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On approximation by truncated max-product Baskakov operators of fuzzy numbers

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Abstract. In this work, we generalize the truncated max-product (non-linear) Baskakov operators to encompass any compact interval $[a, b]$. Our investigation establishes that these operators exhibit the same order of uniform approximation as in the specific case of the interval $[0, 1]$. Furthermore, we demonstrate the preservation of monotonicity and shape properties by these operators on $[a, b]$, rendering them highly valuable for approximating fuzzy numbers. For practical applications, we generate a fuzzy number $\tilde{R}_r^M(\rho; [a, b])(t)$ that preserves the support and core of an arbitrary ρ. This fuzzy number was then employed, utilizing metrics D_C and L^1-type metrics, to enhance convergence estimates. Our analysis yields several direct conclusions. Additionally, when fuzzy numbers ρ are expressed in parametric form, the truncated max-product Baskakov operator produces a sequence of fuzzy numbers $\tilde{R}_r^M(\rho)$. It has been observed that $\tilde{R}_r^M(\rho)$ uniformly converges to ρ. Furthermore, essential characteristics of $\tilde{R}_r^M(\rho)$ also converge to ρ. Finally, we present a comparison and an illustrative graphic, demonstrating how these operators facilitate the convergence of fuzzy functions.

1. Introduction

Approximation theory, a significant branch of mathematics, deals with the development of efficient methods to represent complex mathematical entities through simpler, more manageable ones. The accurate representation of such entities is essential for numerous real-world applications, including decision-making processes, pattern recognition, and optimization. This field traces its roots back to the pioneering work of Weierstrass in 1885, who delved into the problem of approximating continuous functions. Building upon Weierstrass' foundations, Bernstein further contributed to this theory by utilizing polynomials.

Building on these foundations, Korovkin made significant progress in 1953 by proving that linear positive operators uniformly converge to continuous functions. This breakthrough served as a springboard for researchers to explore the approximation properties of various linear positive operators.

Over the last two decades, a range of non-linear approximation operators has been introduced. In the period spanning from 2006 to 2008, B. Bede et al. proposed the use of discrete linear approximation operators to generate nonlinear positive operators. Specifically, they replaced the max-product operation described in [12] and the max-min operation described in [13] with a pair of sum-product operations. This innovation resulted in the development of the so-called nonlinear Shepard-type operators.

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In 2008, S.G. Gal [21] posed an open problem that sparked significant interest among academics, which led to the introduction of the Bernstein of max-product operators. This open problem has attracted the attention of many academics due to the numerous accomplishments in this area. Later, researchers have delved into the estimation of approximation using max-product operators and have successfully attained specific shape preservation characteristics (refer to [1–3, 16, 25, 27]).

In this context, the Truncated Max-Product Baskakov (TMPB) Operators, which are a simplification of the Max-Product Baskakov Operators [14], have emerged as important tools for approximating functions. They were introduced in 2011 by B. Bede et al. [15] as follows:

$$R_r^M(\rho;[0,1])(x)=\frac{\bigvee_{k=0}^{r}\varphi_{r,k}(x)\cdot\rho\left(\frac{k}{r}\right)}{\bigvee_{k=0}^{r}\varphi_{r,k}(x)},\ x\in[0,1]$$

where $\varphi_{r,k}(x)=\binom{r+k-1}{k}\dfrac{x^k}{(1+x)^{r+k}}$, $r\geq 1$.

The concept of fuzzy numbers, introduced by Zadeh in 1965 and further developed by D. Dubois et al. [20], enables the representation of imprecise and ambiguous quantities. Unlike crisp numbers, fuzzy numbers possess a degree of membership that allows for a gradual transition between truth values, making them an ideal candidate for handling uncertain data. The introduction of fuzzy numbers revolutionized the field of approximation theory, paving the way for novel techniques that could effectively deal with vague or imprecise information, which classical methods often fail to handle adequately.

Therefore, over the past decade, there has been significant interest in finding efficient methods to represent and approximate fuzzy numbers. Researchers have explored various approaches to approximate fuzzy numbers using Intervals (refer to [17, 23]), triangles (refer to [7, 8, 11, 22, 29]), trapezoids (refer to [9, 10, 24, 28]) and L-U parametric forms (refer to [5, 6, 30]).

The TMPB operators, originally introduced in classical approximation theory, have demonstrated their effectiveness in handling fuzzy sets and fuzzy numbers. Leveraging the strengths of these operators, this study presents a novel approach to approximating fuzzy numbers that take into consideration their inherent fuzziness and imprecision. By utilizing TMPB operators, this approach offers several advantages over classical approximation techniques, considering the holistic nature of the fuzzy number, including its shape, membership degrees, and other relevant characteristics.

The contributions of this research lie in both theoretical and practical aspects. Theoretical advancements in the TMPB operators deepen our understanding of their mathematical properties and provide a solid basis for their application. Additionally, the practical applications of fuzzy number approximation demonstrate its efficacy in solving complex problems in fields such as decision-making, pattern recognition, optimization, and risk analysis.

In summary, this work aims to highlight the superiority of TMPB operators in approximating fuzzy numbers compared to classical approximation methods. By harnessing the flexibility and expressiveness of fuzzy numbers, these operators offer a more nuanced and accurate representation of uncertain data. Furthermore, this research strives to bridge the gap between fuzzy set theory, approximation theory, and practical applications. Introducing the concept of TMPB operators for approximating fuzzy numbers, contributes to the advancement of fuzzy approximation techniques and provides a valuable tool for handling imprecise data across various domains, paving the way for future research in the realm of uncertain data modeling and analysis.

The present paper is organized as follows: Section 2 presents essential information and basic concepts about fuzzy numbers. Additionally, it redefines $R_r^M(\rho;[0,1])(x)$ in the interval $[a,b]$ and provides some fundamental auxiliary concepts used throughout this paper. In Section 3, we delve into the approximate and shape-preserving properties of these operators on $[a,b]$, discussing and investigating their characteristics. Section 4 of the study focuses on the construction of fuzzy numbers $\tilde{R}_r^M(\rho;[a,b])(t)$ and $\tilde{R}_r^M(\rho)$ and their application in achieving improved approximations, particularly with respect to the D_C and L^1-type meters. Theoretical procedures and illustrative examples effectively demonstrate these findings.

2. Preliminaries

2.1. Basic concepts of fuzzy numbers

Definition 2.1. ([20]) **(fuzzy numbers)** A fuzzy number ρ is a fuzzy subset of the $\mathbb{R}$ with $\mu_\rho(x) : \mathbb{R} \longrightarrow [0,1]$ if: and only if μ_ρ satisfy:

1. $\exists x_0 \in \mathbb{R}$ such that $\mu_\rho(x_0) = 1$ (normal condition);
2. $\mu_\rho(\alpha x_1 + (1-\alpha)x_2) \geq \min\{\mu_\rho(x_1), \mu_\rho(x_2)\}$ (fuzzy convex);
3. μ_ρ is upper semicontinuous;
4. support of ρ is compact.

Therefore, for any fuzzy number ρ we can express a membership function μ_ρ as follows:

$$\mu_\rho(x) = \begin{cases} 0 & if \quad x < t_1 \\ l_\rho(x) & if \quad t_1 \leq x \leq t_2 \\ 1 & if \quad t_2 \leq x \leq t_3 \\ r_\rho(x) & if \quad t_3 \leq x \leq t_4 \\ 0 & if \quad t_4 < x \end{cases}$$

where $l_\rho : [t_1, t_2] \to [0,1]$ is non-decreasing, and known as the left side of ρ and $r_\rho : [t_3, t_4] \to [0,1]$ is non-increasing, and known as the right side of ρ. And we denote the collection of all fuzzy real numbers by $\mathbb{R}_F$.

Definition 2.2. ([19]) **(α -cut)** The α -cut of $\rho \in \mathbb{R}_F$ is the crisp set for some $\alpha \in (0,1]$, defined as $\rho_\alpha = \{x \in \mathbb{R} \mid \mu_\rho(x) \geq \alpha\}$.

Then it is obvious that $\rho_\alpha = \left[\rho^l(\alpha), \rho^u(\alpha)\right]$, $\alpha \in (0,1]$, where
$\rho^l(\alpha) = \inf\{x \in \mathbb{R} \mid \mu_\rho(x) \geq \alpha\}$,
$\rho^u(\alpha) = \sup\{x \in \mathbb{R} \mid \mu_\rho(x) \geq \alpha\}$.

Notation 2.3.
1. If $\alpha = 1$ then , $\rho_1 = \left[\rho^l(1), \rho^u(1)\right]$ is referred to as the core of ρ, and denoted by core(ρ).
2. If $\alpha = 0$ then , $\rho_0 = \left[\rho^l(0), \rho^u(0)\right] = cl\,\{x \in \mathbb{R} \mid \mu_\rho(x) > 0\}$, is referred to as the support of ρ, which will be denoted by supp (ρ).

Here it is necessary to mention some definitions of metric spaces between fuzzy numbers, which are employed in the statement of approximation in this paper.

Definition 2.4. ([19]) Let $\mu, v \in \mathbb{R}_F$, then:
i) Chebyshev-type metric is defined as:
$D_C(\mu, v) = \sup\{\left|\mu(x) - v(x)\right| : x \in \mathbb{R}\}$
and we can denote it $D_C(\mu, v) = \left\|\mu - v\right\|$, for simplicity.
With parametric representations, is defined as:

$$\tilde{D}(\mu, v) = \sup_{\alpha \in [0,1]} \max\left\{\left|\mu^l(\alpha) - v^l(\alpha)\right|, \left|\mu^u(\alpha) - v^u(\alpha)\right|\right\}$$

and denote it $\tilde{D}(\mu, v) = \left\|\mu - v\right\|_{LU}$, for simplicity.
ii) L^p- type metric is defined as:

$$D_p(\mu, v) = \left(\int_{\mathbb{R}} \left|\mu(x) - v(x)\right|^p dx\right)^{\frac{1}{p}}, p \geq 1$$

With parametric representations, is defined as:

$$D_p(\mu, v) = \left(\int_0^1 \left|\mu^l(\alpha) - v^l(\alpha)\right|^p d\alpha + \int_0^1 \left|\mu^u(\alpha) - v^u(\alpha)\right|^p d\alpha\right)^{\frac{1}{p}}, p \geq 1$$

Here are some definitions of some of the fundamental characteristics of the fuzzy number.

i) ([26]) *The expected of* ρ is defined as:

$$EI(\rho)=\left[\int_0^1\rho^l(\alpha)\,d\alpha,\int_0^1\rho^u(\alpha)\,d\alpha\right]$$

ii) ([17]) *The width of* ρ is defined as:

$$wid(\rho)=\int_0^1\left(\rho^u(\alpha)-\rho^l(\alpha)\right)d\alpha$$

iii) ([18]) *The value* of ρ and *the ambiguity* of ρ.

For a non-decreasing reduction function $\delta:[0,1]\to[0,1]$ such that $\delta(0)=0$ and $\delta(1)=1$. *The value* of ρ is given by:

$$Val_\delta(\rho)=\int_0^1\delta(\alpha)\left(\rho^l(\alpha)+\rho^u(\alpha)\right)d\alpha$$

and *the ambiguity* of ρ defined by:

$$Amb_\delta(\rho)=\int_0^1\delta(\alpha)\left(\rho^u(\alpha)-\rho^l(\alpha)\right)d\alpha$$

Also we have the particular case of δ, for $\delta_m(\alpha)=\alpha^m$, $m\in\mathbb{N}$ and $\alpha\in[0,1]$, then we denote $Val_{\delta_m}(\rho)=Val_m(\rho)$ and $Amb_{\delta_m}(\rho)=Amb_m(\rho)$, i.e.

$$Val_m(\rho)=\int_0^1\alpha^m\left(\rho^l(\alpha)+\rho^u(\alpha)\right)d\alpha$$

and

$$Amb_m(\rho)=\int_0^1\alpha^m\left(\rho^u(\alpha)-\rho^l(\alpha)\right)d\alpha$$

2.2. *The operators' construction on compact intervals* $[a,b]$ *and give some basic concept auxiliary*

In this part, $R_r^M(\rho;[0,1])(x)$ is defined on the interval $[a,b]$, which means that the interval $[0,1]$ is expanded to a compact interval $[a,b]$ based on Weierstrass's result, as shown below:

Definition 2.5. Let $\rho\in\mathbb{R}_F$ be a positive continuous on $[a,b]$ such that $a<c\leq d<b$ with core$(\rho)=[c,d]$ and supp$(\rho)=[a,b]$, then for any $t\in[a,b]$ we define:

$$R_r^M(\rho;[a,b])(t)=\frac{\bigvee_{k=0}^{r}\varphi_{r,k}(t)\cdot\rho\left(a+(b-a)\frac{k}{r}\right)}{\bigvee_{k=0}^{r}\varphi_{r,k}(t)}$$

where $\varphi_{r,k}(t)=\binom{r+k-1}{k}\left(\frac{t-a}{b-a}\right)^k\cdot\left(\frac{b-a}{b+t-2a}\right)^{r+k}$, $r\geq1$.

Notation 2.6. For short, from now on throughout this paper, we denote the compact interval $[a,b]$ and the unite interval $[0,1]$, respectively, by the letters $\tilde{I}$ and I.

Theorem 2.7. ([15]) *Let* ρ *be a continuous function on I, then for all* $x\in I$ *we have:*

$$\left|R_r^M(\rho;[0,1])(x)-\rho(x)\right|\leq24\omega_1\left(\rho;\frac{1}{\sqrt{r+1}}\right)_I\quad,r\geq2.$$

Corollary 2.8. ([15]) *Let* ρ *be a non-decreasing concave function on I,then for all* $x\in I$ *we have:*

$$\left|R_r^M(\rho;[0,1])(x)-\rho(x)\right|\leq2\omega_1\left(\rho;\frac{1}{r}\right)_I.$$

Theorem 2.9. ([15]) *Let* ρ *be a non-decreasing function on I, then* $R_r^M(\rho;[0,1])(x)$ *is non-decreasing, for all* $r\in\mathbb{N}$, $r\geq2$.

Corollary 2.10. ([15]) *Let* ρ *be a non-increasing function on I, then* $R_r^M(\rho;[0,1])(x)$ *is non-increasing, for all* $r\in\mathbb{N}$, $r\geq2$.

Corollary 2.11. ([15]) *Let ρ be a quasiconvex continuous function on I, then $R_r^M(\rho;[0,1])(x)$ is quasiconvex on I, for all $r \in \mathbb{N}$, $r \geq 2$.*

Definition 2.12. ([21]) Let ρ be continuous on $\tilde{I}$. The for all $t_1, t_2 \in \tilde{I}$, ρ is known as:
(i) quasi-convex if
$\rho(\alpha t_1 + (1-\alpha)t_2) \leq \max\{\rho(t_1), \rho(t_2)\}$, $\alpha \in I$;
(ii) quasi-concave, if $-\rho$ is quasi-convex.

Remark 2.13. ([16]) Let ρ be a $quasi-convex$ continuous function on $\tilde{I}$, then there is a point $c \in \tilde{I}$ such that ρ is non-increasing on $[a,c]$ and non-decreasing on $[c,b]$. So, the definition above makes it apparent that if ρ is a $quasi-concave$ continuous function on $\tilde{I}$, then there is a point $c \in \tilde{I}$ such that ρ is non-decreasing on $[a,c]$ and non-increasing on $[c,b]$.

3. The approximation and preservation of shape properties

In this section, Subsection 3.1 begins with a remark of paramount importance, which builds upon Weierstrass's result. This remark plays a pivotal role, as the majority of the proofs in this section heavily rely on it.

3.1. The approximation by TMPB operators on compact intervals $[a,b]$

Remark 3.1. Let $\rho : \tilde{I} \to \mathbb{R}^+$be continuous on $\tilde{I}$, for all $a, b \in \mathbb{R}$, and $\upsilon : I \to \mathbb{R}^+$, such that $\upsilon(x) = \rho(a + (b-a)x)$

That is, $\upsilon\left(\frac{k}{r}\right) = \rho\left(a + (b-a)\frac{k}{r}\right), r \geq 1.$

Also, let $x = \frac{t-a}{b-a}$, so, we can say $\upsilon\left(\frac{t-a}{b-a}\right) = \rho(t)$ and $t = a + (b-a)x$, $t \in \tilde{I}$

Hence, from the above and for $\upsilon\left(\frac{k}{r}\right)$ expressions we obtain:

$R_r^M(\rho;[a,b])(t) = R_r^M(\upsilon;[0,1])(x)$.

Theorem 3.2. *Let ρ be continuous on $\tilde{I}$. Then for $r \geq 2$ we have:*

$$\left|R_r^M(\rho;[a,b])(t) - \rho(t)\right| \leq 24([b-a]+1)\,\omega_1\left(\rho; \frac{1}{\sqrt{r+1}}\right)_{\tilde{I}} \quad, t \in \tilde{I}.$$

Proof. By Remark 3.1, we have $R_r^M(\rho;[a,b])(t) = R_r^M(\upsilon;[0,1])(x)$.
Hence, υ is continuous on I. Then, from Theorem 2.7, we get:

$$\left|R_r^M(\rho;[a,b])(t) - \rho(t)\right| = \left|R_r^M(\upsilon;[0,1])(x) - \upsilon(x)\right| \leq 24\omega_1\left(\upsilon; \frac{1}{\sqrt{r+1}}\right)_I.$$

Hence, $\omega_1\left(\upsilon; \frac{1}{\sqrt{r+1}}\right)_I \leq \omega_1\left(\rho, \frac{(b-a)}{\sqrt{r+1}}\right)_{\tilde{I}} \leq ([b-a]+1)\,\omega_1\left(\rho, \frac{1}{\sqrt{r+1}}\right)_{\tilde{I}}$. □

Theorem 3.3. *Let ρ be a non-decreasing concave on $\tilde{I}$. Then we have:*

$$\left|R_r^M(\rho;[a,b])(t) - \rho(t)\right| \leq 2([b-a]+1)\,\omega_1\left(\rho; \frac{1}{r}\right)_{\tilde{I}}, \quad t \in \tilde{I}.$$

Proof. By Remark 3.1, we have, $R_r^M(\rho;[a,b])(t) = R_r^M(\upsilon;[0,1])(x)$.
Therefore, ρ is concave on $\tilde{I}$. So, $\rho(\alpha\ell_1 + (1-\alpha)\ell_2) \geq \alpha\rho(\ell_1) + (1-\alpha)\rho(\ell_2)$ for all $\ell_1, \ell_2 \in \tilde{I}$, $\alpha \in I$.
Therefore, we can write υ as:

$$\upsilon(\alpha\frac{\ell_1 - a}{b-a} + (1-\alpha)\frac{\ell_2 - a}{b-a}) \geq \alpha\upsilon\left(\frac{\ell_1 - a}{b-a}\right) + (1-\alpha)\upsilon\left(\frac{\ell_2 - a}{b-a}\right).$$

That is, υ the concave on I, hence, from Corollary 2.8, we get:

$\left|R_r^M(\rho;[a,b])(t)-\rho(t)\right|=\left|R_r^M(\upsilon;[0,1])(x)-\upsilon(x)\right|\le 2\omega_1\left(\upsilon;\frac{1}{r}\right)_I,$

Hence, $\omega_1\left(\upsilon;\frac{1}{r}\right)_I \le \omega_1\left(\rho,\frac{(b-a)}{r}\right)_{\tilde{I}} \le ([b-a]+1)\,\omega_1\left(\rho;\frac{1}{r}\right)_{\tilde{I}}$. □

Remark 3.4. The order of approximation achieved in Theorem 3.3 can be considered an improvement over the order of approximation obtained in Theorem 3.2 for some subclasses of functions ρ. This can be easily and empirically demonstrated by assuming that ρ possesses a Lipschitz-type property, following a similar approach to the proof of Corollary 4.7 in Section 4 of this work.

3.2. Monotony and shape -preserving properties

Theorem 3.5. *Let ρ be a non-decreasing function on $\tilde{I}$, then for $r \ge 2$, $R_r^M(\rho;[a,b])(t)$ is non-decreasing on $\tilde{I}$.*

Proof. By Remark 3.1, $\upsilon(x)=\rho(a+(b-a)x), x\in I$.
So we obtain $R_r^M(\rho;[a,b])(t_i)=R_r^M(\upsilon;[0,1])(x_i)$, $i=1,2$.
Now, since ρ is a non-decreasing function on $\tilde{I}$. So, if $t_1,t_2\in\tilde{I}$ such that $t_1\le t_2$, we have $\rho(t_1)-\rho(t_2)\le 0$.
Therefore, if $x_1,x_2\in I$ such that $x_1\le x_2$, we get:
$\upsilon(x_1)-\upsilon(x_2)=\rho(a+(b-a)x_1)-\rho(a+(b-a)x_2)=\rho(t_1)-\rho(t_2)\le 0.$
That is υ is a non-decreasing function, hence from Theorem 2.9, we obtain:
$R_r^M(\rho;[a,b])(t_1)\le R_r^M(\rho;[a,b])(t_2)$, and this finishes the proof. □

Corollary 3.6. *Let ρ be a non-increasing function, then for $r\ge 2$, $R_r^M(\rho;[a,b])(t)$ is non-increasing on $\tilde{I}$.*

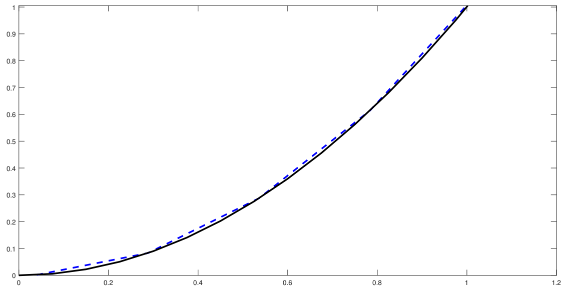
Proof. By the same way as in Theorem 3.5 we have $R_r^M(\rho;[a,b])(t_i)=R_r^M(\upsilon;[0,1])(x_i)$, $i=1,2$.
Now, since ρ is a non-increasing function on $\tilde{I}$. So, if $t_1,t_2\in\tilde{I}$ such that $t_1\le t_2$, we have $\rho(t_1)-\rho(t_2)\ge 0$.
Therefore, if $x_1,x_2\in[0,1]$ such that $x_1\le x_2$, we get:
$\upsilon(x_1)-\upsilon(x_2)=\rho(a+(b-a)x_1)-\rho(a+(b-a)x_2)=\rho(t_1)-\rho(t_2)\ge 0.$
That is υ is a non-increasing function, hence from Corollary 2.10 we obtain:
$R_r^M(\rho;[a,b])(t_1)\ge R_r^M(\rho;[a,b])(t_2)$, and this proof is completed. □

Theorem 3.7. *Let ρ be a quasiconvex continuous function on $\tilde{I}$, then $R_r^M(\rho;[a,b])(t)$ is quasiconvex on $\tilde{I}$, for all $r\ge 2$.*

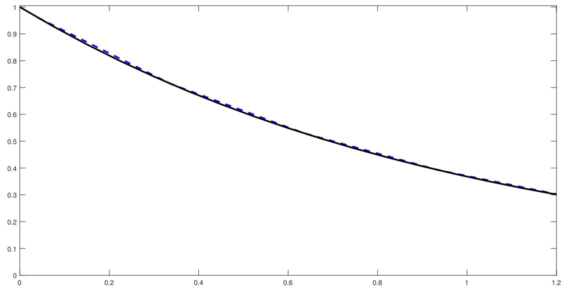
Proof. Since ρ is a quasiconvex on $\tilde{I}$.
So, $\rho(\alpha t_1+(1-\alpha)t_2)\le\max\{\rho(t_1),\rho(t_2)\}$, for all $t_1,t_2\in\tilde{I},\alpha\in I$;
Therefore, by Remark 3.1,we can written υ as:
$\upsilon(\alpha\frac{t_1-a}{b-a}+(1-\alpha)\frac{t_2-a}{b-a})\le\max\{\upsilon\left(\frac{t_1-a}{b-a}\right),\upsilon\left(\frac{t_1-a}{b-a}\right)\}.$
That is υ is a quasiconvex function on I, then from Remark 2.13, we can say there is $c` \in I$ such that $\upsilon$ is non-increasing on $[0,c`]$ and non-decreasing on $[c`,1]$.
Let $c\in\tilde{I}$ such that $c=a+(b-a)c`$
Now, if $t_i\in[a,c]$, $t_1\le t_2$ such that $t_i=a+(b-a)x_i$ for all $x_i\in[0,c`]$, $x_1\le x_2$. Then from Remark 3.1 we obtain:
$R_r^M(\rho;[a,b])(t_i)=R_r^M(\upsilon;[0,1])(x_i)$, for $i=1,2$.
Hence, since υ is non-increasing on $[0,c`]$, so from Corollary 2.10, we obtain:
$R_r^M(\rho;[a,b])(t_1)\ge R_r^M(\rho;[a,b])(t_2)$.
Thus, $R_r^M(\rho;[a,b])(t)$ is non-increasing on $[a,c]$.
By using the same steps above and Theorem 2.9, we obtain that $R_r^M(\rho;[a,b])(t)$ is non-decreasing on $[c,b]$. This finishes the proof, □

Example 3.8. The monotony and shape preservation properties using $R_r^M(\rho;[a,b])(t)$ operators are illustrated in (Fig.1).

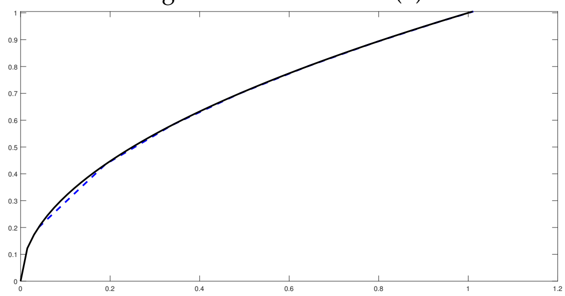
Fig. 1 clearly illustrates the preservation properties exhibited by the dashed-line-marked truncated max-product Baskakov operators. These operators effectively maintain the characteristics of non-decreasing, non-increasing, and the quasiconvexity of the function $\rho(t)$, indicated by the solid line.



(a) Preservation of non-decreasing.



(b) Preservation of non-increasing.



(c) Preservation of quasiconvexity.

TMPB operators Function

Figure 1: Monotony and shape-preserving properties

4. Applications to fuzzy number approximation

4.1. The estimation of approximation using D_C and $\tilde{D}_C$ metrics

Lemma 4.1. ([4]) *Let* $m_{k,r,s}(t) = \dfrac{\varphi_{r,k}(t)}{\varphi_{r,s}(t)}, t \in \left(a + \dfrac{(b-a)s}{r-1}, a + \dfrac{(b-a)(s+1)}{r-1}\right)$. *Then for all* $r \geq 2, s = \{0, 1, \cdots, r-2\}$ *and* $k = \{0, 1, \cdots, r\}/\{s\}$, *we have* $m_{k,r,s}(t) < 1$.

Lemma 4.2. ([4]) *Let* ρ *be a bounded on* $\tilde{I}$, *and* $a, b \in \mathbb{R}$. *Then for all* $r \geq 2, s = \{0, 1, \cdots, r-2\}$, *we have:*

$$R_r^M(\rho; [a,b])\left(a + (b-a)\frac{s}{r}\right) \geq \rho\left(a + (b-a)\frac{s}{r}\right).$$

Remark 4.3. If we take $\rho \in \mathbb{R}_F$, then we can generate a fuzzy number $\tilde{R}_r^M(\rho; [a,b])(t)$ with the same support and core of ρ. So we can introduce the function $\tilde{R}_r^M(\rho; [a,b])(t) : \mathbb{R} \to [0,1]$ and we have $\tilde{R}_r^M(\rho; [a,b])(t) = R_r^M(\rho; [a,b])(t)$. So all the results obtained in Section 3 can be applied to the $\tilde{R}_r^M(\rho; [a,b])(t)$.

Theorem 4.4. ([4]) *Let* $\rho \in \mathbb{R}_F$ *with* $core(\rho) = [c,d]$ *and* $supp(\rho) = [a,b]$.*such that* $a \leq c < d \leq b$ *Then, it follows that* $\tilde{R}_r^M(\rho; [a,b])(t)$ *is a fuzzy number when r is sufficiently large, such that:*

1. $supp(\rho) = supp\left(\tilde{R}_r^M(\rho; [a,b])\right)$;
2. *If* $core(\rho) = [c_r, d_r]$, *then* $|c - c_r| \leq \frac{b-a}{r}$ *and* $|d - d_r| \leq \frac{b-a}{r}$.

Theorem 4.5. *Let* $\rho \in \mathbb{R}_F$ *with* $core(\rho) = [c,d]$ *and* $supp(\rho) = [a,b]$. *Then for all* $r \geq 2$ *we get the estimate:*

$$\left|\tilde{R}_r^M(\rho; [a,b])(t) - \rho(t)\right| \leq 24([b-a]+1)\,\omega_1\left(\rho; \frac{1}{\sqrt{r+1}}\right)_{\tilde{I}}, \ t \in \tilde{I}.$$

Proof. ρ is continuous fuzzy number. Then by Theorem 3.2, the proof is complete. □

Remark 4.6.

1. The conclusion drawn from Theorem 4.5 is that when it comes to approximating fuzzy numbers, the $\tilde{R}_r^M(\rho;[a,b])(t)$ is a preferable choice over the classical Baskakov operators. Although the degree of uniform approximation remains consistent, the $\tilde{R}_r^M(\rho;[a,b])(t)$ better maintains the original shape of the fuzzy number being approximated.
2. In order to address practical aspects, it is valuable to examine the issue of approximating fuzzy numbers characterized by a Lipschitz-type property. Therefore, we present the following Corollary 4.7, based on Theorem 4.5.

Corollary 4.7. *Let $\rho \in \mathbb{R}_F$ with core$(\rho) = [c,d]$ and supp$(\rho) = [a,b]$, such that $\rho \in Lip(\alpha), 0 < \alpha \leq 1$. Then for or all $r \geq 2$, $M > 0$ we have:*
$\left|\tilde{R}_r^M(\rho;[a,b])(t) - \rho(t)\right| \leq 24([b-a]+1)M(r+1)^{\frac{-\alpha}{2}}$.

Proof. From Theorem 4.5, we get:

$$\left|\tilde{R}_r^M(\rho;[a,b])(t) - \rho(t)\right| \leq 24([b-a]+1)\,\omega_1\left(\rho;\frac{1}{\sqrt{r+1}}\right)_{\tilde{I}}.$$

According to the definitions of $\omega_1(\rho;\delta)$ and $Lip(\alpha)$ we get:

$$\begin{aligned}\left|\tilde{R}_r^M(\rho;[a,b])(t) - \rho(t)\right| &\leq 24([b-a]+1)\left[M|x-t|^{\alpha}\right].\\ &\leq 24([b-a]+1)M(r+1)^{\frac{-\alpha}{2}} \quad \square\end{aligned}$$

When aiming to approximate a fuzzy number with another fuzzy number or perform fuzzy arithmetic operations, having a measure of similarity or distance becomes crucial. There are various distance functions defined for fuzzy sets and fuzzy numbers. One of these functions is known as the Chebyshev-type distance. These distance functions take into account the degree of overlap, membership values, and other fuzzy characteristics to determine how similar or dissimilar two fuzzy numbers are.

By utilizing this distance function, we can identify the closest approximation of a fuzzy number using TMPB operators, as shown in Corollary 4.8 below and Figures 2 and 3. This proves to be particularly useful in fuzzy control systems, optimization problems, and decision-making processes where comparing or manipulating fuzzy numbers is necessary.

Corollary 4.8. *Let $\rho \in \mathbb{R}_F$ with core$(\rho) = [c,d]$ and supp$(\rho) = [a,b]$. Then we get:*
$\lim\limits_{r\to\infty} D_c\left(\tilde{R}_r^M(\rho;[a,b]),\rho\right) = 0, \quad r \geq 2.$

Proof. By above Theorem 4.5, we get:

$$\begin{aligned}D_c\left(\tilde{R}_r^M(\rho;[a,b]),\rho\right) &= \sup_{x\in\tilde{I}}\left|\tilde{R}_r^M(\rho;[a,b])(t) - \rho(t)\right|\\ &\leq 24([b-a]+1)\,\omega_1\left(\rho;\frac{1}{\sqrt{r+1}}\right)_{\tilde{I}}\end{aligned}$$

and because ρ is continuous, we obtain $\omega_1\left(\rho;\frac{1}{\sqrt{r+1}}\right)_{\tilde{I}} \to 0.$ $\square$

Finally, the fuzzy numbers represented in the parametric are discussed as follows:

Remark 4.9. Recall, we know the important representation of $\rho \in \mathbb{R}_F$ is that referred to as $\alpha - cut$ (*LU* parametric representation).

That is if $\rho \in \mathbb{R}_F$,we can write it by a pair of functions $\left(\rho^l,\rho^u\right)$where $\rho^l,\rho^u : [0,1] \to \mathbb{R}$ and satisfy the conditions of interval numbers of the real lines.

This means (1) ρ^l is non-decreasing

(2) ρ^u is non-increasing

(3) $\rho^l(1) \leq \rho^u(1)$.

So, let $\rho \in \mathbb{R}_F$, we can construct the fuzzy number $\tilde{R}_r^M(\rho)$ as follows:

We consider $R_r^M\left(\rho^l;[0,1]\right)$ and $R_r^M(\rho^u;[0,1])$ and since $R_r^M(\rho;[0,1])(x)$ preserves the monotonicity. Therefore, $R_r^M\left(\rho^l;[0,1]\right)$ is non-decreasing over $[0,1]$ and $R_r^M(\rho^u;[0,1])$ is non-increasing over $[0,1]$ (because $\rho \in \mathbb{R}_F$).

In addition to that, $R_r^M\left(\rho^{l,u};[0,1]\right)(0) = \rho^{l,u}(0)$ and $R_r^M\left(\rho^{l,u};[0,1]\right)(1) = \rho^{l,u}(1)$ respectively.

Thus, by above we obtain $R_r^M\left(\rho^l;[0,1]\right)(1) \le R_r^M(\rho^u;[0,1])(1)$.

As a result, we get the proper fuzzy number $\tilde{R}_r^M(\rho) = \left(R_r^M\left(\rho^l;[0,1]\right), R_r^M(\rho^u;[0,1])\right)$, which preserves the support and the core of ρ.

Finally, since $\rho^l, \rho^u$are continuous then $R_r^M\left(\rho^{l,u};[0,1]\right)$ are continuous too.

Theorem 4.10. *Let $\rho = \left(\rho^l, \rho^u\right) \in \mathbb{R}_F$ be continuous. Then for all $r \ge 2$ we have:*

$$\tilde{D}_C\left(\tilde{R}_r^M(\rho), \rho\right) \le 24 \max\left\{\omega_1\left(\rho^l; \frac{1}{\sqrt{r+1}}\right)_I, \omega_1\left(\rho^u; \frac{1}{\sqrt{r+1}}\right)_I\right\}, x \in I.$$

Proof. Since $\tilde{D}_C\left(\tilde{R}_r^M(\rho), \rho\right) = \sup\limits_{x\in[0,1]}\left\{\max\left\{\left|R_r^M\left(\rho^l\right)(x) - \rho^l(x)\right|, \left|R_r^M(\rho^u)(x) - \rho^u(x)\right|\right\}\right\}$

from Theorem 2.7, we get:

$$\tilde{D}_C\left(\tilde{R}_r^M(\rho), \rho\right) \le \max\left\{24\omega_1\left(\rho^l; \frac{1}{\sqrt{r+1}}\right)_I, 24\omega_1\left(\rho^u; \frac{1}{\sqrt{r+1}}\right)_I\right\}$$

This finishes the proof. □

Theorem 4.11. *Let $\rho = \left(\rho^l, \rho^u\right) \in \mathbb{R}_F$ and let δ be a reduction function on I. Then for all $r \ge 2$ we have:*

i)$\lim\limits_{r\to\infty}\int_0^1 \delta(\alpha) R_r^M\left(\rho^l\right)(\alpha)\, d\alpha = \int_0^1 \delta(\alpha)\rho^l(\alpha)\, d\alpha$,

ii)$\lim\limits_{r\to\infty}\int_0^1 \delta(\alpha) R_r^M(\rho^u)(\alpha)\, d\alpha = \int_0^1 \delta(\alpha)\rho^u(\alpha)\, d\alpha$

Proof. *i)*
$$\left|\int_0^1 \delta(\alpha) R_r^M\left(\rho^l\right)(\alpha)\, d\alpha - \int_0^1 \delta(\alpha)\rho^l(\alpha)\, d\alpha\right| \le \delta(1)\int_0^1 \left|R_r^M\left(\rho^l\right)(\alpha) - \rho^l(\alpha)\right| d\alpha \le \delta(1)\tilde{D}_C\left(\tilde{R}_r^M(\rho), \rho\right)$$

Hence by Theorem 4.10, and properties of modulus of continuity this proof is finished.

ii) It is proven in the same way as *i*. □

Corollary 4.12. *Let $\rho = \left(\rho^l, \rho^u\right) \in \mathbb{R}_F$ and let δ be a reduction function.on I. Then for all $r \ge 2$ we have:*

1) $\lim\limits_{r\to\infty} El\left(\tilde{R}_r^M(\rho)\right) = El(\rho)$ *and* $\lim\limits_{r\to\infty} wid\left(\tilde{R}_r^M(\rho)\right) = wid(\rho)$

2) $\lim\limits_{r\to\infty} Val_\delta\left(\tilde{R}_r^M(\rho)\right) = Val_\delta(\rho)$ *and* $\lim\limits_{r\to\infty} Amb_\delta\left(\tilde{R}_r^M(\rho)\right) = Amb_\delta(\rho)$

Proof. 1) By taking the particular case of reduction function δ for $\delta(\alpha) = \alpha^m$, $m \in \mathbb{N}$

Hence, if $m = 0$ then $\delta(\alpha) = \alpha^0 = 1$ and by Theorem 4.11 this proof is finished.

2) $\lim\limits_{r\to\infty} Val_\delta\left(\tilde{R}_r^M(\rho)\right) = \lim\limits_{r\to\infty}\int_0^1 \delta(\alpha) R_r^M(\rho^u)(\alpha)\, d\alpha - \lim\limits_{r\to\infty}\int_0^1 \delta(\alpha) R_r^M\left(\rho^l\right)(\alpha)\, d\alpha$

Then from Theorem 4.11, we get:

$$\lim_{r\to\infty} Val_\delta\left(\tilde{R}_r^M(\rho)\right) = \int_0^1 \delta(\alpha)\rho^u(\alpha)\, d\alpha - \int_0^1 \delta(\alpha)\rho^l(\alpha)\, d\alpha = Val_\delta(\rho)$$

In the same way, we can prove

$\lim\limits_{r\to\infty} Amb_\delta\left(\tilde{R}_r^M(\rho)\right) = Amb_\delta(\rho)$ □

4.2. The estimation of approximation using L^1−type metrics

In this part, estimates of approximations with respect to the metrics D_1 are provided. Therefore, before the conclusions related to approximation are started, some basic things that are needed in these conclusions must be given.

Definition 4.13. ([16]) Let ρ be any function on $\tilde{I}$, and let $M > 0$ be any constant. Now, if for any partition of $\tilde{I}$, $a = t_0 < t_1 < \cdots < t_r = b$, we have $\sum_{i=0}^{r} \left|\rho(t_{i+1}) - \rho(t_i)\right| \leq M$. Then ρ is said to be of bounded variation.

The total variation of ρ is the supremum over all the above sums on $\tilde{I}$, and is denoted by the symbol $V_a^b(\rho)$.

In other words, by Jordan's theorem, we can say ρ of bounded variation on $\tilde{I}$ if and only if it can be expressed as the difference between two non-decreasing functions $\rho_1, \rho_2 : \tilde{I} \to \mathbb{R}$. That is, $\rho = \rho_1 - \rho_2$ on $\tilde{I}$.

Noteworthy, is that every fuzzy number exhibits bounded variation within its support.

Recall, Let $\rho \in \mathbb{R}_F$, such that $a < c \leq d < b$ with core$(\rho) = [c, d]$ and supp$(\rho) = [a, b]$. So there exists l_ρ, which is non-decreasing on $[a, c]$, known as the left side of ρ, and r_ρ, which is non-increasing on $[d, b]$, known as the right side of ρ. Such that $\rho(t) = l_\rho$ for $t \in [a, c]$, $\rho(t) = r_\rho$ for $t \in [d, b]$ and $\rho(1) = 1$ for $t \in [c, d]$.

Lemma 4.14. ([16]) *Let $\rho \in \mathbb{R}_F$, so we can say $V_a^b(\rho) \leq 2$ and $\rho(t) = \rho_1(t) - \rho_2(t)$, for all $t \in \tilde{I}$, such that ρ_1 and ρ_2 are non-decreasing, then we get*

$\rho_1(t) = l_\rho(t)$ *if* $t \in [a, c]$, $\rho_1(t) = 1$ *if* $t \in [c, b]$,

$\rho_2(t) = 0$ *if* $t \in [a, d]$, $\rho_2(t) = 1 - r_\rho$ *if* $t \in [d, b]$.

Theorem 4.15. *Let ρ be with bounded variation on I, such that $\xi(\ell) = \dfrac{\rho(\ell)}{\ell}$ is non-increasing on $(0, 1]$. Then for all $r \in \mathbb{N}$, $r \geq 2$ we have:*

$$\int_0^1 \left|R_r^M(\rho; [0, 1])(x) - \rho(x)\right| dx \leq \frac{M}{r-1}$$

where $M = 2\left[V_0^1(\rho_1) + V_0^1(\rho_2) + \|\rho\|\right]$ *and* $\rho = \rho_1 - \rho_2$, *with* ρ_1, ρ_2 *are non-decreasing.*

Proof. This Theorem is proved by applying the same technique as in ([16], Theorem 2.6.9). □

And by extending the results obtained in $L^1 - norm$ to the arbitrary compact interval $\tilde{I}$, $a \leq b$, then we can apply it to the approximation of fuzzy numbers, as follows.

By conclusion of Lemma 4.1 we have:

$$\bigvee_{k=0}^{r} \varphi_{r,k}(t) = \varphi_{r,s}(t) \text{ for all } t \in \left[a + \frac{(b-a)s}{r-1}, a + \frac{(b-a)(s+1)}{r-1}\right].$$

Therefore, if $\rho_{k,r,s} : \left[a + \dfrac{(b-a)s}{r-1}, a + \dfrac{(b-a)(s+1)}{r-1}\right] \to \mathbb{R}$.

Then, $\rho_{k,r,s}(t) = m_{k,r,s}(t) \cdot \rho\left(a + \dfrac{(b-a)k}{r}\right)$, $k = \{0, 1, 2, \cdots, r\}$, $s = \{0, 1, 2, \cdots, r-2\}$

Thus, $\rho_{k,r,s}(t) = \dfrac{\binom{r+k-1}{k}}{\binom{r+s-1}{s}} \left(\dfrac{t-a}{b+t-2a}\right)^{k-s} \cdot \rho\left(a + (b-a)\dfrac{k}{r}\right)$.

As a result of the above, we can show the following:

Theorem 4.16. *Let ρ be with bounded variation on $\tilde{I}$, such that $\dfrac{\rho(t)}{t-a}$ is non-increasing on $(a, b]$. Then for all $r \geq 2$, $s = \{0, 1, 2, \cdots, r-2\}$ there is $M > 0$ that only depends on ρ, such that:*

$$\int_a^b \left|R_r^M(\rho; [a, b])(t) - \rho(t)\right| dt \leq \frac{M}{r-1}, \text{ for all } r \in \mathbb{N}.$$

Proof. By Remark 3.1,we have $\upsilon(x) = \rho(a + (b-a)x)$

Let $x = \dfrac{t-a}{b-a}$, so, we can say $\upsilon\left(\dfrac{t-a}{b-a}\right) = \rho(t)$ and $t = a + (b-a)x$ for all $t \in \tilde{I}$. Then we have:
$\left|R_r^M(\rho;[a,b])(t) - \rho(t)\right| = \left|R_r^M(\upsilon;[0,1])(x) - \upsilon(x)\right|$.
Then, we can easily get:
$\int\limits_a^b \left|R_r^M(\rho;[a,b])(t) - \rho(t)\right| dt = (b-a)\int\limits_0^1 \left|R_r^M(\upsilon;[0,1])(x) - \upsilon(x)\right| dx.$

Now, if $\xi(x) = \dfrac{\upsilon(x)}{x} = \dfrac{\rho\left(a+(b-a)\frac{t-a}{b-a}\right)}{\frac{t-a}{b-a}} = (b-a)\dfrac{\rho(t)}{t-a}$.

Then, directly we can say υ satisfies the hypothesis of Theorem 4.15, that is υ is of bounded variation on I, such that $\xi(x) = \dfrac{\upsilon(x)}{x}$ is non-increasing on $(0,1]$.

Hence, we obtain of a constant M_υ which only depends on υ, such that
$\int\limits_0^1 \left|R_r^M(\upsilon;[0,1])(x) - \upsilon(x)\right| dx \leq \dfrac{M_\upsilon}{r-1},\ r \geq 2.$
But because υ is a function that depends on ρ, we can easily determine that M_υ only depends on ρ.
Thus, if we take $M = (b-a)M_\upsilon$, we get the intended conclusion. □

Theorem 4.17. *Let $\rho \in \mathbb{R}_F$, such that $a < c \leq d < b$ with core$(\rho) = [c,d]$ and supp$(\rho) = [a,b]$, and let the restriction of ρ to its support fulfills the hypotheses of Theorem 4.16, then we have:*
$D_1\left(\tilde{R}_r^M(\rho;[a,b]),\rho\right) \leq \dfrac{6(b-a)}{r-1},\ r \geq 2$

Proof. Since $D_1\left(\tilde{R}_r^M(\rho;[a,b]),\rho\right) = \int\limits_{\mathbb{R}} \left|\tilde{R}_r^M(\rho;[a,b])(t) - \rho(t)\right| dt$,

So, we prove this theorem with the same idea as proving the previous Theorem 4.16.

Therefore, we define the $\xi \in \mathbb{R}_F$, where $\xi(x) = \rho(a + (b-a)x)$, if $x \in I$ and $\xi(x) = 0$ if $x \notin I$ that is supp$(\xi) = [0,1]$.

Hence, we deal with ξ by the same steps that were dealt with υ in the previous Theorem 4.16, so we get:
$\int\limits_{\mathbb{R}} \left|\tilde{R}_r^M(\rho;[a,b])(t) - \rho(t)\right| dt \leq \dfrac{(b-a)M_\xi}{r-1}.$

Where the hypothesis of Theorem 4.15 is fulfilled by the restriction of ξ to its support, we are able to take:
$M_\xi = 2\left[V_0^1(\xi_1) + V_0^1(\xi_2) + \|\xi\|\right]$,

Now we apply the definition of ρ_1 and ρ_2 in Lemma 4.14 to ξ_1 and ξ_2 in the same way, then easily we get:
$V_0^1(\xi_1) = V_0^1(\xi_2) = 1$ and we know the $\|\xi\| = 1$, we obtain that $M_\xi = 6$. This finishes the proof. □

Finally, to clarify all the theoretical conclusions mentioned in this paper, we give the following example.

Example 4.18. The approximation of a fuzzy number ρ was illustrated using both the truncated max-product Baskakov operators and classical operators in (Fig.2 and Fig.3), where:

$$\mu_\rho(t) = \begin{cases} 5t^2 - 0.25 & if \ \ 0 \leq t < 0.5 \\ 1 & if \ \ 0.5 \leq t \leq 0.8 \\ 5 - 5t & if \ \ 0.8 < t \leq 1 \end{cases}$$

In Fig.2 and Fig.3. It's clear to us that the truncated max-product (non-linear) Baskakov operators marked with dashed lines are approximations far better than the classical operators marked with dotted lines. Furthermore, our analysis leads us to the conclusion that as the degree of 'r' increases, the level of approximation by these operators improves significantly.

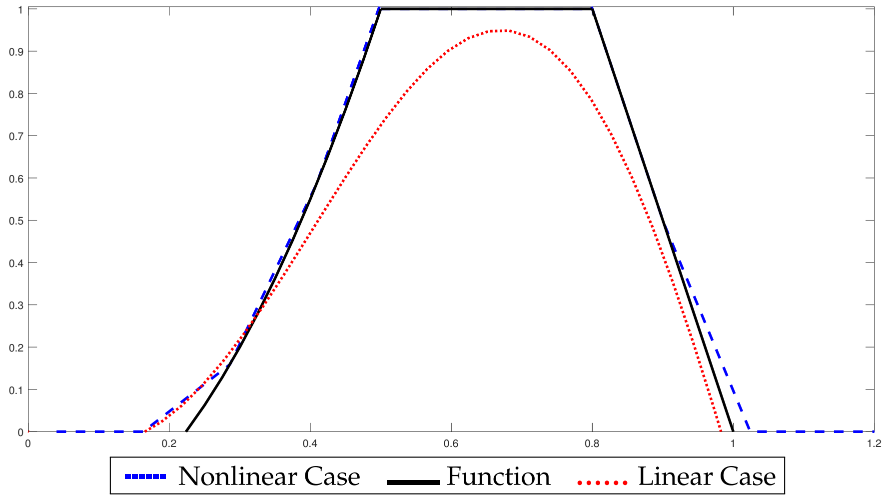


Figure 2: Approximations by classical and the truncated max-product Baskakov operators, r = 20.

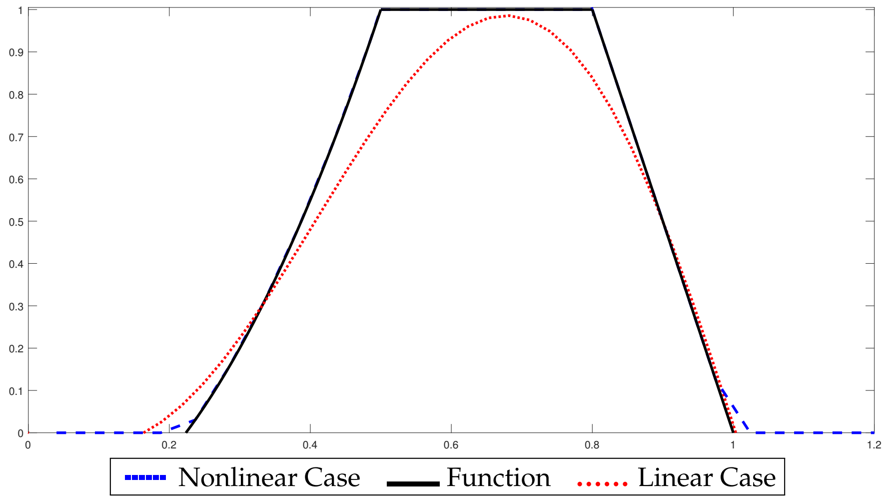


Figure 3: Approximations by classical and the truncated max-product Baskakov operators, r = 50.

5. Conclusion

This study demonstrates the significant potential of utilizing the truncated max-pro Baskakov operators as valuable instruments for approximating fuzzy numbers. Our findings indicate that these operators proficiently preserve both the support and, to a substantial extent, the core of the approximated fuzzy number. From this approximation, several noteworthy conclusions can be drawn:

- The uniform convergence of $\tilde{R}_r^M(\rho; [a, b])(t)$ for the fuzzy number ρ was established when the $\mu_\rho(x)$ is continuous.
- $\tilde{R}_r^M(\rho; [a, b])(t)$ successfully preserves the monotony and shape properties of the approximate $\rho \in \mathbb{R}_F$.
- In the parametric case, it was observed that $\tilde{R}_r^M(\rho)$ uniformly converges to ρ. Additionally, important aspects of $\tilde{R}_r^M(\rho)$, such as widths, expected intervals, and ambiguities, converge to ρ.
- Improved estimates of convergence, in relation to metrics spaces D_C and L^1−type, were obtained through the utilization of $\tilde{R}_r^M(\rho; [a, b])(t)$.

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