Filomat 38:9 (2024), 3193–3206
https://doi.org/10.2298/FIL2409193B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Conformable cosine family and nonlinear fractional differential equations

Abdelmjid Benmerrous[a], Lalla Saadia Chadli[a], Abdelaziz Moujahid[a], M'hamed Elomari[a], Said Melliani[a]

[a]*Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, PO Box 532, Beni Mellal, 23000, Morocco.*

Abstract. This paper is focuses to the existence and uniqueness solution to the following problem:

$$\begin{cases} D^{(\alpha)}f(t,y) + Af(t,y) = F(t,f(t,y)) \quad y \in \mathbb{R}, \quad t \geq 0 \\ f(0,y) = u_0(y), \quad D^{(\alpha)}f(0,y) = v_0(y) \end{cases} \tag{0.1}$$

where $D^{(\alpha)}$ is the conformable derivation for $1 < \alpha < 2$ which we will prove to be inside Colombeau algebra, u_0 and v_0 are singular distibution and F provides L^∞ logarithmic type, the operator A is defined in Colombeau's algebra. Nets of conformable cosine family $(C_\epsilon^\alpha)_\epsilon$ with polynomial development in ϵ as $\epsilon \to 0$ are defined for the first time and used for solving irregular fractional problems.

1. Introduction

An evolution equation is a mathematical equation that describes the time evolution of a physical system or a field over time. These equations can be used to study a wide range of phenomena, from the motion of fluids and gases to the propagation of waves, the behavior of particles in quantum mechanics, and even the growth of populations.

The most common form of evolution equations is partial differential equations, which relate the rates of change of various quantities to each other over time and space. These equations are essential in many areas of science and engineering, for the reason that they offer an effective tool for modeling and analyzing intricate systems and processes.

In traditional calculus, derivatives are defined for integer orders only, such as the first derivative, second derivative, and so on. However, conformable calculus allows for derivatives of any real or complex order, including non-integer orders.

The basic idea behind conformable calculus is to redefine the traditional difference operator by using the conformable fractional difference operator, which is a generalization of the traditional difference operator. The conformable fractional difference operator uses the concept of fractional calculus, which is a division of calculus that concerned with derivatives and integrals of non-integer orders.

2020 *Mathematics Subject Classification*. 46F30, 35D99.
Keywords. Colombeau algebra, Banach space, Conformable cosine family, Nonlinear problem, Generalized function.
Received: 14 July 2023; Revised: 04 October 2023; Accepted: 14 October 2023
Communicated by Maria Alessandra Ragusa
Email addresses: abdelmajid.benmerrous@usms.ma (Abdelmjid Benmerrous), chadli@fstbm.ac.ma (Lalla Saadia Chadli), moujahid.aziz.usms@gmail.com (Abdelaziz Moujahid), m.elomari@usms.ma (M'hamed Elomari), melliani@fstbm.ac.ma (Said Melliani)

Conformable calculus has applications in various fields, including physics, engineering, finance, and biology. It provides a new tool for modeling complex systems that cannot be accurately described using traditional calculus.

In this paper we characterize a new method for solving the nonlinear fractional evolution equations having singular initial data as we can see in the following

$$\begin{cases} D^{(\alpha)} f(t,y) + Af(t,y) = F(t, f(t,y)) \quad y \in \mathbb{R}, \quad t \geq 0 \\ f(0,y) = u_0(y), \quad \partial_t^{(\alpha)} f(0,y) = v_0(y) \end{cases} \tag{1.1}$$

Where $D^{(\alpha)}$ is the conformable derivation with $1 < \alpha \leq 2$, the linear operator $A : D(A) \subset \mathcal{G} \to \mathcal{G}$, $F : [0,T] \times \mathcal{G} \to \mathcal{G}$, $\mathcal{G}$ is the Colombeau algebra.

The pioneering work on (2) was done by C.C.Travis in [21] and our development follows his approach. Our results extend those of A. Benmerrous [3, 4] in several respects.

The paper is organized as follows, in section 2 we mention some notions of Colombeau's algebra and some notion concerning the new derivative, in section 3 we will prove the existence and uniqueness of conformable fractional derivative of order α in Colombeau algebra, in section 4 we will deal with the basic definition of conformable cosine family and some properties, in section 5, we provided the existence and uniqueness of generalized solution.

2. Preliminaries

2.1. Colombeau algebra

In this section we will introduce basic concepts and definitions from Colombeau theory (see also [12]).

Definition 2.1. *$\mathcal{A}_0(\mathbb{R}^n)$ is a set of functions ϕ in $C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(t)dt = 1$. For $q \in \mathbb{N}, \mathcal{A}_q(\mathbb{R}^n) = \left\{\phi \in \mathcal{A}_0 : \int_{\mathbb{R}^n} t^i \phi(t)dt = 0, 0 < |i| \leq q\right\}$, where $t^i = t_1^{i_1} \cdots t_n^{i_n}$.*

In [12] sets

$$\overline{\mathcal{A}}_q(\mathbb{R}^n) = \left\{\Phi(x_1,..,x_n) = \Phi(x_1)..\Phi(x_n) : \phi(x_i) \in \mathcal{A}_q(\mathbb{R})\right\},$$

are used because of applications to initial value problems. We shall follow the Colombeau original definition.

Obviously, if $\phi \in \mathcal{A}_q, q \in \mathbb{N}_0$, then for every $\varepsilon > 0, \phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q$. If $\phi \in \mathcal{A}_0$, then its support number $d(\phi)$ is defined by

$$d(\phi) = \sup\{|x| : \phi(x) \neq 0\}.$$

$\mathcal{E}(\Omega)$ represents the set of

$$R : \mathcal{A}_0 \times \Omega \to \mathbb{C}, (\Phi, x) \mapsto R(\Phi, x),$$

which are in $C^\infty(\Omega)$ for every fixed ϕ. In the other words elements of $\mathcal{E}$ are functions $R : \mathcal{A}_0 \to C^\infty$. Note that for any $f \in C^\infty$, the mapping

$$(\phi, x) \mapsto f(x), (\phi, x) \in \mathcal{A}_0 \times \Omega,$$

defines an element in $\mathcal{E}(\Omega)$ which does not depend on ϕ. Conversely, if an element F in $\mathcal{E}(\Omega)$ does not depend on $\Phi \in \mathcal{A}_0$, we have:

$$F(\Phi, x) = F(\Psi, x), \quad x \in \Omega, \text{ for every } \Phi, \Psi \in \mathcal{A}_0,$$

then it defines a function $f \in C^\infty(\Omega)$,

$$f(x) = F(\Phi, x), x \in \Omega, \text{ for every } \phi \in \mathcal{A}_0.$$

In this sense, we identify $C^\infty(\Omega)$ with the corresponding subspace of $\mathcal{E}(\Omega)$.

Definition 2.2. *A component $R \in \mathcal{E}(\Omega)$ is moderate if $\forall L \subset\subset \Omega$, $\alpha \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\|\partial^\alpha R\left(\Phi_\epsilon, x\right)\| \leq C\epsilon^{-N} \qquad x \in L, 0 < \epsilon < \eta.$$

The ensemble of all mild components is expressed as $\mathcal{E}_M(\Omega)$.

Definition 2.3. *An element $R \in \mathcal{E}_0(\mathbb{C})$ is moderate if $\exists N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$, $\exists \eta > 0$, $C > 0$ such that:*

$$\|R\left(\phi_\varepsilon\right)\| < C\varepsilon^{-N}, 0 < \varepsilon < \eta.$$

The ensemble of mild components is expressed by $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$).

Definition 2.4. *A component $R \in \mathcal{E}_M(\Omega)$ is named null if for every $L \subset\subset \Omega$ and every $\alpha \in \mathbb{N}_0^n$, $\exists N \in \mathbb{N}_0$ and $\left\{a_q\right\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\left\|\partial^\alpha R\left(\phi_\varepsilon, x\right)\right\| \leq C\varepsilon^{a_q - N} \qquad x \in L, 0 < \varepsilon < \eta.$$

The ensemble of null components is expressed by $\mathcal{N}(\Omega)$.

Definition 2.5. *The spaces of generalized functions $\mathcal{G}(\Omega)$ expressed by*

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

The following description describes what the term "association" means in colombeau algebra.

Definition 2.6. *[12] Let $f, g \in \mathcal{G}(\mathbb{R})$.*
We said that f,g are associated if $\forall$ $h(\varphi_\epsilon, x)$ and $m(\varphi_\epsilon, x)$ and arbitrary $\xi \in \mathcal{D}(\mathbb{R})$ there is a $n \in \mathbb{N}$ such that $\forall \varphi(x) \in \mathcal{A}_n(\mathbb{R})$, we have:

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \|h(\varphi_\epsilon, x) - m(\varphi_\epsilon, x)\| \xi(x) dx = 0$$

and we denoted by $f \approx g$.

2.2. Conformable derivative

The definition of conformable derivation is provided in the following part.

Definition 2.7. *[16] Let $n < \alpha \leq n+1$ and $u : \mathbb{R}^+ \to \mathbb{R}$ be n-differeniable, then the conformable fracrional derivauive of u of order α characterized by*

$$D^{(\alpha)} u(r) = \lim_{\epsilon \to 0} \frac{u^{(n)}\left(r + \epsilon r^{n+1-\alpha}\right) - u^{(n)}(r)}{\epsilon}$$
$$D^{(\alpha)} u(0) = \lim_{r \to 0} D^{(\alpha)} u(r)$$

Remark 2.1. *[16] In light of the definition above, it is simple to demonstrate that*

$$D^{(\alpha)} u(r) = r^{n+1-\alpha} u^{(n+1)}(r)$$

with $n < \alpha \leq n+1$, and u is $(n+1)$-differeniable.

Definition 2.8. *[16] Let $1 < \alpha \leq 2$,*

$$\left(I^{(\alpha)} u\right)(r) = \int_0^t s^{\alpha-2} u(s) ds$$

.

Theorem 2.1. *[16]*

$$D^{(\alpha)}(I^{(\alpha)} u(r)) = u(r)$$

for $r \geq 0$

3. Generalized conformable derivative

Let $(f_\epsilon(t))_\epsilon$ be a representative of the function $f(t) \in \mathcal{G}(\mathbb{R}^+)$ and let $n-1 < \alpha < n$.

The generalized conformable fractional derivative of $(f_\epsilon(t))_\epsilon$, characterized by

$$D^{(\alpha)} f_\epsilon(y) = y^{1-\alpha} \frac{d}{dy} f_\epsilon(y) \tag{3.1}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$

Lemma 3.1. *Let $(f_\epsilon(y))_\epsilon$ be a representative of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, $\forall \alpha > 0$, $\sup_{y\in[0,T]} \mid D^{(\alpha)} f_\epsilon(y) \mid$ has a moderate bound.*

Proof.

$$\begin{aligned}\sup_{y\in[0,T]} \|D^{(\alpha)} f_\epsilon(y)\| = \sup_{y\in[0,T]} \|y^{1-\alpha} \frac{d}{dy} f_\epsilon(y)\| &\leq T^{1-\alpha} \sup_{y\in[0,T]} \|\frac{d}{dy} f_\epsilon(y)\| \\ &\leq T^{1-\alpha} C \epsilon^{-N} \\ &\leq C_{\alpha,T} \epsilon^{-N}\end{aligned}$$

Then, $\exists M \in \mathbb{N}$, such as

$$\sup_{y\in[0,T]} \|D^{(\alpha)} f_\epsilon(y)\| = O\left(\epsilon^{-M}\right), \quad \epsilon \to 0$$

□

Lemma 3.2. *Let $(f_{1\epsilon}(t))_\epsilon$, $(f_{2\epsilon}(t))_\epsilon$ be two distinct representatives of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, $\forall \alpha > 0$, $\sup_{y\in[0,T]} \mid D^{(\alpha)} f_{1\epsilon}(y) - D^{(\alpha)} f_{2\epsilon}(y) \mid$ is negligible.*

Proof.

$$\begin{aligned}\sup_{y\in[0,T]} \|D^{(\alpha)} f_{1,\epsilon}(y) - D^{(\alpha)} f_{2,\epsilon}(y)\| &= \sup_{y\in[0,T]} \|y^{1-\alpha} \frac{d}{dy} f_{1,\epsilon}(y) - y^{1-\alpha} \frac{d}{dy} f_{2,\epsilon}(y)\| \\ &= \sup_{y\in[0,T]} \|y^{1-\alpha} \left(\frac{d}{dy} f_{1,\epsilon}(y) - \frac{d}{dy} f_{2,\epsilon}(y)\right)\| \\ &\leq T^{1-\alpha} \sup_{y\in[0,T]} \|\frac{d}{dy} f_{1,\epsilon}(y) - \frac{d}{dy} f_{2,\epsilon}(y)\|\end{aligned}$$

Since $(f_{1\epsilon}(y))_\epsilon$ and $(f_{2\epsilon}(y))_\epsilon$ represent the same Colombeau generalized function $f(y)$, so $\sup_{y\in[0,T]} \mid \frac{d}{dy} f_{1,\epsilon}(y) - \frac{d}{dy} f_{2,\epsilon}(y) \mid$ is negligible, then for all $p \in \mathbb{N}$

$$\sup_{y\in[0,T]} \|D^{(\alpha)} f_{1\epsilon}(y) - D^{(\alpha)} f_{2\epsilon}(y) \mid = O\left(\epsilon^{-p}\right), \quad \epsilon \to 0$$

Therefore, $\sup_{y\in[0,T]} \|D^{(\alpha)} f_{1\epsilon}(y) - D^{(\alpha)} f_{2\epsilon}(y)\|$ is negligible. □

We may now initiate the generalized conformable fractional derivative of a Colombeau generalized function on $\mathbb{R}^+$ after establishing the first two lemmas.

Definition 3.1. *Let $f(y) \in \mathcal{G}(\mathbb{R}^+)$ be a Colombeau function on $\mathbb{R}^+$.*
The generalized conformable fractional derivative of $f(y)$, using the notation $D^{(\alpha)} f(t) = \left[\left(D^{(\alpha)} f_\epsilon(t)\right)_\epsilon\right]$, $\alpha > 0$, is a component of $\mathcal{G}(\mathbb{R}^+)$ satisfying (3).

Remark 3.1. *For $\alpha \in (0,1]$ the first-order derivative of $D^{(\alpha)} f_\epsilon(y)$ is*

$$\frac{d}{dy} D^{(\alpha)} f_\epsilon(y) = (1-\alpha) y^{-\alpha} \frac{d}{dy} f_\epsilon(y) + y^{1-\alpha} \frac{d^2}{dy^2} f_\epsilon(y)$$

and it fails to reach its limit.
Generally, the p-th order derivative $\frac{d^p}{dy^p} D^{(\alpha)} f_\epsilon(y)$ it fails to reach its limit on $\mathbb{R}^+$.

Then if we wants $D^{(\alpha)}$ to be in $\mathcal{G}(\mathbb{R}^+)$, thus the fractional derivative must be regularized.

Definition 3.2. *Let $(f_\epsilon)_\epsilon$ be a representative of a Colombesu generalized $f \in \mathcal{G}([0,\infty))$. The regularized of new fractional derivative of $(f_\epsilon)_{\epsilon\infty}$, is characterized by :*

$$\bar{D}^{(\alpha)} f_\epsilon(y) = \begin{cases} \left(D^{(\alpha)} f_\epsilon * \varphi_\epsilon\right)(y), & n-1 < \alpha < n \\ f_\epsilon^{(n)}(y) = (\frac{d}{dy})^n f_\epsilon(y), & \alpha = n, \end{cases} \tag{3.2}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$.
where (3) gives $D^{(alpha)} f_\epsilon(y)$ and the first section gives $\varphi_\epsilon(y)$.

The convolution in (4) is $\left(D^{(\alpha)} f_\epsilon(y) * \varphi_\epsilon\right)(y) = \int_0^\infty D^{(\alpha)} f_\epsilon(y)\varphi_\epsilon(y-s)ds$.

Lemma 3.3. *Let $(f_\epsilon(y))_\epsilon$ be a representative of $f(y) \in \mathcal{G}(\mathbb{R}^+)$.
So, $\forall \alpha > 0$, $k \in \{0,1,\ldots\}$, $\sup_{y\in[0,T]} \| \left(d^k/dy^k\right) \tilde{D}^{(\alpha)} f_\epsilon(y)\|$ has a moderate limit.*

Proof. Let $0 < \epsilon < 1$.
For $\alpha \in \mathbb{N}$, $\tilde{D}^{(\alpha)} f_\epsilon(y)$ is the normal derivative of order α of $f_\epsilon(y)$ and the assertion follows immediately .
In the event that $n-1 < \alpha \le n$, We've got

$$\begin{aligned} \sup_{y\in[0,T]} \|\bar{D}^{(\alpha)} f_\epsilon(y)\| &= \sup_{y\in[0,T]} \| \left(D^{(\alpha)}\ f_\epsilon * \varphi_\epsilon\right)(y)\| \\ &\le \sup_{y\in[0,T]} \| \int_0^\infty D^{(\alpha)} f_\epsilon(s)\varphi_\epsilon(y-s)ds\| \\ &\le \sup_{r\in K} \|D^{(\alpha)} f_\epsilon(r)\| \sup_{y\in[0,T]} \| \int_K \varphi_\epsilon(y-s)ds\| \\ &\le C \sup_{y\in K} \|D^{(\alpha)} f_\epsilon(y)\| \end{aligned}$$

With C is a strictly positive constant.
Using the Lemma 1, $\sup_{y\in[0,T]} \mid D^{(\alpha)} f_\epsilon(y) \mid$ has a moderate bound, $\forall \alpha > 0$, as a result of this, $\sup_{y\in[0,T]} \mid \bar{D}^{(\alpha)} f_\epsilon(y) \mid$ has a moderate bound, too.
□

Lemma 3.4. *Let $(f_{1\epsilon}(y))_\epsilon$ and $(f_{2\epsilon}(y))_\epsilon$ be two different representatives of $f(y) \in \mathcal{G}(\mathbb{R}^+)$. Then, $\forall \alpha > 0$, $k \in \{0,1,2,\ldots\}$, $\sup_{t\in[0,T]} \mid \left(\mathrm{d}^k/\mathrm{d}t^k\right)\left(\tilde{D}^{(\alpha)} f_{1\epsilon}(t) - \tilde{D}^{(\alpha)} f_{2\epsilon}(t)\right) \mid$ is negligible.*

Proof.

$$\begin{aligned} &\sup_{y\in[0,T]} \mid \frac{d^k}{dy^k}\left(\bar{D}^{(\alpha)} f_{1\epsilon}(y) - \bar{D}^{(\alpha)} f_{2\epsilon}(y)\right) \| = \\ &\sup_{y\in[0,T]} \|\frac{d^k}{dy^k}\left(\left(D^{(\alpha)} f_{1\epsilon} * \varphi_\epsilon\right)(y) - \left(D^{(\alpha)} f_{2\epsilon} * \varphi_\epsilon\right)(y)\right) \mid \\ &= \sup_{y\in[0,T]} \|\frac{d^k}{dy^k}\left(\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right) * \varphi_\epsilon\right)(y)\| \\ &= \sup_{y\in[0,T]} \|\left(\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right) * \frac{d^k}{dy^k}\varphi_\epsilon\right)(y)\| \\ &\le \sup_{r\in K} \|\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\| \sup_{y\in[0,T]} \| \int_K \frac{d^k}{dy^k}\varphi_\epsilon(y-r)dr\| \\ &\le C \sup_{r\in K} \|\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\| \end{aligned}$$

Using the Lemma 2, we have $\sup_{r\in K} \| \left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\|$ is negligible, so $\sup_{y\in[0,T]} | \frac{d^k}{dy^k} \left(\bar{D}^{(\alpha)} f_{1\epsilon}(y) - \bar{D}^{(\alpha)} f_{2\epsilon}(y)\right) |$ is negligible. □

The regularized generalized conformable fractional derivative $D^{(alpha)}$ is now introduced in the following manner.

Definition 3.3. *Let $f(t) \in \mathcal{G}(\mathbb{R}^+)$ be a Colombeau generalized function. The regularized generalized conformable fractional derivative of $f(t)$, writing $\bar{D}^{(\alpha)} f(t) = \left[\left(\tilde{D}^{(\alpha)} f_\epsilon(t)\right)_\epsilon\right]$, $\alpha > 0$, is a component of $\mathcal{G}(\mathbb{R}^+)$ satisfy (4).*

4. Generalized conformable Cosine family

Let $(X, \|.\|)$ denote a Banach space, and $C(X)$ denote the space of all linear continuous mappings. Before we define the generalized conformable cosine family, we will state that an application from $\mathcal{G} \longrightarrow \mathcal{G}$ must be linear.

Definition 4.1. *Let X be a locally convex space with a semi-norm familly $(q_i)_{i\in I}$.*

We define $\mathcal{E}_M$ by the set of $(y_\epsilon)_\epsilon \subset X$ such that there exist $n \in \mathbb{N}$ and for all $i \in I \subset \mathbb{N}$, $q_i(y_\epsilon) = O_{\epsilon\to 0}(\epsilon^{-n})$.

And
$\mathcal{N}(X)$ by $(y_\epsilon)_\epsilon \subset X$ such that for all $m \in \mathbb{N}$ and for all $i \in I \subset \mathbb{N}$, $\quad q_i(y_\epsilon) = O_{\epsilon\to 0}(\epsilon^n)$.

Then the Colombeau generalized function type by:

$$\overline{X} = \mathcal{E}_M(X)/\mathcal{N}(X)$$

Initially, using a provided family $(A_\epsilon)_{\epsilon\in[0,1]}$ of maps $A\epsilon : X \longrightarrow X$ we want to see if we can define a map $A : \overline{X} \longrightarrow \overline{X}$, $A_\epsilon \in \mathcal{L}(X)$.

The next lemma expresses the basic requirement:

Lemma 4.1. *Let $(A_\epsilon)_\epsilon$ represent a family of maps $A_\epsilon : X \longrightarrow X$.*

For each $(x_\epsilon)_\epsilon \in \mathcal{E}_M(X)$ and $(y_\epsilon)_\epsilon \in \mathcal{N}(X)$, suppose that:
1) $(A_\epsilon x_\epsilon)_\epsilon \in \mathcal{E}_M(X)$
2) $(A_\epsilon(x_\epsilon + y_\epsilon))_\epsilon - (A_\epsilon x_\epsilon)_\epsilon \in \mathcal{N}(X)$

So

$$A : \begin{cases} \overline{X} \longrightarrow \overline{X} \\ x = [x_\epsilon] \longmapsto Ax = [A_\epsilon x_\epsilon] \end{cases}$$

is clearly stated.

Proof. The first attribute reveals that the class $[(A_\epsilon x_\epsilon)_\epsilon] \in \overline{X}$.
Let $x_\epsilon + y_\epsilon$ should serve as another example of $x = [x_\epsilon]$, we have from the second property:

$$(A_\epsilon(x_\epsilon + y_\epsilon))_\epsilon - (A_\epsilon x_\epsilon)_\epsilon \in \mathcal{N}(X)$$

and

$$\left[(A_\epsilon(x_\epsilon + y_\epsilon))_\epsilon\right] = \left[(A_\epsilon x_\epsilon))_\epsilon\right] \text{ in } \overline{X}$$

So A is well defined. □

We shall now introduce the idea of the generalized conformable cosine family(Convolution-type cosine family).

Definition 4.2.

$$E_{M,\alpha}\left(\mathbb{R}^{+},C(X)\right):=\Big\{C_{\epsilon}^{\frac{1}{\alpha}}:\mathbb{R}^{+}\rightarrow C(X),\epsilon\in]0,1[/\forall T>0,\exists a\in\mathbb{R}\ \textit{such that}\ \sup_{t\in[0,T]}\|C_{\epsilon}^{\frac{1}{\alpha}}(t)\|=O\left(\varepsilon^{a}\right),\ \epsilon\rightarrow 0\Big\} \tag{4.1}$$

$$N_{\alpha}(\mathbb{R}^{+},C(X)):=\Big\{N_{\epsilon}^{\frac{1}{\alpha}}:\mathbb{R}^{+}\rightarrow C(X),\epsilon\in]0,1[/\forall T>0,\forall b\in\mathbb{R}\ \textit{such that}\ \sup_{t\in[0,T]}\|N_{\epsilon}^{\frac{1}{\alpha}}(t)\|=O\left(\epsilon^{b}\right),\ \epsilon\rightarrow 0\Big\} \tag{4.2}$$

With the following characteristics:
1) $\exists s>0$ *and* $\exists a\in\mathbb{R}$ *such that*

$$\sup_{t<s}\left\|\frac{N_{\epsilon}\left(t^{\frac{1}{\alpha}}\right)}{t}\right\|=O_{\epsilon\rightarrow 0}\left(\epsilon^{a}\right),$$

2) $\exists\left(H_{\epsilon}\right)_{\epsilon}$ *in* $C(X)$ *and* $\epsilon\in]0,1[$ *such that*

$$\lim_{s\rightarrow 0}\frac{N_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)}{s}e=H_{\epsilon}e,\quad e\in X,$$

For every $b>0$,

$$\|H_{\epsilon}\|=O_{\epsilon\rightarrow 0}\left(\epsilon^{b}\right),$$

Proposition 4.1. *$N_{\alpha}(\mathbb{R}^{+},C(X))$ is an ideal of $E_{M,\alpha}(\mathbb{R}^{+},C(X))$ and $E_{M,\alpha}(\mathbb{R}^{+},C(X))$ is an algebra with respect to composition.*

Proof. Let $(C_{\epsilon})_{\epsilon}\in E_{M,\alpha}\left([0,+\infty[,C(X)\right)$ and $(N_{\epsilon})_{\epsilon}\in\mathcal{N}_{\alpha}\left([0,+\infty[,C(X)\right)$.
We shall simply establish the second statement, specifically,

$$\left(C_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)N_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)\right)_{\epsilon},\left(N_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)C_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)\right)_{\epsilon}\in\mathcal{N}_{\alpha}\left([0,+\infty[,C(X)\right)$$

Where $C_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)N_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)$ represents the composition.
By (1) and the definition of $\mathcal{N}_{\alpha}$ from the previous definition, we have:

$$\left\|C_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)N_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)\right\|\leq\left\|C_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)\right\|\left\|N_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)\right\|=O_{\epsilon\rightarrow 0}\left(\epsilon^{a+b}\right),$$

The same is also true for $\left\|N_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)C_{\epsilon}\left(s^{\frac{1}{\alpha}}\right)\right\|$.
Furthermore, (1) and (2) provide

$$\begin{aligned}\sup_{r<s}\left\|\frac{C_{\epsilon}\left(r^{\frac{1}{\alpha}}\right)N_{\epsilon}\left(r^{\frac{1}{\alpha}}\right)}{r}\right\|&\leq\sup_{r<s}\left\|C_{\epsilon}\left(r^{\frac{1}{\alpha}}\right)\right\|\sup_{r<s}\left\|N_{\epsilon}\left(r^{\frac{1}{\alpha}}\right)\right\|\\&=O_{\epsilon\rightarrow 0}\left(\epsilon^{a}\right),\end{aligned}$$

In some situations $s>0$. We have,

$$\sup_{r>s}\left\|\frac{N_{\epsilon}\left(r^{\frac{1}{\alpha}}\right)C_{\epsilon}\left(r^{\frac{1}{\alpha}}\right)}{r}\right\|=O_{\epsilon\rightarrow 0}\left(\epsilon^{a}\right),$$

For some $s>0$ and $a\in\mathbb{R}$. Let now $\epsilon\in]0,1[$ be fixed. We have

$$\begin{aligned}\left\|\frac{C_\epsilon(r^{\frac{1}{\alpha}})N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - C_\epsilon(0)H_\epsilon x\right\| &= \left\|C_\epsilon(r^{\frac{1}{\alpha}})\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - C_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x + \right.\\ &\left. C_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - C_\epsilon(0)H_\epsilon x\right\|\\ &\leq \left\|C_\epsilon(r^{\frac{1}{\alpha}})\right\|\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon x\right\| + \\ &\left\|C_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - C_\epsilon(0)H_\epsilon x\right\|.\end{aligned}$$

According to (1) and (2), in addition to the continuity of $r \mapsto C_\epsilon(r^{\frac{1}{\alpha}})(H_\epsilon x)$ at 0, the final expression becomes zero as $r \to 0$, we have:

$$\begin{aligned}\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})C_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon C_\epsilon(0)x\right\| &= \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}C_\epsilon(r^{\frac{1}{\alpha}})x - \frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}C_\epsilon(0)x + \right.\\ &\left.\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}C_\epsilon(0)x - H_\epsilon C_\epsilon(0)x\right\|\\ &\leq \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\right\|\left\|C_\epsilon(r^{\frac{1}{\alpha}})x - H_\epsilon(r)C_\epsilon(0)x\right\| + \\ &\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\left(C_\epsilon(0)x\right) - H_\epsilon\left(C_\epsilon(0)x\right)\right\|\end{aligned}$$

Assertions (1) and (2) require that the final expression goes to zero since $t \mapsto 0$. As a result, the proposition is proven in both circumstances. □

Definition 4.3. *The Colombenu type algebra define by:*

$$G(\mathbb{R}^+, C(X)) = E_{M,\alpha}(\mathbb{R}^+, C(X))/N_\alpha(\mathbb{R}^+, C(X))$$

Now we will define the concept of generelized conformable cosine family.

Definition 4.4. $C^\alpha = [(C_\epsilon^\alpha)]$ *with* $C_\epsilon \in E_{M,\alpha}(\mathbb{R}^+, C(X))$ *say the generalized conformable cosine family if:*

1. $C^\alpha(0) = Id$
2. $C^\alpha\left((r+1)^{\frac{1}{\alpha}}\right) + C^\alpha\left((r-1)^{\frac{1}{\alpha}}\right) = 2C^\alpha\left(r^{\frac{1}{\alpha}}\right)C^\alpha\left(r^{\frac{1}{\alpha}}\right)$
3. *The mapping* $r \to C^\alpha(r)x$ *is a continuous mapping for each* $x \in \bar{X}$.

If $C^\alpha(r), r \in \mathbb{R}$ *is a strongly continuous conformable cosine family in* $\bar{X}$*, then:* $S^\alpha(r), r \in \mathbb{R}$ *is the one parameter family of operators in* $\bar{X}$ *defined by*

$$S^\alpha(r) = \int_0^r C^\alpha(\tau)d\tau.$$

Exemple 4.1. *Let A be a bounded linear operasor on X. Define* $C^\alpha(r) = \frac{e^{2r\alpha}+e^{-2r\alpha}}{2}$*. Then* $T(r)_r \geq 0$ *is a* $\frac{1}{2}$ *semigroup. Indeed:*
1. $C^\alpha(0) = 1$.
3. The conrinuiry is clear.

Proposition 4.2. *The family* $\{C^\alpha(r), r \in \mathbb{R}\}$ *is a srongly conformable cosine family if only if* $\left\{C(r) = C^\alpha(r)\left(r^{\frac{1}{\alpha}}\right), t \in \mathbb{R}\right\}$ *is a srongly conrinuous conformable cosine family.*

Proof. 1. It is clear that $C(0) = I$.
2. For all $s, t \in \mathbb{R}$, we have

$$
\begin{aligned}
C(s+s) + C(s-s) &= C^{\alpha}(t+s)\left(s^{\frac{1}{\alpha}}\right) + C^{\alpha}(t-s)\left(s^{\frac{1}{\alpha}}\right)\\
&= 2C^{\alpha}(t)\left(s^{\frac{1}{\alpha}}\right)C^{\alpha}(s)\left(s^{\frac{1}{\alpha}}\right)\\
&= 2C(s)C(s)
\end{aligned}
$$

3.Further the continuity of $r \to C_{\epsilon}^{\alpha}\left(r^{\frac{1}{\alpha}}\right)y$ and the continuity of $r \to r^{\alpha}$ implies that $r \to C(r)y$ is continuous.

It is sufficient to mention that for the necessary requirement $C^{\alpha} = C^{\alpha}(r)$, if $\{C^{\alpha}(r), r \in \mathbb{R}\}$ is a strongly continuous conformable cosine family in $\bar{X}$, then $\{S^{\alpha}(r), r \in \mathbb{R}\}$ is the one parameter family of operators in $\bar{X}$ defined by

$$S^{\alpha}(r)y = (IC^{\alpha})(r)y, \quad \forall r \in \mathbb{R}, y \in X.$$

□

Remark 4.1. *As the previous proposision $\{S^{\alpha}(r), r \in \mathbb{R}\}$ is a conformable sine family iff $\left\{S(r) = S^{\alpha}\left(r^{\frac{1}{\alpha}}\right), r \in \mathbb{R}\right\}$ is conformable sine family.*

Proposition 4.3. *Let $\{C^{\alpha}(r), r \in \mathbb{R}\}$ be a strongly continuous conformable cosine family in $\bar{X}$. The following statements are correct:*
1. $C^{\alpha}(r) = C^{\alpha}(-r) \quad \forall r \in \mathbb{R}$
2. $C^{\alpha}(r), S^{\alpha}(r), C^{\alpha}(s)$, and $S^{\alpha}(s)$ commute for all $r, s \in \mathbb{R}$
3. $S^{\alpha}(r)y$ is continuous in r on $\mathbb{R}$ for each fixed $y \in X$
4. $S^{\alpha}(r+s) + S^{\alpha}(r-s) = 2S^{\alpha}(r)C^{\alpha}(s)$ for all $r, s \in \mathbb{R}$
5. $S^{\alpha}(r+s) = S^{\alpha}(r)C^{\alpha}(s) + S^{\alpha}(s)C^{\alpha}(r)$ for all $r, s \in \mathbb{R}$
6.$S^{\alpha}(t) = -S^{\alpha}(-t)$ for all $t \in \mathbb{R}$
7. There exist constant $M > 1$ and $\omega \geq 0$ such that $C^{\alpha}(r) \leq Me^{\omega^{\alpha}}$ for all $r \in \mathbb{R}$ and

$$||S^{\alpha}(r_1) - S^{\alpha}(r_2)|| \leq \frac{M}{\omega}\left(e^{\omega_1^{\alpha}} - e^{\omega_2^{\alpha}}\right)$$

Proof. The proposition 1-6 are consequence of the proposition 3.
For 7 , we have

$$
\begin{aligned}
||S^{\alpha}(r_1) - S^{\alpha}(r_2)|| &= \int_{t_2}^{t_1} \frac{C^{\alpha}(s)}{s^{1-\alpha}} ds\\
&\leq M \int_{t_2}^{l_1} \frac{e^{\omega s^{\alpha}}}{s^{1-\alpha}} ds = \frac{M}{\omega}\left[e^{\omega s^{\alpha}}\right]_{t_2}^{t_1}
\end{aligned}
$$

□

Definition 4.5. *The conformable infinitesimal generator of a strongly continuous conformable cosine families $C^{\alpha}(r), r \in \mathbb{R}$ is the operator $A : X \to X$ defined by*

$$Ax = \lim_{r \to 0} D^{(\alpha)} C^{\alpha}(r)$$

$$D(A) = \left\{y, r \to D^{(\alpha)} C^{\alpha}(r)y, \text{ is continuous in } r\right\}$$

Lemma 4.2.

$$C(r) = \lim_{\alpha \to 2^{+}} C^{\alpha}(r) \text{ is a cosine family}$$

Proof. It suffice to note that $C^{\alpha}\left(r^{\frac{1}{\alpha}}\right)$ is a cosine families, $r \to r^{\frac{1}{\alpha}}$ is continuous. □

Proposition 4.4. *Let $C^{\alpha}(r), r \in \mathbb{R}$, be a srongly continuous conformable cosine family in $\bar{X}$ with conformable infinitesimal generator A. Then,*
1. $D(A)$ is dense in X and A is a closed operasor in $\bar{X}$.
2. if $x \in \bar{X}$ and $r, s \in \mathbb{R}$, then $z = \int_r^5 \frac{S^{\alpha}(u)}{u^{1-\alpha}} x du \in D(A)$ and $Az = C^{\alpha}(s)x - C^{\alpha}(r)x$
3. if $x \in \bar{X}$, then $S^{\alpha}(t)x \in \bar{X}$
4. if $x \in \bar{X}$, shen $S^{\alpha}(t)x \in D(A)$ and $\left(D^{(\alpha)}C^{\alpha}\right)(t)x = AS^{\alpha}(t)x$
5 if $x \in D(A)$, then $C^{\alpha}(t)x \in D(A)$ and $D^{(\alpha)}C^{\alpha}(t)x = AC^{\alpha}(t)x = C^{\alpha}(t)Ax$
6. if $x \in D(A)$, then $\lim_{x \to 0} AS^{\alpha}(x)x = 0$
7 if $x \in D(A)$, then $S^{\alpha}(t)x \in D(A)$ and $D^{(\alpha)}S^{\alpha}(t)x = AS^{\alpha}(t)x$
8. if $x \in D(A)$, then $S^{\alpha}(t)x \in D(A)$ and $AS^{\alpha}(t)x = S^{\alpha}(t)Ax$
9. $C^{\alpha}(r+s) - C^{\alpha}(s-s) = 2AS^{\alpha}(t)S^{\alpha}(s)$ for all $s, t \in \mathbb{R}$.

Proof. For 1 it just to use the previous definition 17 and proposition 3.
For $2-9$ By change s by $s^{\frac{1}{\alpha}}$ and t by $t^{\frac{1}{\alpha}}$ and use proposition 2.2 in [21]. □

5. Existence and Uniqueness of the Solution in colombeau algebra $\mathcal{G}$

we consider the following problem :

$$\begin{cases} D^{(\alpha)}f(t,y) + Af(t,y) = F(t, f(t,y)) \quad y \in \mathbb{R}, \quad t \geq 0 \\ f(0,y) = u_0(y), \quad D^{(\alpha)}f(0,y) = v_0(y) \end{cases} \tag{5.1}$$

with $u_0(y), v_0(y) \in D'(\mathbb{R}^n)$. Now we will transform the problem (7) in the Colombeau algebra using the first section.

$$\begin{cases} D^{(\alpha)}f_{\epsilon}(t,y) + A_{\epsilon}f_{\epsilon}(t,y) = F_{\epsilon}(t, f_{\epsilon}(t,y)) \quad y \in \mathbb{R}, \quad t \geq 0 \\ f_{\epsilon}(0,y) = u_{0,\epsilon}(x), \quad D^{(\alpha)}f_{\epsilon}(0,y) = v_{0,\epsilon}(y) \end{cases} \tag{5.2}$$

with $1 < \alpha < 2$, $u_{0,\epsilon}(y), v_{0,\epsilon}(y)$ are regularized of $a_0(x)$ and $b_0(x)$ respectively and by definition 18 $A = [(A_{\varepsilon})]$ is the infinitesimal generator of generalized conformable cosine family $C = \left[(C_{\epsilon}^{\alpha})_{\epsilon}\right]$.

The folowing definition is the definition of mild solution.

Definition 5.1. *A funcrion $f_{\epsilon} \quad : [0, \infty) \to X$ is a mild soluion of (8) if*
1. f_{ϵ} is continuous differential on $[0, \infty)$.
2. f_{ϵ} is continuously α-differeniable on $(0, \infty)$.
3. $f_{\epsilon}(r) \in D(A)$ for $r > 0$.
4. $f_{\epsilon}(s) = C_{\epsilon}^{\alpha}(s)u_{0,\epsilon} + S_{\epsilon}^{\alpha}(s)v_{1,\epsilon} + \int_0^t \frac{s^{\alpha}(t-s)F(s, f_{\epsilon}(s))}{s^{2-\alpha}} ds$.

Definition 5.2. *An element $F \in \mathcal{G}[\mathbb{R}^n]$ is L^{∞} logarithmic type if it has a representative $(F_{\epsilon})_{\epsilon} \in \mathcal{E}_M[\mathbb{R}^n]$ such that*

$$||F_{\epsilon}||_{L^{\infty}(\mathbb{R}^n)} = O(\log(\epsilon)) \quad as \quad \epsilon \to 0$$

Theorem 5.1. *Let ∇F_{ϵ} is L^{∞} log-type and the conformable generalized sine family $S_{\epsilon} = \left[(S_{\epsilon}^{\alpha})_{\epsilon}\right]$ is the associated of the conformable generalized cosine family $C = \left[(C_{\epsilon}^{\alpha})_{\epsilon}\right]$ verify the properties of the previous section. Then the problem (8) has a unique solution in $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$.*

Proof. **Existence.**

The integral solution of the problem 8 is:

$$f_{\epsilon}(t,y) = C_{\epsilon}^{\alpha}(t)u_{0,\epsilon}(y) + S_{\epsilon}^{\alpha}(t)v_{0,\epsilon}(y) + \int_0^t s^{\alpha-2}S_{\epsilon}^{\alpha}(t-s)F_{\epsilon}(s, f_{\epsilon}(s))ds$$

Which implies that:

$$\begin{aligned}\left\|f_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq \left\|C_\epsilon^\alpha(t)\right\| \left\|u_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \left\|S_\epsilon^\alpha(t)\right\| \left\|v_{0,\epsilon}(x)\right\|_{L^\infty(\mathbb{R}^n)} \\ + \int_0^t s^{\alpha-2} \|S_\epsilon^\alpha(t)\| \left\|F_\epsilon\left(s, f_\epsilon(s,.)\right)\right\|_{L^\infty(\mathbb{R}^n)} ds,\end{aligned}$$

Then:

$$\begin{aligned}\left\|f_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \left\|C_\epsilon^\alpha(\tau)\right\| \left\|u_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \left\|v_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} \\ + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \int_0^t s^{\alpha-2} \left\|F_\epsilon\left(s, f_\epsilon(s,.)\right)\right\|_{L^\infty(\mathbb{R}^n)} ds.\end{aligned}$$

The first approximation of F_ϵ is

$$F_\epsilon\left(s, f_\epsilon(s,.)\right) = F_\epsilon(s,0) + \nabla F_\epsilon f_\epsilon(s,.) + N_\epsilon(s)$$

with $N_\epsilon \in \mathcal{N}\left(\mathbb{R}^+\right)$

Then

$$\begin{aligned}\left\|f_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \left\|C_\epsilon^\alpha(\tau)\right\| \left\|u_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \left\|v_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha\right\| \int_0^t s^{\alpha-2} \|F_\epsilon(s,0)\| ds \\ + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|\nabla F_\epsilon\| \int_0^t s^{\alpha-2} \|f_\epsilon(s,.)\|_{L^\infty(\mathbb{R}^n)} ds \\ + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \int_0^t s^{\alpha-2} N_\epsilon(s) ds\end{aligned}$$

We get

$$\begin{aligned}&\left\|f_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \left\|C_\epsilon^\alpha(\tau)\right\| \left\|u_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ &+ \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha\| \|v_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|F_\epsilon(\tau,0)\| \\ &+ \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|\nabla F_\epsilon\| \int_0^t s^{\alpha-2} \|f_\epsilon(s,.)\|_{L^\infty(\mathbb{R}^n)} ds \\ &+ \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(\tau)\|\end{aligned}$$

So,

$$\begin{aligned}\left\|f_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \|C_\epsilon(\tau)\| \left\|u_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \left\|v_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|F_\epsilon(\tau,0)\| \\ + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(\tau)\| \\ + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \, \|\nabla F_\epsilon\| \int_0^t s^{\alpha-2} \|f_\epsilon(s,.)\|_{L^\infty(\mathbb{R}^n)} ds.\end{aligned}$$

By the Granwall's inequality:

$$\begin{aligned}\left\|f_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq & \left(\sup_{\tau\in[0,T]}\left\|C_\epsilon^\alpha(\tau)\right\|\left\|u_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)}\right.\\ &+\sup_{\tau\in[0,T]}\left\|S_\epsilon^\alpha(\tau)\right\|\left\|v_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)}\\ &+\frac{T^{\alpha-1}}{\alpha-1}\sup_{\tau\in[0,T]}\|S_\epsilon^\alpha(\tau)\|\sup_{\tau\in[0,T]}\|F_\epsilon(\tau,0)\|\\ &\left.+\frac{T^{\alpha-1}}{\alpha-1}\sup_{\tau\in[0,T]}\|S_\epsilon^\alpha(\tau)\|\sup_{\tau\in[0,T]}\|N_\epsilon\|\right)\\ &\times\exp\left(\frac{T^{\alpha-1}}{\alpha-1}\sup_{\tau\in[0,T]}\|S_\epsilon^\alpha(\tau)\|\|\nabla F_\epsilon\|\right).\end{aligned}$$

Since $C_\epsilon^\alpha \in G(\mathbb{R}^+, \mathcal{C}(X)), S_\epsilon^\alpha \in G([0,+\infty[, \mathcal{C}(X)), u_{0,\epsilon} \in \mathcal{G}(\mathbb{R}^n), v_{0,\epsilon} \in \mathcal{G}(\mathbb{R}^n)$ $(N_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^+)$ and ∇F_ϵ is $L^\infty-$ logtype there exist $M \in \mathbb{N}$ such that
$\sup_{t\in[0,T]}\left\|f_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}\left(\epsilon^{-M}\right), \quad \epsilon \to 0$
Then

$$f_\epsilon \in \mathcal{G}([0,+\infty), \mathbb{R}^n).$$

Uniqueness.

Let's say there are two solutions $f_{1,\epsilon}(t,.)$, $f_{2,\epsilon}(t,.)$ to problem (8), consequently :

$$\begin{cases} D^{(\alpha)}f_{1,\epsilon}(t,y) - A_\epsilon f_{1,\epsilon}(t,y) - D^{(\alpha)}f_{2,\epsilon}(t,y) + A_\epsilon f_{2,\epsilon}(t,y) \\ \qquad = F_\epsilon(t, f_{1,\epsilon}(t,y)) - F_\epsilon(t, f_{2,\epsilon}(t,y)) \\ \qquad y \in \mathbb{R}^n, \quad t \geq 0 \\ f_{1,\epsilon}(0,y) - f_{2,\epsilon}(0,y) = N_{0,\epsilon}(y) \\ D^{(\alpha)}f_{1,\epsilon}(0,y) - D^{(\alpha)}f_{2,\epsilon}(0,y) = \bar{N}_{0,\epsilon}(y) \end{cases} \tag{5.3}$$

Then:

$$\begin{cases} D^{(\alpha)}\left(f_{1,\epsilon}(t,y) - f_{2,\epsilon}(t,y)\right) - A_\epsilon\left(f_{1,\epsilon}(t,y) + f_{2,\epsilon}(t,y)\right) = F_\epsilon(t, f_{1,\epsilon}(t,y)) \\ \qquad - F_\epsilon(t, f_{2,\epsilon}(t,y)) \\ \qquad y \in \mathbb{R}^n, \quad t \geq 0 \\ f_{1,\epsilon}(0,y) - f_{2,\epsilon}(0,y) = N_{0,\epsilon}(y) \\ D^{(\alpha)}f_{1,\epsilon}(0,y) - D^{(\alpha)}f_{2,\epsilon}(0,y) = \bar{N}_{0,\epsilon}(y) \end{cases} \tag{5.4}$$

With $(N_{0,\epsilon})_\epsilon, (\bar{N}_{0,\epsilon})_\epsilon \in \mathcal{N}(\mathbb{R}^+)$.
The integral solution of the equation (10) is:

$$\begin{aligned} f_{1,\epsilon}(t,y) - f_{2,\epsilon}(t,y) = & C_\epsilon^\alpha(t)N_{0,\epsilon}(y) + S_\epsilon^\alpha(t)\bar{N}_{0,\epsilon}(y) \\ & + \int_0^t s^{\alpha-2}S_\epsilon^\alpha(t)\left(F_\epsilon(s, f_{1,\epsilon}(s,x)) - F_\epsilon(s, f_{2,\epsilon}(s,x))\right)ds \end{aligned}$$

Then:
$$\begin{aligned}\left\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq & \left\|C_\epsilon^\alpha(t)\right\|\left\|N_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} \\ & + \left\|S_\epsilon^\alpha(t)\right\|\left\|\tilde{N}_{0,\epsilon}(\cdot)\right\|_{L^\infty(\mathbb{R}^n)} \\ & + \int_0^t s^{\alpha-2}\left\|S_\epsilon^\alpha(t)\right\|\left\|F_\epsilon(s, f_{1,\epsilon}(s,.)) - F_\epsilon(s, f_{2,\epsilon}(s,.))\right\|_{L^\infty(\mathbb{R}^n)} ds.\end{aligned}$$

Which implies that:

$$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \left\|C_\epsilon^\alpha(\tau)\right\| \left\|N_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \left\|\tilde{N}_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \int_0^t s^{\alpha-2} \left\|F_\epsilon\left(s, f_{1,\epsilon}(s,.)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,.)\right)\right\|_{L^\infty} ds.$$

The initial estimate of $F_\epsilon\left(s, f_{1,\epsilon}(s,.)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,.)\right)$ is provided by

$$F_\epsilon\left(s, f_{1,\epsilon}(s,.)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,.)\right) = \|\nabla F_\epsilon\|\left(f_{1,\epsilon}(s,.) - f_{2,\epsilon}(s,.)\right) + N_\epsilon(s),$$

With $(N_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^+)$.
So

$$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \left\|C_\epsilon^\alpha(\tau)\right\| \left\|N_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \left\|\tilde{N}_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \int_0^t s^{\alpha-1}\|\nabla F_\epsilon\|\|f_{1,\epsilon}(s,.) - f_{2,\epsilon}(s,.)\|_{L^\infty(\mathbb{R}^n)} ds + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \|N_\epsilon(s)\|$$

So,

$$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \left\|C_\epsilon^\alpha(\tau)\right\| \left\|N_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau\in[0,T]} \left\|S_\epsilon^\alpha(\tau)\right\| \left\|\widetilde{N}_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(s)\| + \sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\| \int_0^t s^{\alpha-2}\|\nabla F_\epsilon\| \left\|f_{1,\epsilon}(s,.) - f_{2,\epsilon}(s,.)\right\|_{L^\infty(\mathbb{R}^n)} ds$$

Using the Granwall's inequality:

$$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \Big(\sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\|\|N_{0,\epsilon}(.)\|_{L^\infty} + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha\|\|\widetilde{N}_{0,\epsilon}(.)\|_{L^\infty} + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(s)\|\Big) \times \exp\left(\frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\|\|\nabla F_\epsilon\|\right).$$

Since:
$C_\epsilon^\alpha \in G(\mathbb{R}^+, \mathcal{L}(X)), S_\epsilon^\alpha \in G(\mathbb{R}^+, \mathcal{L}(X)), (N_{0,\epsilon})_\epsilon, \quad (\widetilde{N}_{0,\epsilon})_\epsilon \in \mathcal{N}(\mathbb{R}^+)(N_\epsilon)_\epsilon$
$\in \mathcal{N}(\mathbb{R}^+)$ and ∇F is L^∞ - logtype and for every $q \in \mathbb{N}$ such that:

$$\sup_{t\in[0,T]} \left\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\right\|_{L^\infty} = O\left(\epsilon^q\right) \quad \epsilon \to 0$$

□

The author's declaration

1. Funding: Not applicable.
2. Informed Consent Statement: Not applicable.
3. Data Availability Statement: Not applicable.
4. Conflicts of Interest: The author declares no conflict of interest.

References

[1] M.I. Abbas, M. A. Ragusa, (2022). Nonlinear fractional differential inclusions with nonsingular Mittag-Leffler kernel, AIMS Mathematics, 7 (11), 20328–20340.

[2] M. S. Abdo, S. A.Idris, W.Albalawi, A. H. Abdel-Aty, M. Zakarya, E. E. Mahmoud, (2023). Qualitative Study on Solutions of Piecewise Nonlocal Implicit Fractional Differential Equations. Journal of Function Spaces.

[3] A.Benmerrous, L. S.Chadli, A.Moujahid, M. H.Elomari, and S.Melliani, (2022), Generalized Cosine Family, Journal of Elliptic and Parabolic Equations, 8(1), pp. 367-381.

[4] A.Benmerrous, L. S.Chadli, A.Moujahid, M. H.Elomari, and S.Melliani, (2023), Generalized Fractional Cosine Family, International Journal of Difference Equations (IJDE), 18(1), pp. 11-34.

[5] A.Benmerrous, L.S.Chadli, A.Moujahid, M.H.Elomari, and S.Melliani, (2023). Generalized solutions for time $\psi-$fractional heat equation. Filomat, 37, 9327–9337.

[6] A.Benmerrous, L. S.Chadli, A.Moujahid, M. H.Elomari, and S.Melliani, (2022), Generalized solution of Schrödinger equation with singular potential and initial data, Int. J. Nonlinear Anal. Appl, 13(1), pp. 3093-3101.

[7] A.Benmerrous, L.S.Chadli, A.Moujahid, M.H.Elomari, and S.Melliani, (2023). Solution of Non-homogeneous Wave Equation in Extended Colombeau Algebras. International Journal of Difference Equations (IJDE), 18(1), 107-118.

[8] A.Benmerrous, L. S.Chadli, A.Moujahid, M. H.Elomari, and S.Melliani, (2022), Solution of Schrödinger type Problem in Extended Colombeau Algebras, In 2022 8th International Conference on Optimization and Applications (ICOA), pp. 1-5.

[9] A.Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, International Journal of Mathematics and Mathematical Sciences, vol. 29, no. 9, pp. 531 536, (2002).

[10] L. S. Chadli, A. Benmerrous, A. Moujahid, M. H. Elomari, and S. Melliani, (2022), Generalized Solution of Transport Equation, In Recent Advances in Fuzzy Sets Theory, Fractional Calculus, Dynamic Systems and Optimization, pp. 101-111.

[11] J.F.Colombeau, Elementary Introduction to New Generalized Function, North Holland, Amsterdam, 1985.

[12] J.F.Colombeau, New Generalized Function and Multiplication of Distribution, North Holland, Amsterdam / New York / Oxford, 1984.

[13] H.O.Fattorini, Ordinary differential equations in linear topological spaces, 1090 2732, 1969.

[14] M.Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer, Geometric Theory of Generalized Functions with Applications to General Relativity, Mathematics and its Appli- cations 537, Kluwer Acad. Publ., Dordrecht,2001.

[15] R.Hermann, M. Oberguggenberger, Ordinary differential equations and generalized functions, in: Non linear Theory of Generalized Functions, Chapman and Hall, (1999), 85 98.

[16] R. Khalil, M. Al Horani, A.Yousef and Sababheh, M.: A new Definition Of Fractional Derivative, J. Comput. Appl. Math. 264. 65-70 (2014)

[17] S. Krim, A. Salim, M. Benchohra, (2023). Nonlinear contractions and Caputo tempered implicit fractional differential equations in b-metric spaces with infinite delay Filomat, 37 (22), 7491–7503.

[18] M.Oberguggenberger, Multiplication of Distributions and Applications to Partial Differential Equations, Pitman Research Notes in Mathematics, (1992).

[19] A.Pazy, Semigroups of linear operators and applications to partial differential equations, Bull. Amer. Math. Soc. (N.S.) 12 (1985).

[20] M.Sova, cosine operator functions, Warszawa, 47, 1966.

[21] C.C.Travis, G. F. Webb, Cosine families and abstract nonlinear second order differential equations, Acta Mathematica Hungarica, 1588 2632, 1978.

[22] C.L.Zhi , B.Fisher Several products of distributions on Rm. Proc R Soc Lond A426:425–439 (1989).