)$, the existence of which proved in Theorem 4.4, is continuous on $X$ as a function of $x \neq 0$ and $y$.*

*Proof.* Let $x, y \in X$ and $x \neq 0$. Consider the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $x_n \to x$, $y_n \to y$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Then since $|\tau_\pm(x,y)| \leq \|y\|$, the sequence $\{\alpha_n = \alpha(x_n, y_n)\}$ is bounded and thus any subsequence $\{\alpha_{n_k}\}$ of which has a further subsequence $\{\alpha_{n_{k_\ell}}\}$ which is convergent to a real number $\alpha$. Since $(\alpha_{n_{k_\ell}} x_{n_{k_\ell}} + y_{n_{k_\ell}}) \perp^p_{mA} x_{n_{k_\ell}}$ for every $\ell \in \mathbb{N}$, by continuity of the mean $p$-angular distance orthogonality (Proposition 2.2), we get that $(\alpha x + y) \perp^p_{mA} x$ and therefore, the whole sequence $\{\alpha_n\}$ converges to $\alpha = \alpha(x,y)$ by uniqueness of $\alpha$. □

The following corollary is an immediate consequence of the above lemma and Proposition 4.1.

**Corollary 4.6.** *Let $(X, \|\cdot\|)$ be a real normed linear space. If the mean $p$-angular distance orthogonality satisfies property $(H)$, then the norm of $X$ is Gâteaux differentiable.*

Now, let us consider any two vectors $x, y \in X$ with $x \neq 0$. Suppose that the mean $p$-angular distance orthogonality in $X$ satisfies property (H). Then, Gâteaux differentiability of the norm and the assumption $y \perp^p_{mA} x$ entail by Theorem 4.4 to $\tau(x,y) = 0$. As a result, $x \perp_{BJ} y$. The opposite implication is also true.

**Corollary 4.7.** *Let $(X, \|\cdot\|)$ be a real normed linear space. Suppose that the mean $p$-angular distance orthogonality in $X$ satisfies property (H). Then $y \perp^p_{mA} x$ if and only if $x \perp_{BJ} y$ for every $x, y \in X$ with $x \neq 0$.*

Combining Theorem 1.1 and the previous Corollary, leads us to the following characterization of inner product spaces.

**Corollary 4.8.** *Let $(X, \|\cdot\|)$ be a real normed linear space the dimension of which is at least 3. Suppose that the mean $p$-angular distance orthogonality in $X$ satisfies property (H). Then the following statements are equivalent:*

- *$X$ is an inner product space.*
- *$\perp^p_{mA} \subseteq \perp_{BJ}$ (i.e., $x \perp^p_{mA} y \Rightarrow x \perp_{BJ} y$ for all $x, y \in X$).*

In particular, if the mean $p$-angular distance orthogonality in $X$ is symmetric and satisfies property (H), then $\perp^p_{mA} = \perp_{BJ}$ and so $X$ is an inner product space.

**Corollary 4.9.** *Let $(X, \|\cdot\|)$ be a real normed linear space the dimension of which is at least 3. Suppose that the mean $p$-angular distance orthogonality in $X$ is symmetric. Then the following conditions are equivalent:*

- *$X$ is an inner product space;*
- *$\perp^p_{mA}$ is additive;*
- *$\perp^p_{mA}$ is homogeneous.*

Koldobsky [21] proved that a linear mapping $T : X \to X$ on a real normed linear space $X$ preserving Birkhoff-James orthogonality, has to be an isometry. As a direct consequence of this and Corollary 4.7, we have the following result.

**Corollary 4.10.** *Suppose that $(X, \|\cdot\|)$ is a real normed linear space and the mean $p$-angular distance orthogonality in $X$ satisfies property (H). Let $T : X \longrightarrow X$ be a bounded linear operator satisfying*

$$x \perp_{BJ} y \Longrightarrow Ty \perp^p_{mA} Tx \qquad (\forall x, y \in X).$$

*Then $T$ is a scalar multiple of an isometry.*

## 5. Declarations

**Ethical Approval**

This work has not been published in or submitted for publication to any other journals. All the works consulted have been properly cited and mentioned in the references. There is no any ethical issue regarding the publication of the article.

**Competing interests**

The authors declare that they do not have any conflict of interest for the publication of the article.

**Author's contributions**

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