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Some properties of proximal homotopy theory

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Abstract. Nearness theory comes into play in homotopy theory because the notion of closeness between points is essential in determining whether two spaces are homotopy equivalent. While nearness theory and homotopy theory have different focuses and tools, they are intimately connected through the concept of a metric space and the notion of proximity between points, which plays a central role in both areas of mathematics. This manuscript investigates some concepts of homotopy theory in proximity spaces. Moreover, these concepts are taken into account in descriptive proximity spaces.

1. Introduction

Topological perspective first appears in the scientific works of Riemann and Poincaré in the 19th century[21, 22]. The concept reveals that the definitions of topological space emerge either through Kuratowski's closure operator[4] or through the use of open sets. Given the Kuratowski's closure operator, there are many strategies and approaches that seem useful in different situations and are worth developing, as in nearness theory. Proximity spaces are created by reflection of the concept of being near/far on sets. For instance, one can consider a nearness relation as follows: Given a nonempty set X, and any subsets $E, F \subset X$, we say that E is near F if $E \cap F \neq \emptyset$. A method based on the idea of near sets is first proposed by Riesz, is revived by Wallace, and is axiomatically elaborated it by Efremovic[2, 23, 25]. Let X be a nonempty set. A proximity is a binary relation (actually a nearness relation) defined on subsets of X and generally denoted by δ. One can construct a topology on X induced by the pair (X, δ) using the closure operator (named a proximity space). Indeed, for any point $x \in X$, if $\{x\}$ is near E, then $x \in clE$. In symbols, if $\{x\}\ \delta\ E$, then $x\ \delta\ clE$. It should be noted that the notation $x\ \delta\ E$ is sometimes used instead of $\{x\}\ \delta\ E$ for abbreviation (in particular in a metric space). It appears that several proximities may correspond in this way to the same topology on X. Moreover, several topological conclusions can be inferred from claims made about proximity spaces.

The near set theory is reasonably improved by Smirnov's compactification, Leader's non-symmetric proximity, and Lodato's symmetric proximity[5, 6, 24]. Peters also contributes to the theory of nearness by introducing the concept of spatial nearness and descriptive nearness[11, 12]. In addition, the strong structure of proximity spaces stands out in the variety of application areas: In [8], it is possible to see the construction of proximity spaces in numerous areas such as cell biology, the topology of digital images,

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visual marketing, and so on. In a broader context, the application areas of near spaces are listed along with the history of the subject in [13]. According to this, some near set theory-related topics are certain engineering problems, image analysis, and human perception. The main subject of this article, algebraic topology approaches in proximity spaces, is a work in progress in the literature. Mapping spaces, one of the fundamental concepts in homotopy theory, is examined in proximity spaces in [10]. The proximal setting of the notion of fibration is first defined in [20]. Peters and Vergili have recently published interesting research on descriptive proximal homotopy, homotopy cycles, path cycles, and Lusternik-Schnirelmann theory of proximity spaces[17–20].

This paper is primarily concerned with the theory of proximal homotopy and is organized as follows. In Section 2, we discuss the general properties of proximity and descriptive proximity spaces. Section 3 covered four main topics in proximity spaces: Mapping spaces, covering spaces, fibrations, and cofibrations. They provide different types of examples and frequently used algebraic topology results in proximal homotopy cases. Next, the descriptive proximal homotopy theory in Section 5 discusses the ideas from Section 3 with presenting some interesting examples. Finally, the last section establishes a direction for future works by clearly emphasizing the application areas of homotopy theory.

2. Preliminaries

Before proceeding with the main results, it is critical to remember the fundamental characteristics of proximity and descriptive proximity spaces.

2.1. On proximity spaces

Consider a nonempty space X. A binary relation δ on the collection of the subsets of X which satisfies

(a) $E\,\delta\,F \;\Rightarrow\; F\,\delta\,E$,

(b) $(E \cup F)\,\delta\,G \;\Leftrightarrow\; E\,\delta\,G \,\vee\, F\,\delta\,G$,

(c) $E\,\delta\,F \;\Rightarrow\; E \neq \emptyset \,\wedge\, F \neq \emptyset$,

(d) $E \cap F \neq \emptyset \;\Rightarrow\; E\,\delta\,F$,

(e) $E\,\delta\,F \,\wedge\, \{f\}\,\delta\,G$ for all $f \in F \;\Rightarrow\; E\,\delta\,G$

is said to be a *Lodato Proximity* (denoted by L-proximity)[6]. δ is a nearness relation and $E\,\delta\,F$ is read as "*E is near F*". Otherwise, the notation $E\,\underline{\delta}\,F$ means that "*E* is *far from F*". If the nearness relation δ satisfies only the axioms **(a)**-**(d)**, then δ is said to be a *Čech proximity* (denoted by C-proximity)[1].

Definition 2.1. ([2, 7, 24]) The nearness relation δ for the subsets of X is said to be an *Efremovic proximity* (simply denoted by *EF-proximity* or *proximity*) provided that δ satisfies **(a)**-**(d)**, and in addition,

(f) $E\,\underline{\delta}\,F \;\Rightarrow\; \exists H \subset X : E\,\underline{\delta}\,H \,\wedge\, (X - H)\,\underline{\delta}\,F$.

Then the pair (X, δ) is said to be an *EF-proximity* (or *proximity*) *space*.

As an example of a proximity space, the *discrete proximity* δ on a (nonempty) set X is defined by $E\,\delta\,F \;\Leftrightarrow\; E \cap F \neq \emptyset$ for $E, F \subset X$. Also, the *indiscrete proximity* δ' on a (nonempty) set X is given by $E\,\delta'\,F$ for every nonempty subsets E and F in X. A subset E of X with a proximity δ is a closed set if $x\,\delta\,E \;\Rightarrow\; x \in E$. The converse is also valid. Therefore, given a proximity δ on X, the topology $\tau(\delta)$ is defined by the family of complements of all closed sets the Kuratowski closure operator [7].

Theorem 2.2. ([7]) *For a proximity δ and a topology $\tau(\delta)$ on a set X, we have that the closure clE coincides with $\{x : x\,\delta\,E\}$.*

Recall that a set E is closed in a proximity space if and only if $E = cl(E)$. Given any proximities δ and δ' on respective sets X and X', a map h from X to X' is called *proximally continuous* if $E\ \delta\ F \Rightarrow h(E)\ \delta'\ h(F)$ for $E, F \subset X$[2, 24]. We denote a proximally continuous map by "pc-map". If we have two pc-maps, then their composition is also a pc-map. Given a proximity δ on X and a subset $E \subset X$, a *subspace proximity* δ_E is defined on the subsets of E as follows[7]: $E_1\ \delta_E\ E_2 \Leftrightarrow E_1\ \delta\ E_2$ for $E_1, E_2 \subset E$. Let (X, δ) be a proximity space and (E, δ_E) a subspace proximity. A pc-map $k : (X, \delta) \to (E, \delta_E)$ is a *proximal retraction* provided that $k \circ j$ is an identity map on 1_E, where $j : (E, \delta_E) \to (X, \delta)$ is an inclusion map.

Lemma 2.3. ([17][Gluing Lemma]) *Assume that $f_1 : (X', \delta'_1) \to (Y', \delta'_2)$ and $f_2 : (X'', \delta''_1) \to (Y', \delta'_2)$ are pc-maps with the property that they agree on the intersection of X' and X''. If X and X' are closed subsets, then the map $f_1 \cup f_2 : (X' \cup X'', \delta) \to (Y', \delta'_2)$, defined by $f_1 \cup f_2(s) = \begin{cases} f_1(s), & s \in X' \\ f_2(s), & s \in X'' \end{cases}$ for any $s \in X' \cup X''$, is a pc-map.*

We say that h is a *proximity isomorphism* provided that h is a bijection and both of h and h^{-1} are pc-maps[7]. According to this, (X, δ) and (X', δ') are said to be *proximally isomorphic spaces*. Another important proximity relation is given on the subsets of the cartesian product of two proximity spaces as follows[5]: Let δ and δ' be any proximities on respective sets X and X'. For any subsets $E_1 \times E_2$ and $F_1 \times F_2$ of $X \times X'$, $E_1 \times E_2$ *is near* $F_1 \times F_2$ if $E_1\ \delta\ F_1$ and $E_2\ \delta'\ F_2$.

Definition 2.4. ([17, 19]) Given two pc-maps h_1 and h_2 from X to X', if there is a pc-map F from $X \times I$ to X' with the properties $F(x, 0) = h_1(x)$ and $F(x, 1) = h_2(x)$, then h_1 and h_2 are called *proximally homotopic maps*.

The map F in Definition 2.4 is said to be a *proximal homotopy between h and h'*. We simply denote a proximal homotopy by "prox-hom". Similar to topological spaces, prox-hom is an equivalence relation on proximity spaces[19]. Let δ be a proximity on X and $E \subset X$. E is called a *$\delta-$neighborhood of F*, denoted by $F \ll_\delta E$, provided that $F\ \underline{\delta}\ (X - E)$[7]. The proximal continuity of any function $h : (X, \delta) \to (X', \delta')$ can also be expressed as

$$E \ll_{\delta'} F \Rightarrow h^{-1}(E) \ll_\delta h^{-1}(F)$$

for any $E, F \subset X'$.

Theorem 2.5. ([7]) *Let $E_k \ll_\delta F_k$ for all $k = 1, \cdots, r$. Then*

$$\bigcap_{k=1}^{r} E_k \ll_\delta \bigcap_{k=1}^{r} F_k \quad \text{and} \quad \bigcup_{k=1}^{r} E_k \ll_\delta \bigcup_{k=1}^{r} F_k.$$

Definition 2.6. ([17]) For any two elements x_1 and x_2 in X with a proximity δ, a *proximal path from x_1 to x_2* in X is a pc-map h from $I = [0, 1]$ to X for which $h(0) = x_1$ and $h(1) = x_2$.

Recall that X is a *connected proximity space* if and only if for all nonempty subsets $E, F \in 2^X$, $E \cup F = X$ implies that $E\ \delta\ F$[9]. Let δ be a proximity on X. Then X is called a *path-connected proximity space* if, for any points x_1 and x_2 in X, there exists a proximal path from x_1 to x_2 in X.

Lemma 2.7. *Proximal path-connectedness implies proximal connectedness as in the same as topological spaces.*

Proof. Let δ be a path-connected proximity on X. Suppose that (X, δ) is not proximally connected. Then there exist two nonempty subsets E, F in X such that $E \cup F = X$ and $E\ \underline{\delta}\ F$. Since X is proximally path-connected, there is a pc-map $h : I \to X$ with $h(0) = E$ and $h(1) = F$. Consider the subsets $h^{-1}(E)$ and $h^{-1}(F) \in I$. They are nonempty sets because $0 \in h^{-1}(E)$ and $1 \in h^{-1}(F)$. Their union is $[0, 1]$, and by the proximal continuity of h, $h^{-1}(E)\ \underline{\delta}\ h^{-1}(F)$. This contradicts with the fact that $[0, 1]$ is proximally connected. Consequently, X is proximally connected. □

Theorem 2.8. *Proximal path-connectedness coincides with proximal connectedness when we consider the proximity relation δ on a metric space X as follows: $E\ \delta\ F$ if and only if $D(E, F) = 0$ for all E, F in 2^X.*

Proof. Given a proximity δ on X, by Lemma 2.7, it is enough to prove that any connected proximity space is a path-connected proximity space. Suppose that X is not a path-connected proximity space. Then any map $h : (I, \delta') \to (X, \delta)$ with $h(0) = x$ and $h(1) = y$ is not proximally continuous, i.e., if $E\ \delta'\ F$ for all $E, F \in [0, 1]$, then $h(E)\ \underline{\delta}\ h(F)$. Take $E = \{0\} \subset I$ and $F = (0, 1] \subset I$. Since $D(E, F) = \inf\{d(0, z) : z \in F\} = 0$, we have that $E\ \delta\ F$. It follows that $h(E) = \{x\}$ is not near to $h(F) = X \setminus \{x\}$. On the other hand,

$$h(E) \cup h(F) = \{x\} \cup X \setminus \{x\} = X.$$

Thus, X is not proximally connected and this is a contradiction. □

2.2. On descriptive proximity spaces

Assume that X is a nonempty set and $x \in X$. Consider the set $\Phi = \{\phi_1, \cdots, \phi_m\}$ of maps (generally named as probe functions) $\phi_j : X \to \mathbb{R}$, $j = 1, \cdots, m$, where $\phi_j(x)$ denotes a feature value of x. Let $E \subset X$. Then the set of descriptions of a point e in E, denoted by $\mathcal{Q}(E)$, is given by the set $\{\Phi(e) : e \in E\}$, where $\Phi(e)$ (generally called a feature vector for e) equals $(\phi_1(e), \cdots, \phi_m(e))$. For $E, F \subset X$, the binary relation δ_Φ is defined by

$$E\ \delta_\Phi\ F \iff \mathcal{Q}(E) \cap \mathcal{Q}(F) \neq \emptyset,$$

and $E\ \delta_\Phi\ F$ is read as "E is *descriptively near* F"[11, 12, 14]. Also, $E\ \overline{\delta_\Phi}\ F$ is often used to state "E is *descriptively far from* F". The *descriptive intersection of E and F* and the *descriptive union of E and F* are defined by

$$E \bigcap_\Phi F = \{x \in E \cup F : \Phi(x) \in \mathcal{Q}(E)\ \wedge\ \Phi(x) \in \mathcal{Q}(F)\},$$

and

$$E \bigcup_\Phi F = \{x \in E \cup F : \Phi(x) \in \mathcal{Q}(E)\ \vee\ \Phi(x) \in \mathcal{Q}(F)\},$$

respectively[14].

Then a binary relation δ_Φ given in by (1)[8] satisfies

(f) $E\ \delta_\Phi\ F \Rightarrow E \neq \emptyset \wedge F \neq \emptyset$,

(g) $E \bigcap_\Phi F \neq \emptyset \Rightarrow E\ \delta_\Phi\ F$,

(h) $E \bigcap_\Phi F \neq \emptyset \Rightarrow F \bigcap_\Phi E$,

(i) $E\ \delta_\Phi\ (F \cup G) \iff E\ \delta_\Phi\ F\ \vee\ E\ \delta_\Phi\ G$,

(k) $E\ \underline{\delta_\Phi}\ F \Rightarrow \exists G \subset X : E\ \underline{\delta_\Phi}\ G\ \wedge\ (X - G)\ \underline{\delta_\Phi}\ F.$

δ_Φ is a descriptive nearness relation.

Definition 2.9. ([8]) The nearness relation δ_Φ for the subsets of X is said to be an *descriptive Efremovic proximity* (simply denoted by *descriptive EF-proximity* or *descriptive proximity*) if δ_Φ satisfies **(f)-(k)**. (X, δ_Φ) is said to be a *descriptive EF-proximity* (or *descriptive proximity*) *space*.

A map $h : (X, \delta_\Phi) \to (X, \delta_{\Phi'})$ is called a *descriptive proximally continuous* provided that

$$E\ \delta_\Phi\ F \Rightarrow h(E)\ \delta_{\Phi'}\ h(F)$$

for E, $F \subset X$[15, 17]. We denote a descriptive proximally continuous map by "dpc-map". Let δ_Φ be a descriptive proximity on X, and $E \subset X$ a subset. Then a *descriptive subspace proximity* δ_Φ^E is defined on the subsets of E as follows:

$$E_1\ \delta_\Phi\ E_2 \iff E_1\ \delta_\Phi^E\ E_2$$

for E_1, $E_2 \subset E$. Given a descriptive proximity δ_Φ on X, a descriptive subspace proximity (E, δ_Φ^E), and the inclusion $j : (E, \delta_\Phi^E) \to (X, \delta_\Phi)$, a dpc-map $k : (X, \delta_\Phi) \to (E, \delta_\Phi^E)$ is called a *descriptive proximal retraction* if $k \circ j = 1_E$.

To state Gluing Lemma in descriptive proximity spaces, we need to refer to descriptively closed sets. In [16], a descriptive closure $cl_\Phi(E)$ of a set $E \in 2^X$ is given by the set $\{x \in X : \Phi(x) \in \mathcal{Q}(cl(E))\}$. Thus, a set E is descriptively closed in a descriptive proximity space if and only if $E = cl_\Phi(E)$.

Lemma 2.10. ([17, 19][Descriptive Gluing Lemma]) *Let X' and X'' be two descriptively closed subsets and assume that $f_1 : (X', \delta_{\Phi_1'}) \to (Y', \delta_{\Phi_2'})$ and $f_2 : (X'', \delta_{\Phi_1''}) \to (Y', \delta_{\Phi_2'})$ are two dpc-maps with the property that they agree on the intersection of X' and X''. Then the map $f_1 \cup f_2$ from $(X' \cup X'', \delta_\Phi)$ to $(Y', \delta_{\Phi_2'})$, defined by* $f_1 \cup f_2(s) = \begin{cases} f_1(s), & s \in X' \\ f_2(s), & s \in X'' \end{cases}$ *for any $s \in X' \cup X''$, is a dpc-map.*

h is a *descriptive proximity isomorphism* if h is a bijection and each of h and h^{-1} is a dpc-map [7]. Hence, (X, δ_Φ) and $(X', \delta_{\Phi'})$ are called *descriptive proximally isomorphic spaces*. A descriptive proximity relation on the cartesian product of descriptive proximity spaces is defined as follows[5]: Assume that δ_Φ and $\delta_{\Phi'}$ are any descriptive proximities on X and X', respectively. For two subsets $E \times F$ and $E' \times F'$ of $X \times X'$, we say that $(E \times F)\ \delta_\Phi\ (E' \times F')$ if and only if $E\ \delta_\Phi\ E'$ and $F\ \delta_{\Phi'}\ F'$.

Definition 2.11. ([17, 19]) Let $h_1, h_2 : (X, \delta_\Phi) \to (X', \delta_{\Phi'})$ be any map. Then h_1 and h_2 are said to be *descriptive proximally homotopic maps* provided that there exists a dpc-map $G : X \times I \to X'$ with $G(x, 0) = h_1(x)$ and $G(x, 1) = h_2(x)$.

In Definition 2.11, G is a *descriptive proximal homotopy between h_1 and h_2*. We simply denote a descriptive proximal homotopy by "dprox-hom". Given a descriptive proximity δ_Φ on X and a subset $F \subset X$, F is said to be a δ_Φ*−neighborhood of E*, denoted by $E \ll_{\delta_\Phi} F$, if $E\ \underline{\delta_\Phi}\ (X - F)$[15].

Theorem 2.12. ([7]) *Let $E_j \ll_{\delta_\Phi} F_j$ for $j = 1, \cdots, m$. Then*

$$\bigcap_{j=1}^{m} E_j \ll_{\delta_\Phi} \bigcap_{j=1}^{m} F_j \ \ textand \ \ \bigcup_{j=1}^{m} E_j \ll_{\delta_\Phi} \bigcup_{j=1}^{m} F_j.$$

Definition 2.13. ([17]) Let x_1 and x_2 be any two elements in X with a descriptive proximity δ_Φ. Then a *descriptive proximal path from x_1 to x_2* in X is a dpc-map h from $I = [0, 1]$ to X for which $h(0) = x_1$ and $h(1) = x_2$.

A descriptive proximity space (X, δ_Φ) is *connected* if and only if for all nonempty $E, F \in 2^X$, $E \cup F = X$ implies that $E\ \delta_\Phi\ F$[9]. A descriptive proximity space (X, δ_Φ) is *path-connected* if, for any points x_1 and x_2 in X, there exists a descriptive proximal path from x_1 to x_2 in X.

Theorem 2.14. *In a descriptive proximity space, path-connectedness coincides with connectedness.*

Proof. Follow the method in the proof of Theorem 2.8. □

3. Homotopy theory on proximity spaces

This section, one of the main parts (Section 3 and Section 4) of the paper, examines the projection of the homotopy theory elements in parallel with the proximity spaces. First, we start with the notion of proximal mapping spaces. Then we have proximal covering spaces. The last two parts are related to proximal fibrations and its dual notion of proximal cofibrations. Results on these four topics that we believe will be relevant to future proximity space research are presented.

3.1. Proximal mapping spaces

The work of mapping spaces in nearness theory starts with [10] and is still open to improvement. Note that the study of discrete invariants of function spaces is essentially homotopy theory in algebraic topology, and recall that depending on the nature of the spaces, it may be useful to attempt to impose a topology on the space of continuous functions from one topological space to another. One of the best-known examples of this is the compact-open topology.

Definition 3.1. Let δ_1 and δ_2 be two proximities on X and Y, respectively. The proximal mapping space Y^X is defined as $\{\alpha : X \to Y \mid \alpha$ is a pc-map$\}$ having the following proximity relation δ on itself: Let $E, F \subset X$ and $\{\alpha_i\}_{i\in I}$ and $\{\beta_j\}_{j\in J}$ be any subsets of pc-maps in Y^X. We say that $\{\alpha_i\}_{i\in I}\ \delta\ \{\beta_j\}_{j\in J}$ if the fact $E\ \delta_1\ F$ implies that $\alpha_i(E)\ \delta_2\ \beta_j(F)$ for all $i \in I$ and $j \in J$.

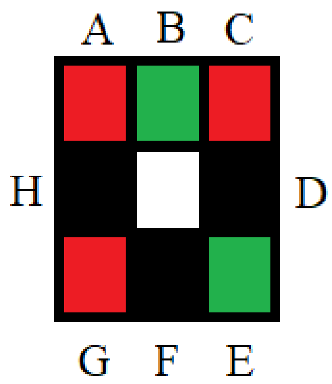


Figure 1: The picture represented by X is given by the set $\{A, B, C, D, E, F, G, H\}$.

Example 3.2. Consider the set $X = \{A, B, C, D, E, F, G, H\}$ of cells in Figure 1 with the proximity δ defined by $\mathrm{bd}(E_1) \cap E_2 \neq \emptyset$ for any subsets E_1 and E_2 in 2^X, where $\mathrm{bd}(E_1)$ denotes the boundary of E_1 (see [19] for more details of the boundary of a set in proximity spaces). Define three proximal paths α_1, α_2, and $\alpha_3 \in X^I$ by

$$\alpha_1 : A \mapsto B \mapsto C \mapsto D \mapsto E \mapsto F \mapsto G \mapsto H,$$

$$\alpha_2 : H \mapsto A \mapsto B \mapsto C \mapsto D \mapsto E \mapsto F \mapsto G,$$

$$\alpha_3 : A \mapsto H \mapsto G \mapsto F \mapsto E \mapsto D \mapsto C \mapsto B.$$

For all $t \in I$, $\alpha_1(t)\ \delta\ \alpha_2(t)$. This means that α_1 is near α_2. On the other hand, for $t \in [2/8, 3/8]$, we have that $\alpha_1(t) = C$ and $\alpha_3(t) = G$, that is, α_1 and α_3 are not near in X.

Remark 3.3. For the proximal continuity of a map $H : (X, \delta_1) \to (Z^Y, \delta')$, we say that the fact $E\ \delta_1\ F$ implies that $H(E)\ \delta'\ H(F)$ for any subsets $E, F \subset X$.

Proposition 3.4. *Let δ_1, δ_2, and δ_3 be any proximities on the sets X, Y, and Z, respectively. Then the map $G : (X \times Y, \delta'') \to (Z, \delta_3)$ is pc-map if and only if the map $H : (X, \delta_1) \to (Z^Y, \delta')$ defined by $H(E)(F) := G(E \times F)$ is pc-map for $E \subset X$ and $F \subset Y$.*

Proof. Assume that $E_1\ \delta_1\ F_1$ for $E_1, F_1 \subset X$. If $E_2\ \delta_2\ F_2$ for $E_2, F_2 \subset Y$, then we find $(E_1 \times E_2)\ \delta''\ (F_1 \times F_2)$. Since G is a pc-map, we get $G(E_1 \times E_2)\ \delta_3\ G(F_1 \times F_2)$. It follows that $H(E_1)(E_2)\ \delta_3\ H(F_1)(F_2)$. This shows that $H(E_1)\ \delta'\ H(F_1)$, i.e., H is a pc-map. Conversely, assume that $(E_1 \times E_2)\ \delta''\ (F_1 \times F_2)$. Then we get $E_1\ \delta_1\ F_1$ in X and $E_2\ \delta_2\ F_2$ in Y. Since H is a pc-map, we get $H(E_1)\ \delta'\ H(F_1)$. So, we have that $H(E_1)(E_2)\ \delta_3\ H(F_1)(F_2)$. This leads to the fact that $G(E_1 \times E_2)\ \delta''\ G(F_1 \times F_2)$, namely that, G is a pc-map. □

Theorem 3.5. *Let δ_1, δ_2, and δ_3 be any proximities on X, Y, and Z, respectively. Then $(Z^{X\times Y}, \delta_4)$ and $((Z^Y)^X, \delta_5)$ are proximally isomorphic spaces.*

Proof. Define a bijective map $f : Z^{X\times Y} \to (Z^Y)^X$ by $f(G) = H$. For any pc-maps G, $G' \subset Z^{X\times Y}$ such that $G\ \delta_4\ G'$, we have that $f(G)\ \delta_5\ f(G')$. Indeed, for $E_1\times E_2$, $F_1\times F_2 \subset X\times Y$, we have that $G(E_1\times E_2)\ \delta_3\ G(F_1\times F_2)$. This means that $H(E_1)(E_2)\ \delta_3\ H(F_1)(F_2)$. Another saying, we find $H\ \delta_5\ H'$. Therefore, f is a pc-map. For the proximal continuity of f^{-1}, assume that $H\ \delta_5\ H'$. Then we have that $H(E_1)$ and $H'(F_1)$ are near in Z^Y for E_1, $F_1 \subset X$. If $E_2\ \delta_2\ F_2$ in Y, then we have that $H(E_1)(E_2)\ \delta_3\ H'(F_1)(F_2)$. It follows that $G(E_1\times E_2)\ \delta_3\ G'(F_1\times F_2)$. Thus, we obtain that $G\ \delta_4\ G'$, which means that $f^{-1}(H)\ \delta_4\ f^{-1}(H')$. As a result, f is a proximity isomorphism. □

Theorem 3.6. *Let δ_1, δ_2, and δ_3 be any proximities on X, Y, and Z, respectively. Then $((Y\times Z)^X, \delta_4)$ and $(Y^X\times Z^X, \delta_5)$ are proximally isomorphic spaces.*

Proof. The proximal isomorphism is given by the map

$$f : ((Y \times Z)^X, \delta_4) \to (Y^X \times Z^X, \delta_5)$$

with $f(\alpha) = (\pi_1 \circ \alpha, \pi_2 \circ \alpha)$, where π_1 and π_2 are the $i-$th projection maps from $Y\times Z$ to the respective spaces. For any $\{\alpha_i\}_{i\in I}$, $\{\beta_j\}_{j\in J} \subset (Y\times Z)^X$ such that $\{\alpha_i\}_{i\in I}$ is near $\{\beta_j\}_{j\in J}$, we obtain that $\pi_k \circ \{\alpha_i\}_{i\in I}$ is near $\pi_k \circ \{\beta_i\}_{j\in J}$ for each $k \in \{1, 2\}$. Therefore, we have that $(\pi_1 \circ \{\alpha_i\}_{i\in I}, \pi_2 \circ \{\alpha_i\}_{i\in I})$ is near $(\pi_1 \circ \{\beta_j\}_{j\in J}, \pi_2 \circ \{\beta_j\}_{j\in J})$. Thus, $f(\{\alpha_i\}_{i\in I})$ is near $f(\{\beta_j\}_{j\in J})$, i.e., f is a pc-map. For the pc-map

$$g : (Y^X \times Z^X, \delta_5) \to ((Y \times Z)^X, \delta_4)$$

with $g(\beta, \gamma) = (\beta \times \gamma) \circ \Delta_X$, where $\Delta_X : (X, \delta_1) \to (X \times X, \delta_1')$ is a diagonal map of proximity spaces on X, we have that $g \circ f$ and $f \circ g$ are identity maps on respective proximity spaces $(Y\times Z)^X$ and $Y^X\times Z^X$. Note that Δ_X is also pc (similarly, in descriptive proximity spaces, the diagonal map is dpc) because $A\ \delta_1\ B$ implies that $(A, A)\ \delta_1'\ (B, B)$ by the property of cartesian product proximity. Consequently, $((Y\times Z)^X, \delta_4)$ and $(Y^X\times Z^X, \delta_5)$ are proximally isomorphic spaces. □

Definition 3.7. Let δ_1 and δ_2 be any proximities on X and Y, respectively. Then the proximal evaluation map

$$e_{X,Y} : (Y^X \times X, \delta) \to (Y, \delta_2)$$

is defined by $e(\alpha, x) = \alpha(x)$.

To show that the evaluation map $e_{X,Y}$ is a pc-map, we first assume that $(\{\alpha_i\}_{i\in I}\times E)\ \delta\ (\{\beta_j\}_{j\in J}\times F)$ in $Y^X\times X$. This means that $\{\alpha_i\}_{i\in I}\ \delta'\ \{\beta_j\}_{j\in J}$ for a proximity relation δ' on Y^X and $E\ \delta_1\ F$ in X. It follows that $\alpha_i(E)\ \delta_2\ \beta_j(F)$ in Y for any $i \in I$ and $j \in J$. Finally, we conclude that

$$e_{X,Y}(\{\alpha_i\}_{i\in I} \times E)\ \delta_2\ e_{X,Y}(\{\beta_j\}_{j\in J} \times F).$$

Example 3.8. Consider the proximal evaluation map $e_{I,X} : (X^I \times I, \delta) \to (X, \delta_1)$. Since $X^I \times \{0\}$ is proximally isomorphic to X^I by the map $(\alpha, 0) \mapsto \alpha(0)$, the restriction

$$e^0_{I,X} = e_{I,X}|_{(X^I\times\{0\})} : (X^I, \delta') \to (X, \delta_1),$$

defined by $e^0_{I,X}(\alpha) = \alpha(0)$, is a pc-map.

Example 3.9. Let $e_{I,X\times X} : ((X \times X)^I \times I, \delta) \to (X, \delta_1)$ be the proximal evaluation map. By Theorem 3.6, the restriction

$$e^0_{I,X\times X} = e_{I,X\times X}|_{(X^I\times\{0\})} : (X^I, \delta') \to (X \times X, \delta'),$$

defined by $e^0_{I,X\times X}(\alpha) = (\alpha(0), \alpha(1))$, is a pc-map.

Note that, in topological spaces, the map $X^I \to X \times X$, $\alpha \mapsto (\alpha(0), \alpha(1))$, is a path fibration. Similarly, the map $X^I \to X$, $\alpha \mapsto \alpha(0)$, is the path fibration with a fixed initial point at $t = 0$.

3.2. *Proximal covering spaces*

A covering space of a topological space and the fundamental group are tightly related. One can categorize all the covering spaces of a topological space using the subgroups of its fundamental group. Covering spaces are not only useful in algebraic topology but also in complex dynamics, geometric group theory, and the theory of Lie groups.

Definition 3.10. A surjective pc-map $p : (X,\delta) \to (X',\delta')$ is a proximal covering map if the followings hold:

- Let $\{x'\} \subseteq X'$ be any subset with $\{x'\} \ll_{\delta'} Y'$. Then there is an index set I satisfying that

$$p^{-1}(Y') = \bigcup_{i\in I} Y_i$$

  with $V_i \ll_{\delta} Y_i$, where $V_i \in p^{-1}(\{x'\})$ for each $i \in I$.
- $Y_i \neq Y_j$ when $i \neq j$ for $i, j \in I$.
- $p|_{Y_i} : Y_i \to Y'$ is a proximal isomorphism for every $i \in I$.

In Definition 3.10, (X,δ) is called a proximal covering space of (X',δ'). For $i \in I$, Y_i is said to be a proximal sheet. For any $x' \in X'$, $p^{-1}(\{x'\})$ is called a proximal fiber of x'. The map $p|_{Y_i} : Y_i \to Y'$ is a proximal isomorphism if the map $p : (X,\delta) \to (X',\delta')$ is a proximal isomorphism. However, the converse is not generally true. Given any proximity δ on X, it is obvious that the identity map on X is always a proximal covering map.

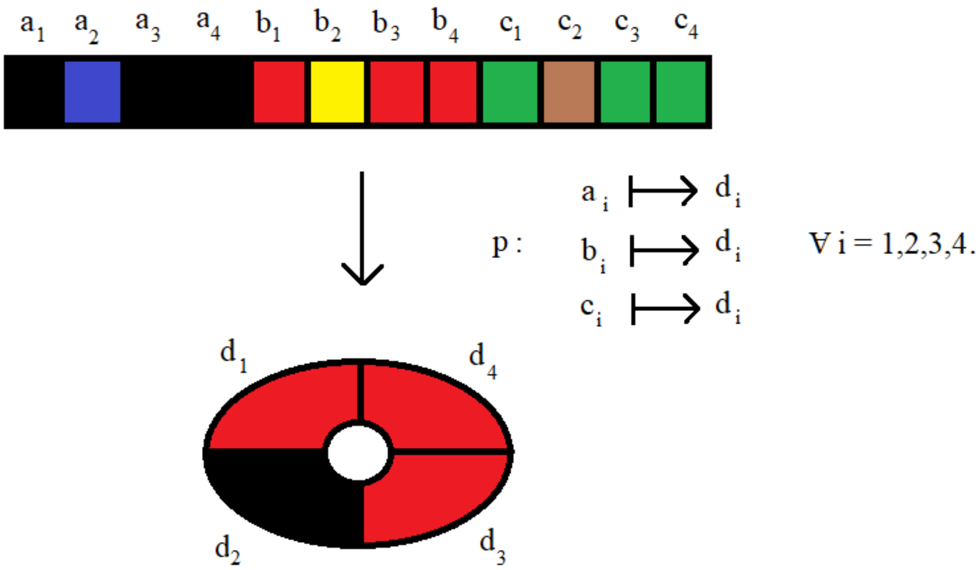


Figure 2: A map p from $\{a_1,a_2,a_3,a_4\}\cup\{b_1,b_2,b_3,b_4\}\cup\{c_1,c_2,c_3,c_4\}$ to $\{d_1,d_2,d_3,d_4\}$ defined by $p(a_i) = p(b_i) = p(c_i) = d_i$ for any $i = 1,2,3,4$.

Example 3.11. Assume that $X = \{a_1,a_2,a_3,a_4\} \cup \{b_1,b_2,b_3,b_4\} \cup \{c_1,c_2,c_3,c_4\}$ and $X' = \{d_1,d_2,d_3,d_4\}$ are two proximity spaces such that $p : (X,\delta) \to (X',\delta')$ is a surjective and pc-map defined by $p(a_i) = p(b_i) = p(c_i) = d_i$ for each $i = 1,2,3,4$ (see Figure 2). Let $\{d_1\} \subset X'$ and $Y' = \{d_1,d_2,d_4\}$ a proximal δ'−neighborhood of $\{d_1\}$. For $V_1 = \{a_1\}$, $V_2 = \{b_1\}$, and $V_3 = \{c_1\}$, we have $p^{-1}(Y') = \bigcup_{i=1}^{3} Y_i$, where $Y_1 = \{a_1,a_2,a_4\}$, $V_2 = \{b_1,b_2,b_4\}$, and $V_3 = \{c_1,c_2,c_4\}$. Note that $V_1 \underline{\delta} (X-Y_1)$, $V_2 \underline{\delta} (X-Y_2)$, and $V_3 \underline{\delta} (X-Y_3)$, i.e., Y_i is a proximal δ−neighborhood of V_i for each $i \in \{1,2,3\}$. Moreover, for $i, j \in \{1,2,3\}$ with $i \neq j$, we have that Y_i is not near Y_j, and $p|_{Y_1} : \{a_1,a_2,a_4\} \to \{d_1,d_2,d_4\}$, $p|_{Y_2} : \{b_1,b_2,b_4\} \to \{d_1,d_2,d_4\}$, and $p|_{Y_3} : \{c_1,c_2,c_4\} \to \{d_1,d_2,d_4\}$ are proximal isomorphisms. For other points d_2, d_3, and d_4, a similar process is done. This shows that p is a proximal covering map.

Example 3.12. Suppose that δ is any proximity relation on the subsets of a nonempty set X. Let $\{0,1,2\cdots\}$ admit a discrete proximity such that the subsets of $X \times \{0,1,2\cdots\}$ have the cartesian product proximity. Consider the surjective and pc-map $p : (X \times \{0,1,2\cdots\}, \delta') \to (X, \delta)$ with $p(x,t) = x$. For a proximal $\delta-$neighborhood Y of any subset $\{x\} \subset X$, we have that

$$p^{-1}(Y) = Y \times \mathbb{Z}^+ \subset X \times \{0,1,2\cdots\}$$

for a proximal $\delta'-$neighborhood Y' of $Y \times \mathbb{Z}^+$. Moreover, $p|_{Y\times\mathbb{Z}^+} : Y \times \mathbb{Z}^+ \to Y$, $p|_{Y\times\mathbb{Z}^+}(x,t) = x$, is a proximal isomorphism. Thus, p is a proximal covering map.

Proposition 3.13. *Any proximal isomorphism is a proximal covering map.*

Proof. Let $p : (X, \delta) \to (X', \delta')$ be a proximal isomorphism. Then p is a pc-map. Lemma 4.1 of [8] says that p is continuous with respect to compatible topologies. Therefore, we get $p^{-1}(Y') = Y \subset X$ for an open neighborhood Y' of any subset $\{x'\} \subseteq X'$. Combining with the fact that a proximal neighborhood of a set is also a neighborhood, we conclude that $p^{-1}(Y') = Y$ for a proximal $\delta'-$neighborhood Y' of $\{x'\}$ in X' and a proximal $\delta-$neighborhood Y of V, where $V \in p^{-1}(\{x'\})$ in X. Furthermore, $p|_Y : Y \to Y'$ is an isomorphism of proximity spaces because p is an isomorphism of proximity spaces. Finally, p is a proximal covering map. □

The following diagram illustrates two ways to prove that any proximal isomorphism $p : (X, \delta) \to (X', \delta')$ is a covering map between respective compatible topologies on both (X, δ) and (X', δ').

$$\begin{array}{ccc} \text{proximal isomorphism} & \longrightarrow & \text{proximal covering map} \\ \downarrow & & \downarrow \\ \text{homeomorphism} & \longrightarrow & \text{covering map.} \end{array}$$

Theorem 3.14. *The cartesian product of two proximal covering maps is a proximal covering map.*

Proof. Let $p : (X, \delta_1) \to (X', \delta'_1)$ and $q : (Y, \delta_2) \to (Y', \delta'_2)$ be two proximal covering maps. Then for a proximal δ'_1-neighborhood M'_1 of $\{x'_1\} \subset X'$, we have that

$$p^{-1}(M'_1) = \bigcup_{i\in I} M_i$$

for a proximal δ_1-neighborhood M_1 of V_i, where $V_i \in p^{-1}(\{x'_1\})$. We also have that $M_i \neq M_k$ with any $k \in I$ when $i \neq k$. Similarly, for a proximal δ'_2-neighborhood N'_2 of $\{x'_2\} \subset Y'$, we have that

$$q^{-1}(N'_2) = \bigcup_{j\in J} N_j$$

for a proximal δ_2-neighborhood N_j of W_j, where $W_j \in q^{-1}(\{x'_2\})$. Also, we have that $N_i \neq N_l$ with any $l \in J$ when $j \neq l$. For a proximal neighborhood $M'_1 \times N'_2$ of $\{x'_1\} \times \{x'_2\} \subset X' \times Y'$, we get

$$(p \times q)^{-1}(M'_1 \times N'_2) = p^{-1}(M'_1) \times q^{-1}(N'_2) = \bigcup_{i\in I} M_i \times \bigcup_{j\in J} N_j = \bigcup_{\substack{i\in I \\ j\in J}} (M_i \times N_j).$$

It is clear that $M_i \times N_j \neq M_k \times N_l$ when $(i, j) \neq (k, l)$ for any $i, k \in I$ and $j, l \in J$. Moreover, since $p|_{M_i} : M_i \to M'_1$ and $q|_{N_j} : N_j \to N'_2$ are proximal isomorphisms, the map $(p \times q)|_{M_i \times N_j} : M_i \times N_j \to M'_1 \times N'_2$ is a proximal isomorphism. Consequently, $p \times q : X \times Y \to X' \times Y'$ is a proximal covering map. □

3.3. Proximal fibrations

Some topological problems can be conceptualized as lifting or extension problems. In the homotopy-theoretic viewpoint, fibrations and cofibrations deal with them, respectively (see Section 3.4 for the detail of cofibrations). Postnikov systems, spectral sequences, and obstruction theory, which are important tools constructed on homotopy theory, involve fibrations. On the other hand, the notion of proximal fibration of proximity spaces is first mentioned in [20], and we extend this with useful properties in proximity cases.

Definition 3.15. A pc-map $p : (X, \delta) \to (X', \delta')$ is said to have the proximal homotopy lifting property (PHLP) with respect to a proximity space (X'', δ'') if for an inclusion map $i_0 : (X'', \delta'') \to (X'' \times I, \delta_1)$, $i_0(x'') = (x'', 0)$, for every pc-map $k : (X'', \delta'') \to (X, \delta)$, and prox-hom $G : (X'' \times I, \delta_1) \to (X', \delta')$ with $p \circ k = G \circ i_0$, then there exists a prox-hom $G' : (X'' \times I, \delta_1) \to (X, \delta)$ for which $G'(x'', 0) = k(x'')$ and $p \circ G'(x'', t) = G(x'', t)$.

$$\begin{array}{ccc} X'' & \xrightarrow{k} & X \\ {\scriptstyle i_0}\downarrow & \overset{G'}{\nearrow} & \downarrow{\scriptstyle p} \\ X'' \times I & \xrightarrow[G]{} & X'. \end{array}$$

Definition 3.16. A pc-map $p : (X, \delta) \to (X', \delta')$ is said to be a proximal fibration if it has the PHLP for any proximity space (X'', δ'').

Example 3.17. For any proximity spaces (X, δ) and (X', δ'), we shall show that the projection map

$$\pi_1 : (X \times X', \delta_2) \to (X, \delta)$$

onto the first factor is a proximal fibration. Consider the diagram

$$\begin{array}{ccc} X'' & \xrightarrow{(k_X, k_{X'})} & X \times X' \\ {\scriptstyle i_0}\downarrow & \overset{G'}{\nearrow} & \downarrow{\scriptstyle \pi_1} \\ X'' \times I & \xrightarrow[G]{} & X' \end{array}$$

with $\pi_1 \circ (k_X, k_{X'}) = G \circ i_0$. Then there is a map $G' : (X'' \times I, \delta_1) \to (X \times X', \delta_2)$ defined by $G' = (G, F)$, where $F : (X'' \times I, \delta_1) \to (X', \delta')$ is the composition of the first projection map $(X'' \times I, \delta_1) \to (X'', \delta'')$ and $k_{X'}$. Since $k_{X'}$ and the first projection map are pc-maps, it follows that F is a pc-map. Moreover, we get $F(x'', 0) = F(x'', 1) = k_{X'}(x'')$, which means that F is a (constant) prox-hom. Combining this result with the fact that H is a prox-hom, we have that G' is a prox-hom. Moreover, we get

$$G' \circ i(x'') = G'(x'', 0) = (G(x'', 0), F(x'', 0)) = (k_X(x''), k_{X'}(x'')) = (k_X, k_{X'})(x''),$$

and

$$\pi_1 \circ G'(x'', t) = \pi_1(G(x'', t), F(x'', t)) = G(x'', t).$$

This shows that π_1 is a proximal fibration.

Example 3.18. Let $c : (X, \delta) \to (\{x_0\}, \delta_0)$ be the constant map of proximity spaces. Given the diagram

$$\begin{array}{ccc} X'' & \xrightarrow{k} & X \\ {\scriptstyle i_0}\downarrow & \overset{G'}{\nearrow} & \downarrow{\scriptstyle p} \\ X'' \times I & \xrightarrow[G]{} & \{x_0\} \end{array}$$

with the condition $p \circ k(x'') = G \circ i_0(x'') = \{x_0\}$. Then there exists a (constant) prox-hom $G' : (X'' \times I, \delta_1) \to (X, \delta)$ defined by $G'(x'', t) = k(x'')$ satisfying that

$$p \circ G'(x'', t) = p(G'(x'', t)) = \{x_0\} = G(x'', t),$$

$$G' \circ i_0(x'') = G'(x'', 0) = k(x'').$$

This proves that p is a proximal fibration.

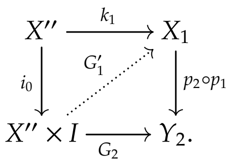
Proposition 3.19. ***i)*** *The composition of two proximal fibrations is also a proximal fibration.*
ii) *The cartesian product of two proximal fibrations is also a proximal fibration.*

Proof. **i)** Let $p_1 : (X_1, \delta_1) \to (Y_1, \delta_1')$ and $p_2 : (Y_1, \delta_1') \to (Y_2, \delta_2')$ be any proximal fibrations. Then for the inclusion map $i_0 : (X'', \delta'') \to (X'' \times I, \delta_3)$, pc-maps $k_1 : (X'', \delta'') \to (X_1, \delta_1)$, $k_2 : (X'', \delta'') \to (Y_1, \delta_1')$, and proximal homotopies $G_1 : (X'' \times I, \delta_3) \to (Y_1, \delta_1')$, $G_2 : (X'' \times I, \delta_3) \to (Y_2, \delta_2')$ with the property $p_1 \circ k_1 = G_1 \circ i_0$ and $p_2 \circ k_2 = G_2 \circ i_0$, there exist two proximal homotopies $G_1' : (X'' \times I, \delta_3) \to (X_1, \delta_1)$ and $G_2' : (X'' \times I, \delta_3) \to (Y_1, \delta_1')$ satisfying that

$$G_1' \circ i_0 = k_1, p_1 \circ G_1' = G_1,$$

$$G_2' \circ i_0 = k_2, p_2 \circ G_2' = G_2.$$

If we take $G_2' = G_1$, then we have the following commutative diagram:



Thus, we get

$$G_1' \circ i_0 = k_1,$$

$$(p_2 \circ p_1) \circ G_1' = G_2.$$

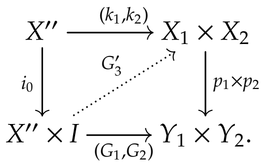
This shows that the composition $p_2 \circ p_1$ is a proximal fibration.

ii) Let $p_1 : (X_1, \delta_1) \to (Y_1, \delta_1')$ and $p_2 : (X_2, \delta_2) \to (Y_2, \delta_2')$ be any proximal fibrations. Then for the inclusion map $i_0 : (X'', \delta'') \to (X'' \times I, \delta_3)$, pc-maps $k_1 : (X'', \delta'') \to (X_1, \delta_1)$, $k_2 : (X'', \delta'') \to (X_2, \delta_2)$, and proximal homotopies $G_1 : (X'' \times I, \delta_3) \to (Y_1, \delta_1')$, $G_2 : (X'' \times I, \delta_3) \to (Y_2, \delta_2')$ with the property $p_1 \circ k_1 = G_1 \circ i_0$ and $p_2 \circ k_2 = G_2 \circ i_0$, there exist two proximal homotopies $G_1' : (X'' \times I, \delta_3) \to (X_1, \delta_1)$ and $G_2' : (X'' \times I, \delta_3) \to (X_2, \delta_2)$ satisfying that

$$G_1' \circ i_0 = k_1, p_1 \circ G_1' = G_1,$$

$$G_2' \circ i_0 = k_2, p_2 \circ G_2' = G_2.$$

Consider the map $G_3' = (G_1', G_2')$. Then G_3' is clearly a prox-hom and we have the following commutative diagram.



Thus, we get

$$G_3' \circ i_0 = (k_1, k_2),$$

$$(p_1 \times p_2) \circ G_3' = (G_1, G_2).$$

This proves that the cartesian product $p_1 \times p_2$ is a proximal fibration. □

Let $f : (X, \delta_1) \to (Y, \delta_2)$ be a pc-map. Then for any pc-map $g : (Z, \delta_3) \to (Y, \delta_2)$, a proximal lifting of f is a pc-map $h : (X, \delta_1) \to (Z, \delta_3)$ satisfying that $f = g \circ h$.

Proposition 3.20. *Let $p : (X, \delta_1) \to (Y, \delta_2)$ be a proximal fibration. Then*

i) *The pullback $g^*p : (P, \delta) \to (Y', \delta_2')$ is a proximal fibration for any pc-map $g : (Y', \delta_2') \to (Y, \delta_2)$.*

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & X \\ {\scriptstyle g^*p}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

ii) *For any proximity space (Z, δ_3), the map $p_* : (X^Z, \delta_3') \to (Y^Z, \delta_3'')$ is a proximal fibration.*

Proof. **i)** Let

$$P = \{(x, y') \mid g(y') = p(e)\} \subseteq X \times Y'$$

be a proximity space with the proximity δ_0 on itself. p is a proximal fibration. Then, for an inclusion map $i_0 : (X'', \delta'') \to (X'' \times I, \delta_3)$, for any pc-map k_1 from (X'', δ'') to (X, δ_1), and prox-hom $G_1 : (X'' \times I, \delta_3) \to (X, \delta)$ with $p \circ k_1 = G_1 \circ i_0$, there exists a prox-hom

$$G_1' : (X'' \times I, \delta_3) \to (X, \delta_1)$$

for which $G_1'(x'', 0) = k_1(x'')$ and $p \circ G_1'(x'', t) = G_1(x'', t)$. Assume that a map k_2 from (X'', δ'') to (P, δ_0) is a pc-map and $G_2 : (X'' \times I, \delta_3) \to (Y', \delta_2')$ is a prox-hom with $g^*p \circ k_2 = G_2 \circ i_0$. If we define $G_2' : (X'' \times I, \delta_3) \to (P, \delta_0)$ by $G_2' = (G_1', G_2)$, then we observe that

$$G_2' \circ i_0 = k_2,$$

$$g^*p \circ G_2' = G_2.$$

This gives the desired result.

ii) Consider the following diagrams:

$$\begin{array}{ccc} Z \times X'' & \xrightarrow{k_1} & X \\ {\scriptstyle i_0}\downarrow & {\scriptstyle G_1'} & \downarrow{\scriptstyle p} \\ Z \times X'' \times I & \xrightarrow[G_1]{} & Y \end{array}$$

and

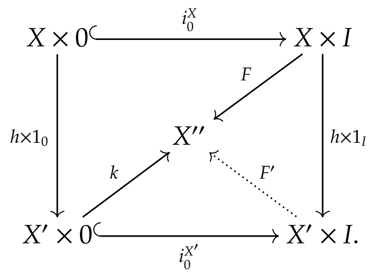
$$\begin{array}{ccc} X'' & \xrightarrow{k_2} & X^Z \\ {\scriptstyle i_0}\downarrow & {\scriptstyle G_2'} & \downarrow{\scriptstyle p_*} \\ X'' \times I & \xrightarrow[G_2]{} & Y^Z. \end{array}$$

Since p is a proximal fibration, we have $H_1' : (Z \times X'' \times I, \delta'') \to (X, \delta_1)$ as the prox-hom in the upper diagram. $Z \times X'' \times I$ is proximally isomorphic to $X'' \times I \times Z$ and we can think of G_1' as the prox-hom $(X'' \times I \times Z, \delta'') \to (X, \delta_1)$. By Proposition 3.4, we have the prox-hom $G_2' : (X'' \times I, \delta') \to (X^Z, \delta_3')$ in the lower diagram. This map satisfies the desired conditions, and thus, we conclude that p_* is a proximal fibration. □

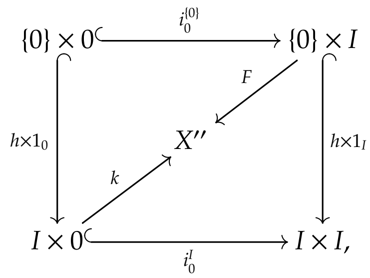
3.4. *Proximal cofibrations*

Similar to the proximal fibration, we currently deal with the notion of proximal cofibration of proximity spaces. We first study the problem of extension in homotopy theory, and then present the definition of proximal cofibration with its basic results.

Definition 3.21. Given two proximity spaces (X, δ) and (X', δ'), a pc-map $h : X \to X'$ is said to have a proximal homotopy extension property (PHEP) with respect to a proximity space (X'', δ'') provided that for inclusion maps $i_0^X : (X, \delta) \to (X \times I, \delta_1)$ and $i_0^{X'} : (X', \delta') \to (X' \times I, \delta_1')$, for every pc-map $k : (X', \delta') \to (X'', \delta'')$, and prox-hom $F : (X \times I, \delta_1) \to (X'', \delta'')$ with $k \circ (h \times 1_0) = F \circ i_0^X$, then there exists a prox-hom $F' : (X' \times I, \delta_1') \to (X'', \delta'')$ satisfying $F' \circ i_0^{X'} = k$ and $F' \circ (h \times 1_I) = F$.



Example 3.22. Let $X'' = \{A, B, C, D\}$ be a set with the proximity δ'' on itself as in Figure 3. Let γ_1 and γ_2 be proximal paths on X'' such that $\gamma_1(0) = B$, $\gamma_1(1) = A$, $\gamma_2(0) = B$, and $\gamma_2(1) = C$. Consider the following diagram for an inclusion map $h : (\{0\}, \delta) \to (I, \delta')$, where $k : (I, \delta') \to (X'', \delta'')$ is the map γ_2 and $F : (\{0\} \times I, \delta_1) \to (X'', \delta'')$ is defined by $F(0, t) = \gamma_1(t)$ for all $t \in I$:

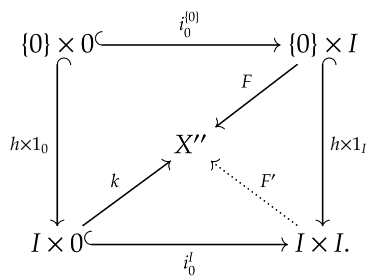


i.e., the equality $k \circ (h \times 1_0) = F \circ i_0^{\{0\}}$ holds. Then, by Gluing Lemma, there exists a prox-hom $F' : (I \times I, \delta_1') \to (X'', \delta'')$ defined by $F'(0, t_1) = F(0, t_1)$ and $F'(t_2, 0) = k(t_2)$ for all $(t_1, t_2) \in I \times I$ which satisfy

$$F' \circ (h \times 1_I) = F,$$

$$F' \circ i_0^I = k.$$

Schematically, we have the diagram



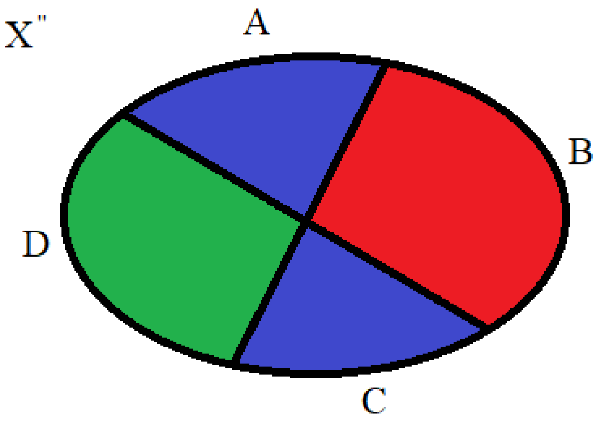


Figure 3: The picture is represented by $X'' = \{A, B, C, D\}$.

Definition 3.23. A pc-map $h : (X, \delta) \to (X', \delta')$ is said to be a proximal cofibration if it has the PHEP with respect to any proximity space (X'', δ'').

Example 3.24. Let $h : (X, \delta) \hookrightarrow (X', \delta')$ be an inclusion map such that $X \subset X'$. Then h is a natural proximal cofibration, since there exists a prox-hom

$$F' = F|_{X'} : (X' \times I, \delta_1') \to (X'', \delta'')$$

satisfying the conditions of PHEP with respect to any proximity space (X'', δ'').

The notion of coproduct proximity is studied in [3]. Assume that $\{(X_j, \delta_j)\}_{j \in J}$ is a collection of proximity spaces and $\sqcup_j X_j$ denotes the disjoint union

$$X_1 \sqcup X_2 \sqcup \cdots \sqcup X_j.$$

Then one defines a relation δ on the subsets of $\sqcup_j X_j$ as follows: Let $E_k = \sqcup_j X_j \cap X_k$. For all $E, F \in 2^{\sqcup_j X_j}$,

$$E \, \delta \, F \quad \Leftrightarrow \quad E_k \, \delta_k \, F_k$$

for some $k \in J$[3, 20].

Proposition 3.25. ***i)*** *Let $h : (X, \delta) \to (X', \delta')$ and $h' : (Y, \delta_1) \to (Y', \delta_1')$ be two maps such that X and X' are proximally isomorphic to Y and Y', respectively, and the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\approx_\delta} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[\approx_\delta]{} & Y'. \end{array}$$

Then h is a proximal cofibration if and only if h' is a proximal cofibration.

ii) *The composition of two proximal cofibrations is also a proximal cofibration.*

iii) *The coproduct of two proximal cofibrations is also a proximal cofibration.*

iv) *Let $h : (X, \delta) \to (X', \delta')$ be a proximal cofibration and the following is a pushout diagram.*

$$\begin{array}{ccc} X & \xrightarrow{l} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[l']{} & Y'. \end{array}$$

Then h' is a proximal cofibration.

Proof. **i)** Let h be a proximal cofibration. By Definition 3.21, there is a prox-hom $F' : (X' \times I, \delta'_2) \to (X'', \delta'')$ such that

$$F' \circ i_0^{X'} = k \text{ and } F' \circ (h \times 1_I) = F$$

for any pc-map $k : (X', \delta') \to (X'', \delta'')$, and prox-hom F from $(X \times I, \delta_2)$ to (X'', δ'') with $k \circ (h \times 1_0) = F \circ i_0^X$. Assume that $\beta_1 : X \to Y$ and $\beta_2 : X' \to Y'$ are two proximal isomorphisms. Since the diagram commutes, we know that $h' \circ \beta_1 = \beta_2 \circ h$. Let $i_0^Y : (Y, \delta_1) \to (Y \times I, \delta_3)$ and $i_0^{Y'} : (Y', \delta'_1) \to (Y' \times I, \delta'_3)$ be two inclusion maps, $k' := k \circ (\beta_2)^{-1} : (Y', \delta'_1) \to (X'', \delta'')$ a pc-map, and $F'' := F' \circ (\beta_1^{-1} \times 1_I)$ from $(Y \times I, \delta_3)$ to (X'', δ'') a prox-hom for which

$$k' \circ (h' \times 1_0) = F' \circ i_0^Y.$$

Then there exists a prox-hom

$$F^{'''} := F' \circ ((\beta_2)^{-1} \times 1_I) : (Y' \times I, \delta'_3) \to (X'', \delta'')$$

such that $F'' \circ i_0^{Y'} = k'$ and $F'' \circ (h' \times 1_I) = F'$.

Conversely, assume that h' is a proximal cofibration. Similarly, for a prox-hom F' from $(Y' \times I, \delta'_3)$ to (X'', δ'') that makes h' a proximal cofibration, there exists a prox-hom

$$F'' := F' \circ (f' \times 1_I) : (X' \times I, \delta'_2) \to (X'', \delta'')$$

that makes h a proximal cofibration.

ii) Let $h : (X, \delta) \to (X', \delta')$ and $h' : (X', \delta') \to (Y, \delta_1)$ be two proximal cofibrations. Then for any pc-map $k : (X', \delta') \to (X'', \delta'')$ and prox-hom F from $(X \times I, \delta_2)$ to (X'', δ'') with

$$k \circ (h \times 1_0) = F \circ i_0^X,$$

there is a prox-hom $F' : (X' \times I, \delta'_2) \to (X'', \delta'')$ such that $F' \circ i_0^{X'} = k$ and $F' \circ (h \times 1_I) = F$, similarly, for any pc-map $k' : (Y, \delta_1) \to (X'', \delta'')$ and prox-hom $G' : (X' \times I, \delta'_2) \to (X'', \delta'')$ with

$$k' \circ (h' \times 1_0) = G' \circ i_0^{X'},$$

there is a prox-hom $F'' : (Y \times I, \delta'_3) \to (X'', \delta'')$ such that $F'' \circ i_0^Y = k'$ and $F'' \circ (h' \times 1_I) = G'$. Combining these results with the fact $F' = G'$, we have the following: For a pc-map $k'' := k'$ and prox-hom $G'' := F$ with

$$k'' \circ ((h' \circ h) \times 1_0) = G'' \circ i_0^X,$$

there is a prox-hom $F^{'''} := F''$ such that

$$F^{'''} \circ i_0^Y = k'' \text{ and } F^{'''} \circ ((h' \circ h) \times 1_I) = G''.$$

This proves that $h' \circ h$ is a proximal cofibration.

iii) Let $h_j : (X_j, \delta_j) \to (X'_j, \delta'_j)$ be a family of proximal cofibrations for all $j \in J$. Then we shall show that $\sqcup_j h_j : (\sqcup_j X_j, \delta) \to (\sqcup_j X'_j, \delta')$ is a proximal cofibration. Since for all $j \in J$, $h_j : (X_j, \delta_j) \to (X'_j, \delta'_j)$ is proximal cofibration, we have that for any pc-map k_j from (X'_j, δ') to (X'', δ'') and prox-hom $F_j : (X_j \times I, \delta_2) \to (X'', \delta'')$ with

$$k_j \circ (h_j \times 1_0) = F_j \circ i_0^{X_j},$$

there is a prox-hom $F'_j : (X'_j \times I, \delta'_2) \to (X'', \delta'')$ such that $F'_j \circ i_0^{X'_j} = k_j$ and $F'_j \circ (h_j \times 1_I) = F_j$. Now assume that for a pc-map $\sqcup k_j : (\sqcup_j X'_j, \delta') \to (X'', \delta'')$ and prox-hom $\sqcup_j F_j : (\sqcup_j X_j \times I, \delta_4) \to (X'', \delta'')$ with

$$\sqcup_j k_j \circ (\sqcup_j h_j \times 1_0) = \sqcup_j F_j \circ i_0^{\sqcup_j X'_j}.$$

Then there exists a map $\sqcup_j F'_j : (\sqcup_j X'_j \times I, \delta_5) \to (X'', \delta'')$ of proximity spaces for which $F'_j = \sqcup_j F'_j \circ i_j$ for a map $i_j : (X'_j \times I, \delta'_2) \to (\sqcup_j X'_j \times I, \delta_5)$. If we define i'_j as $i_j \circ i_0^{X'_j}$, then we find that $i_0^{\sqcup_j X'_j} = \sqcup_j i'_j$. It follows that

$$\sqcup_j F'_j \circ i_0^{\sqcup_j X'_j} = \sqcup_j k_j$$

and

$$\sqcup_j F'_j \circ (\sqcup_j h_j \times 1_I) = \sqcup_j F_j.$$

Finally, we have that $\sqcup_j h_j$ is a proximal cofibration.

iv) Let $h : (X, \delta) \to (X', \delta')$ be a proximal cofibration, i.e., there is a prox-hom $F' : (X' \times I, \delta'_2) \to (X'', \delta'')$ such that

$$F' \circ i_0^{X'} = k \text{ and } F' \circ (h \times 1_I) = F$$

for any pc-map $k : (X', \delta') \to (X'', \delta'')$ and prox-hom F from $(X \times I, \delta_2)$ to (X'', δ'') with $k \circ (h \times 1_0) = F \circ i_0^X$. Since we have a pushout diagram, it follows that $l' \circ h = h' \circ l$ holds. Now assume that $k' : (Y', \delta'_1) \to (X'', \delta'')$ is a pc-map, and F'' from $(Y' \times I, \delta_3)$ to (X'', δ'') is a prox-hom with $k' \circ l' = k$, $F'' \circ (l \times 1_I)$, and

$$k' \circ (h' \times 1_0) = F'' \circ i_0^{Y'}.$$

Then there exists a prox-hom

$$F''' : (Y' \times I, \delta_3) \to (X'', \delta'')$$

such that $F''' \circ (l' \times 1_I) = F'$. Moreover, we have that

$$\begin{aligned} F''' \circ (l' \times 1_I) = F' \quad &\Rightarrow \quad F''' \circ (l' \times 1_I) \circ i_0^{X'} = F' \circ i_0^{X'} \\ &\Rightarrow \quad F''' \circ i_0^{Y'} \circ l' = k \\ &\Rightarrow \quad F''' \circ i_0^{Y'} \circ l' = k' \circ l' \\ &\Rightarrow \quad F''' \circ i_0^{Y'} = k', \end{aligned}$$

and

$$\begin{aligned} F' \circ (h \times 1_I) = F' \quad &\Rightarrow \quad F'' \circ (l' \times 1_I) \circ (h \times 1_I) = F'' \circ (l \times 1_I) \\ &\Rightarrow \quad F''' \circ ((l' \circ h) \times 1_I) = F'' \circ (l \times 1_I) \\ &\Rightarrow \quad F''' \circ ((h' \circ l) \times 1_I) = F'' \circ (l \times 1_I) \\ &\Rightarrow \quad F''' \circ (h' \times 1_I) \circ (l \times 1_I) = F'' \circ (l \times 1_I) \\ &\Rightarrow \quad F''' \circ (h' \times 1_I) = F''. \end{aligned}$$

As a consequence, h' is a proximal cofibration. □

Theorem 3.26. *$h : (X, \delta) \to (X', \delta')$ is a proximal cofibration if and only if $(X' \times 0) \cup (X \times I)$ is a proximal retract of $X' \times I$.*

Proof. Let $X'' = (X' \times 0) \cup (X \times I)$. If f is a proximal cofibration, then for any pc-map $k : (X', \delta') \to (X'', \delta'')$ and prox-hom $F : (X \times I, \delta_2) \to (X'', \delta'')$ with $k \circ (h \times 1_0) = F \circ i_0^X$, there is a prox-hom $F' : (X' \times I, \delta'_2) \to (X'', \delta'')$ such that $F' \circ i_0^{X'} = k$ and $F' \circ (h \times 1_I) = F$. Hence, F' is a proximal retraction of $X' \times I$. Conversely, let $h : (X' \times I, \delta'_2) \to (X'', \delta'')$ be a proximal retraction. Assume that $k : (X', \delta') \to (Y, \delta)$ is a pc-map and $F : (X \times I, \delta_2) \to (Y, \delta)$ is a prox-hom with

$$k \circ (h \times 1_0) = F \circ i_0^X.$$

Define a map $F'' : (X'', '') \to (Y, \delta)$ by $F''(x', t) = F(x', t)$ and $F''(x', 0) = k(x)$. By Lemma 2.3, F'' is a pc-map. Therefore, the map $F' = F'' \circ k$ is a proximal fibration satisfying that $F' \circ i_0^{X'} = k$ and $F' \circ (h \times 1_I) = F$. This shows that h is a proximal cofibration. □

4. Basic constructions for descriptive proximity

This section is dedicated to describing the concepts given in Section 3 on descriptive proximity spaces. Recall that a (spatial) proximity is also a descriptive proximity, and note that, in the examples of this section, descriptions of feature vectors consider the colors of boxes or some parts of balls (see Example 4.2, Example 4.7, and Example 4.12).

Definition 4.1. Let (X, δ_Φ^1) and (Y, δ_Φ^2) be two descriptive proximity spaces. The descriptive proximal mapping space Y^X is defined as the set

$$\{\alpha : X \to Y \mid \alpha \text{ is a dpc-map}\}$$

having the following descriptive proximity relation δ_Φ on itself: Let E, $F \subset X$ and $\{\alpha_i\}_{i\in I}$ and $\{\beta_j\}_{j\in J}$ be any subsets of dpc-maps in Y^X. We say that $\{\alpha_i\}_{i\in I}\ \delta_\Phi\ \{\beta_j\}_{j\in J}$ if the fact $E\ \delta_\Phi^1\ F$ implies that $\alpha_i(E)\ \delta_\Phi^2\ \beta_j(F)$ for all $i \in I$ and $j \in J$.

Example 4.2. Consider the set $X = \{A, B, C, D, E, F, G, H\}$ in Figure 1 with the descriptive proximity δ_Φ, where Φ is a set of probe functions that admit colors of given boxes. Define three descriptive proximal paths γ_1, γ_2, and $\gamma_3 \in X^I$ by

$$\gamma_1 : A \mapsto B \mapsto C \mapsto D,$$
$$\gamma_2 : C \mapsto B \mapsto A \mapsto H,$$
$$\gamma_3 : A \mapsto H \mapsto G \mapsto F.$$

For all $t \in I$, $\Phi(\gamma_1(t))\ \delta_\Phi\ \Phi(\gamma_2(t))$. Indeed,

$$\Phi(\gamma_1(t)) = \begin{cases} \text{red}, & t \in [0, 1/4] \text{ and } [3/4, 1] \\ \text{green}, & t \in [1/4, 2/4] \\ \text{black}, & t \in [2/4, 3/4] \end{cases} = \Phi(\gamma_2(t)),$$

namely that, γ_1 is descriptively near γ_2. However, for $t \in [1/4, 2/4]$, we have that $\Phi(\alpha_1(t))$ equals green and $\Phi(\alpha_3(t))$ equals black, that is, α_1 and α_3 are not descriptively near in X.

Remark 4.3. We say that a map $H : (X, \delta_\Phi^1) \to (Z^Y, \delta_{\Phi'})$ is descriptive proximally continuous if the fact $E\ \delta_\Phi^1\ F$ implies that $H(E)\ \delta_{\Phi'}\ H(F)$ for any subsets $E, F \subset X$.

Definition 4.4. For any descriptive proximity spaces (X, δ_Φ^1) and (Y, δ_Φ^2), the descriptive proximal evaluation map

$$e_{X,Y} : (Y^X \times X, \delta_\Phi) \to (Y, \delta_\Phi^2)$$

is defined by $e(\alpha, x) = \alpha(x)$.

Proposition 4.5. *The descriptive proximal evaluation map $e_{X,Y}$ is a dpc-map.*

Proof. We shall show that for any E, $F \subset X$ and $\{\alpha_i\}_{i\in I}$, $\{\beta_j\}_{j\in J} \subset Y^X$, $(\{\alpha_i\}_{i\in I} \times E)\ \delta_\Phi\ (\{\beta_j\}_{j\in J} \times F)$ implies $e_{X,Y}(\{\alpha_i\}_{i\in I} \times E)\ \delta_\Phi^2\ e_{X,Y}(\{\beta_j\}_{j\in J} \times F)$.

$$\begin{aligned} (\{\alpha_i\}_{i\in I} \times E)\ \delta_\Phi\ (\{\beta_j\}_{j\in J} \times F) \quad &\Rightarrow \quad \{\alpha_i\}_{i\in I}\ \delta_{\Phi'}\ \{\beta_j\}_{j\in J} \text{ and } E\ \delta_\Phi^1\ F \\ &\Rightarrow \quad \alpha_i(E)\ \delta_\Phi^2\ \beta_j(F), \ \ \forall i \in I,\ \forall j \in J \\ &\Rightarrow \quad e_{X,Y}(\{\alpha_i\}_{i\in I} \times E)\ \delta_\Phi^2\ e_{X,Y}(\{\beta_j\}_{j\in J} \times F), \end{aligned}$$

where Y^X has a descriptive proximity $\delta_{\Phi'}$. □

Definition 4.6. A surjective and dpc-map $p : (X, \delta_\Phi) \to (X', \delta_{\Phi'})$ between any descriptive proximity spaces (X, δ_Φ) and $(X', \delta_{\Phi'})$ is a descriptive proximal covering map if the following hold:

- Let $\{x'\} \subseteq X'$ be any subset with $\{x'\} \ll_{\delta_{\Phi'}} Y'$. Then there is an index set I satisfying that

$$p^{-1}(Y') = \bigcup_{i \in I} Y_i$$

with $V_i \ll_{\delta_\Phi} Y_i$, where $V_i \in p^{-1}(\{x'\})$ for each $i \in I$.

- $Y_i \neq Y_j$ when $i \neq j$ for $i, j \in I$.
- $p|_{Y_i} : Y_i \to Y'$ is a descriptive proximal isomorphism for every $i \in I$.

In Definition 4.6, (X, δ_Φ) is called a descriptive proximal covering space of $(X', \delta_{\Phi'})$. For $i \in I$, Y_i is said to be a descriptive proximal sheet. For any $x' \in X'$, $p^{-1}(\{x'\})$ is called a descriptive proximal fiber of x'. The map $p|_{Y_i} : Y_i \to Y'$ is a descriptive proximal isomorphism if the map $p : (X, \delta_\Phi) \to (X', \delta_{\Phi'})$ is a descriptive proximal isomorphism. However, the converse is not generally true. Given any descriptive proximity space (X, δ_Φ), it is obvious that the identity map on X is always a descriptive proximal covering map.

Example 4.7. Consider the surjective and dpc-map $p : (X, \delta_\Phi) \to (X', \delta_{\Phi'})$, defined by $p(a_i) = p(b_i) = p(c_i) = d_i$ for any $i = 1, 2, 3, 4$, in Figure 2, where Φ is a set of probe functions that admits colors of given shapes. Let $\{d_1\} \subset X'$ and $Y' = \{d_1, d_3, d_4\}$ a $\delta_{\Phi'}$−neighborhood of $\{d_1\}$. For $V_1 = \{a_1\}$, $V_2 = \{b_1\}$, and $V_3 = \{c_1\}$, we have that $p^{-1}(Y') = \bigcup_{i=1}^{3} Y_i$, where $Y_1 = \{a_1, a_3, a_4\}$, $Y_2 = \{b_1, b_3, b_4\}$, and $Y_3 = \{c_1, c_3, c_4\}$. This gives us that for all $i \in \{1, 2, 3\}$, Y_i is a δ_Φ−neighborhood of V_i. We also observe that $Y_i \neq Y_j$ if $i \neq j$ for $i, j \in \{1, 2, 3\}$. In addition, $p|_{Y_i} : Y_i \to Y'$ is a descriptive proximal isomorphism for each i. If one considers d_3 and d_4, the same process can be repeated. Let $\{d_2\} \ll_{\delta_{\Phi'}} \{d_2\} = Y'$ in X'. Then $p^{-1}(Y') = Y_1 \cup Y_2 \cup Y_3$, where $Y_1 = \{a_2\}$, $Y_2 = \{b_2\}$, and $Y_3 = \{c_2\}$. We observe that $V_1 = \{a_2\} \ll_{\delta_\Phi} Y_1$, $V_2 = \{b_2\} \ll_{\delta_\Phi} Y_2$, and $V_3 = \{c_2\} \ll_{\delta_\Phi} Y_3$. Note that $Y_1 \neq Y_2 \neq Y_3$. Furthermore, $p|_{Y_i} : Y_i \to Y'$ is a descriptive proximal isomorphism for each $i = 1, 2, 3$. This proves that p is a descriptive proximal covering map.

Definition 4.8. A dpc-map $p : (X, \delta_\Phi) \to (X', \delta_{\Phi'})$ is said to have the descriptive proximal homotopy lifting property (DPHLP) with respect to a descriptive proximity space $(X'', \delta_{\Phi''})$ if, for an inclusion map $i_0 : (X'', \delta_{\Phi''}) \to (X'' \times I, \delta^1_\Phi)$ defined by $i_0(x'') = (x'', 0)$, for every dpc-map $h : (X'', \delta_{\Phi''}) \to (X, \delta_\Phi)$, and dprox-hom $G : (X'' \times I, \delta^1_\Phi) \to (X', \delta_{\Phi'})$ with $p \circ h = G \circ i_0$, then there exists a dprox-hom $G' : (X'' \times I, \delta^1_\Phi) \to (X, \delta_\Phi)$ for which $G'(x'', 0) = h(x'')$ and $p \circ G'(x'', t) = G(x'', t)$.

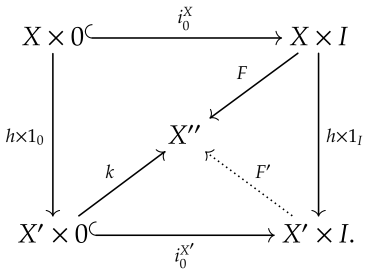
$$\begin{array}{ccc} X'' & \xrightarrow{h} & X \\ {\scriptstyle i_0}\downarrow & \overset{G'}{\nearrow} & \downarrow{\scriptstyle p} \\ X'' \times I & \xrightarrow[G]{} & X'. \end{array}$$

Definition 4.9. A map $p : (X, \delta_\Phi) \to (X', \delta_{\Phi'})$, which is a dpc-map, is said to be a descriptive proximal fibration if it has the DPHLP for any descriptive proximity space $(X'', \delta_{\Phi''})$.

Definition 4.10. Given two descriptive proximity spaces (X, δ_Φ) and $(X', \delta_{\Phi'})$, a dpc-map $h : X \to X'$ is said to have a dprox-hom extension property (DPHEP) with respect to a descriptive proximity space $(X'', \delta_{\Phi''})$ if there exists a dprox-hom

$$F' : (X' \times I, \delta_{\Phi^{1'}}) \to (X'', \delta_{\Phi''})$$

satisfying the conditions $F' \circ i_0^{X'} = k$ and $F' \circ (h \times 1_I) = F$ for any dpc-map $k : (X', \delta_{\Phi'}) \to (X'', \delta_{\Phi''})$, and dprox-hom $F : (X \times I, \delta_{\Phi^1}) \to (X'', \delta_{\Phi''})$ with the equality $k \circ (h \times 1_0) = F \circ i_0^X$, where the maps $i_0^X : (X, \delta_\Phi) \to (X \times I, \delta_{\Phi^1})$ and $i_0^{X'} : (X', \delta_{\Phi'}) \to (X' \times I, \delta_{\Phi^{1'}})$ are inclusions.



Definition 4.11. A dpc-map $f : (X, \delta_\Phi) \to (X', \delta_{\Phi'})$ is said to be a descriptive proximal cofibration if it has the DPHEP with respect to any descriptive proximity space $(X'', \delta_{\Phi''})$.

Example 4.12. Let $(X'', \delta_{\Phi''})$ be a descriptive proximity space as in Figure 3, where Φ is a set of probe functions that admit colors of given rounds. Assume that γ_1 and γ_2 are descriptive proximal paths on X'' such that γ_1 is from B to A and γ_2 is from B to C. Let $h : (\{0\}, \delta_\Phi) \to (I, \delta_{\Phi'})$ be an inclusion map. For a dpc-map $k : (I, \delta_{\Phi'}) \to (X'', \delta_{\Phi''})$ defined as $k = \gamma_2$, and a dprox-hom $F : (\{0\} \times I, \delta_{\Phi^1}) \to (X'', \delta_{\Phi''})$ defined by $F(0, t) = \alpha(t)$ for all $t \in I$ with the property $k \circ (h \times 1_0) = F \times i_0^{\{0\}}$, there exists a dprox-hom

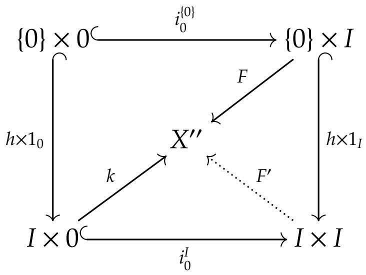
$$F' : (I \times I, \delta_{\Phi^{1'}}) \to (X'', \delta_{\Phi''})$$

defined by $F'(0, t_1) = F(0, t_1)$ and $F'(t_2, 0) = k(t_2)$ for all $(t_1, t_2) \in I \times I$ which satisfy

$$F' \circ (h \times 1_I) = F,$$

$$F' \circ i_0^I = k.$$

In another saying, the diagram



holds.

5. Conclusion

A subfield of topology called homotopy theory investigates spaces up to continuous deformation. Although homotopy theory began as a topic in algebraic topology, it is currently studied as an independent discipline. For instance, algebraic and differential nonlinear equations emerging in many engineering and scientific applications can be solved using homotopy approaches. As an example, these equations include a set of nonlinear algebraic equations that model an electrical circuit. In certain studies, the aging process of the human body is presented using the algebraic topology notion of homotopy. In addition to these examples, one can easily observe homotopy theory once more when considering the algorithmic problem of robot motion planning. In this sense, this research is planned to accelerate homotopy theory studies within proximity spaces that touch many important application areas. Moreover, this examination encourages not only homotopy theory but also homology and cohomology theory to take place within proximity spaces. The powerful concepts of algebraic topology always enrich the proximity spaces and thus it becomes possible to see the topology even in the highest-level fields of science such as artificial intelligence and medicine.

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