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Spectral Turán problem on Berge-$K_{2,t}$ hypergraphs

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Abstract. It is well-known that Turán problem is a classical problem in combinatorics, and the spectral Turán-type problem is the special form of Turán problem. Given a graph F, a hypergraph is called *Berge-F* if it can be obtained by replacing each edge in F by a hyperedge containing it. In this paper, we investigate the spectral Turán-type problem on linear r-uniform hypergraphs without Berge-$K_{2,t}$, and attain an upper bound of its spectral radius.

1. Introduction

It is well-known that Turán type problem is a classical problem in combinatorics, that is, for a given graph H (or a family graphs $\mathcal{H}$), what is the maximal size of an H-free ($\mathcal{H}$-free) graph of order n? The extremal value is called Turán number of $H(\mathcal{H})$ and denoted by $ex(n, H)(ex(n, \mathcal{H}))$. In 2013, Füredi and Simonovits [7] extended Turán type problem to the following general form.

(P1) For a class $\mathcal{G}$ of graphs, $G \in \mathcal{G}$ does not contain some subgraph H (or subgraph family $\mathcal{H}$), there are two parameters on $\mathcal{G}$ (for example, order and size), to maximize the second parameter under the condition that G is H-free ($\mathcal{H}$-free) and the first parameter is given.

In 1986, Brualdi and Solheid [2] proposed the following problem, which became one of the classic problems in spectral graph theory.

(P2) Given a set $\mathcal{G}$ of graphs, find $\min\{\lambda_1(G) : G \in \mathcal{G}\}$ and $\max\{\lambda_1(G) : G \in \mathcal{G}\}$, and characterize the graphs which achieve the minimum or maximum value.

If the first parameter is the order $n(G)$ of G and the second is spectral radius $\lambda_1(G)$ of G in **P1**, Nikiforov [21] proposed the following problem, which is named as **Spectral Turán-type problem**:

(P3) What is the maximal spectral radius $\lambda_1(G)$ of an H-free ($\mathcal{H}$-free) graph G of order $n(G)$?

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These problems attract many researchers interesting and there are many elegant results on these fields and they are still a very active research topics, for examples, please see [2, 4, 7, 19, 21] and references there in.

Hypergraphs model more general types of relations than graphs do. It is natural to generalize the above problems to hypergraphs. Indeed the theory of hypergraphs attract more and more researchers interesting. Since Lim [17] and Qi [22] independently introduced the notions of eigenvalue and eigenvector for tensors, and Cooper and Dutle [3] gave the definition of adjacent tensor of hypergraphs, many results on spectral radius of hypergraphs and on hypergraph Turán problems, which are similar to **P2** and **P1** respectively, are obtained, for examples, [1, 3, 6, 11, 12, 14, 15, 18, 23] and references therein. Recently, a handful of results on **Spectral Turán-type problem** relating to hypergraphs, which is similar to **P3**, are attained, such as [8–10, 13, 20] and references therein. In this work, we will continue to study on problems for hypergraphs spectral Turán-type problem.

The rest of this work is organized as following. In the next section, some necessary notions and terminologies are given. In section 3, we will study the spectral Turán-type problem on linear r-uniform hypergraphs without Berge-$K_{2,t}$.

2. Preliminaries

An order r dimension n complex tensor $\mathbb{A} = (a_{i_1 \ldots i_r})$ is a multi-array of entries $a_{i_1 \ldots i_r} \in \mathbb{C}$, where $i_j \in [n] = \{1, 2, \ldots, n\}$ and $j \in [r]$. If elements $a_{i_1 \ldots i_r}$ are invariant under any permutation of indices $i_1, \ldots, i_r$, $\mathbb{A}$ is said symmetric. If there exists a subset $\mathcal{I}$ with $\emptyset \subsetneq \mathcal{I} \subsetneq [n]$, satisfying $a_{i_1 i_2 \ldots i_r} = 0$ for all $i_1 \in \mathcal{I}$ and some $i_j \notin \mathcal{I}$ for $j \in \{2, 3, \ldots, r\}$, $\mathbb{A}$ is called a weakly reducible, otherwise we say $\mathbb{A}$ a weakly irreducible. If every element $a_{i_1 \ldots i_r} \geq 0$, $\mathbb{A}$ is called an nonnegative tensor.

For a vector $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{C}^n$, $\mathbb{A}x^{r-1}$ is an n-dimensional vector with its i-th component being

$$(\mathbb{A}x^{r-1})_i = \sum_{i_2, \cdots, i_r = 1}^{n} a_{i i_2 \cdots i_r} x_{i_2} \cdots x_{i_r}.$$

If there exists a $\lambda \in \mathbb{C}$ and $0 \neq x = (x_1, x_2, ..., x_n)^T \in \mathbb{C}^n$ satisfying $\mathbb{A}x^{r-1} = \lambda x^{[r-1]}$, where $(x^{[r-1]})_i = x_i^{r-1}$ for $i \in [n]$, λ is called an eigenvalue of $\mathbb{A}$ and x the eigenvector of $\mathbb{A}$ associated with λ [17, 22]. $\rho(\mathbb{A})(= max\{|\lambda| : \lambda$ is an eigenvalue of $\mathbb{A}\})$ is the spectral radius of $\mathbb{A}$.

A general hypergraph $H = (V(H), E(H))$ consists of a vertex set $V(H) = \{v_1, v_2, \cdots, v_n\}$ and a hyperedge (or edge for simplicity) set $E(H) = \{e_1, e_2, \cdots, e_m\}$, where $E(H) \subseteq P(V) \backslash \{\emptyset\}$ and $P(V)$ stands for the power set of V. Denote $r(H) = \max_{e \in E(H)} |e|$ (resp. $cr(H) = \min_{e \in E(H)} |e|$) be the rank (resp. co-rank) of H. If $r(H) = cr(H) = r$, H is called an r-uniform hypergraph, and it is an ordinary graph for $r(H) = cr(H) = 2$. A hypergraph H is called a linear hypergraph if there is at most one common vertex between any two edges. For a fixed vertex $u \in V(H)$, let

$$\begin{aligned} N_u &= \{v \in V(H) \setminus \{u\} \mid v, u \in e \in E(H)\}, \\ \overline{N_u} &= \{v \in V(H) \backslash (N_u \cup \{u\}) \mid v \in e \in E(H), e \cap N_u \neq \emptyset\}. \end{aligned}$$

Let $E_u = \{e \in E(H) | u \in e \in E(H)\}$, the degree of u denoted by $d(u)$ and $d(u) = |E_u|$. For two vertices u, v, let N_{uv} be the set of common neighbors of u and v. The codegree of u and v, denoted by $d(u, v)$, is the number of edges containing both u and v in H. Denote Δ, Δ_2 be the maximum degree, the maximum codegree of H, respectively.

For a graph $F = (V(F), E(F))$, a hypergraph H is called a Berge-F if it can be obtained by replacing each edge in F by a hyperedge containing it. Given a class of graphs $\mathcal{F}$, we say that a hypergraph H is Berge-$\mathcal{F}$ free if for every $F \in \mathcal{F}$, the hypergraph H does not contain a Berge-F as a subhypergraph. We call the maximum possible number $ex(n, \mathcal{F})$ of hyperedges in a Berge-$\mathcal{F}$ free hypergraph on n vertices as Turán number of Berge-$\mathcal{F}$.

In [16], F. Lazebnik and J. Verstraëte obtained an upper bound of Turán number on r-uniform hypergraphs H without cycles of length less than five. This result was improved by Ergemlidze, Győri and

Methuku in [5]. In [8], Gerbner, Methuku and Vizer attained an asymptotics for the Turán number of Berge-$K_{2,t}$. In [13], Hou, Chang and Cooper generalized these problems to spectral version of Turán-type problem and given an upper bound for spectral radius of linear hypergraphs without Berge-C_4, at the same time, they put forward a problem to study spectral version of Turán-type problem of linear hypergraphs without Berge-$K_{2,t}$. In this paper, we will deal with this problem.

The following lemma is a useful tool in our main result.

Lemma 2.1. *[13] Let H be a connected r-uniform linear hypergraph and ρ be the spectral radius of the adjacency tensor of H. Let u be the vertex with maximum eigenvector entry. Then $\rho^2 \leq \frac{1}{r-1}\sum_{v\in N_u} d(v)$.*

3. Spectral Turán-type problem on linear uniform hypergraphs without Berge $K_{2,t}$

Let $H = (V(H), E(H))$ be an r-uniform linear hypergraph and $X \subseteq V(H)$, let

$$\begin{aligned} E_s(X) &= \{e|e \in E(H) \text{ and } |e \cap X| = s\},\ \ e_s(X) = |E_s(X)|; \\ E_s^v(X) &= \{e|v \in e \in E(H) \text{ and } |e \cap X| = s\},\ \ e_s^v(X) = |E_s^v(X)|. \end{aligned}$$

Lemma 3.1. *For $r \geq 3$, let H be a r-uniform linear hypergraph without Berge-$K_{2,t}$. Then for any $v \in N_u$,*

$$\sum_{s=2}^{r} e_s^v(N_u) \leq (r-1)(t-1)+1.$$

Proof. In order to obtain a contradiction, we assume that there exists a vertex $v \in N_u$ such that the number of hyperedges in $\cup_{s=2}^{r} E_s^v(N_u)$ is at least $(r-1)(t-1)+2$. Since H is linear, it is easy to see that only one hyperedge, say h_1, in $\cup_{s=2}^{r} E_s^v(N_u)$ contains u, and let $h_2, \dots, h_l$ be the remaining hyperedges in $\cup_{s=2}^{r} E_s^v(N_u)$, where $l = \sum_{s=2}^{r} e_s^v(N_u)$. Then $l \geq (r-1)(t-1)+2 \geq t+1$.

For each hyperedge $h_i (2 \leq i \leq l)$, we can choose out a vertex $x_i \in (h_i \cap N_u) \setminus \{v\}$ since $|h_i \cap N_u| \geq 2$. Obviously, for any distinct $i, j \in [l] \setminus \{1\}$, $x_i \neq x_j$ and both are adjacent to u. It is easy to see that we can select a vertex set W consisting of $(r-1)(t-1)+1$ distinct vertices from $\{x_i | i \in [l] \setminus \{1\}\}$, without loss of generality, say $W = \{x_2, x_3, \dots, x_{(r-1)(t-1)+2}\}$. Furthermore we know that there exist at least t distinct hyperedges obtained from $W \cup \{u\}$, which are incident to u, without loss of generality, say $l_1^u, l_2^u, \cdots, l_t^u$. Let $y_i \in l_i^u \cap W$, it is obvious that y_i is adjacent to v for $i \in [t]$. Denote l_i^v be the hyperedge containing v and y_i. By the linearity of H, we know that $l_1^v, l_2^v, \cdots, l_t^v$ are distinct each other. Then the $2t$ hyperedges $l_1^u, l_2^u, \cdots, l_t^u$ and $l_1^v, l_2^v, \cdots, l_t^v$ form a Berge-$K_{2,t}$ in H, a contradiction. □

For $i \in [d(u)]$ and $j \in [r-1]$, let $h_i^u = \{u\} \cup \{u_{j,i} | j \in [r-1]\}$ be a hyperedge of H in the following

$$\begin{aligned} U_{u_{j,i}} &= \{x \in \overline{N_u} \mid x \in e \in E_1^{u_{j,i}}(N_u)\}, && (1) \\ U_i &= \cup_{j=1}^{r-1} U_{u_{j,i}}. && (2) \end{aligned}$$

Then $|U_{u_{j,i}}| = (r-1)|E_1^{u_{j,i}}(N_u)| = (r-1)e_1^{u_{j,i}}(N_u)$, $|N_u| = (r-1)d(u)$. Further by Lemma 3.1, we have

$$d(u_{j,i}) = e_1^{u_{j,i}}(N_u) + \sum_{s=2}^{r} e_s^{u_{j,i}}(N_u) \leq e_1^{u_{j,i}}(N_u) + (r-1)(t-1)+1.$$

Therefore,

$$\begin{aligned}
e_1^{u_{j,i}}(N_u) &\geq d(u_{j,i}) - (r-1)(t-1) - 1, \\
|U_{u_{j,i}}| &= (r-1)e_1^{u_{j,i}}(N_u) \geq (r-1)d(u_{j,i}) - (r-1)^2(t-1) - (r-1) \qquad (3) \\
\sum_{u_{j,i}\in N_u} |U_{u_{j,i}}| &\geq \sum_{u_{j,i}\in N_u} ((r-1)d(u_{j,i}) - (r-1)^2(t-1) - (r-1)) \\
&= \sum_{u_{j,i}\in N_u} (r-1)d(u_{j,i}) - (r-1)^2(t-1)|N_u| - (r-1)|N_u| \\
&= \sum_{u_{j,i}\in N_u} (r-1)d(u_{j,i}) - (r-1)^3(t-1)d(u) - (r-1)^2 d(u) \qquad (4)
\end{aligned}$$

Lemma 3.2. *For $r \geq 3$, let H be a r-uniform linear hypergraph without Berge-$K_{2,t}$ and $h_i^u = \{u\} \cup \{u_{j,i} | j \in [r-1]\}$ be its a hyperedge. For $i \in [d(u)]$, $j, k \in [r-1]$, let $u_{j,i}, u_{k,i} \in h_i^u$ be any two distinct vertices. Then $|N_{u_{j,i}u_{k,i}}| \leq t(r-1) - 1$.*

Proof. For simplicity, let $u_{j,i} = u_j$, $u_{k,i} = u_k$, then $N_{u_{j,i}u_{k,i}} = N_{u_j u_k}$. If $d(u_k) \leq t$ or $d(u_j) \leq t$, it is easy to that

$$|N_{u_j u_k}| \leq (t-1)(r-1) + (r-2) = t(r-1) - 1.$$

The equality holds only for the case that all vertices which are adjacent to u_k (resp. u_j) also belong to N_{u_j} (resp. N_{u_k}).

Now we only need to consider the case that $d(u_k) \geq t+1$ and $d(u_j) \geq t+1$. Suppose for the sake of a contradiction that $|N_{u_j u_k} \setminus \{h_i^u\}| \geq t(r-1)$. Then there are at least $(t-1)(r-1)+1$ vertices in $N_{u_j u_k} \setminus \{h_i^u\}$. Since H is r-uniform, there must exist t vertices $v_1, v_2, \cdots, v_t \in N_{u_j u_k} \setminus \{h_i^u\}$ such that the pairs $u_k v_1, \cdots, u_k v_t$ are contained in t distinct edges which are not the edge h_i^u.

(1.1). If there are two pairs $u_j v_p, u_j v_q$ contained in one edge incident to u_j, there must exist a vertex $v'_q \in N_{u_j u_k} \setminus \{h_i^u\}$ such that $u_j v_1, \cdots, u_j v'_q, u_j v_p(v_q), \cdots, u_j v_t$ in t distinct edges incident to u_j. Then the edges containing $u_j v_1, \cdots, u_j v'_q, u_j v_p(v_q), \cdots, u_j v_t$ and $u_k v_1, \cdots, u_k v'_q, u_j v_p(v_q), \cdots, u_k v_t$ form a Berge-$K_{2,t}$ in H, a contradiction.

(1.2). If there are three pairs $u_j v_p, u_j v_q, u_j v_m$ contained in one edge incident to u_j, there must exist two vertices $v'_q, v'_m \in N_{u_j u_k} \setminus \{h_i^u\}$ such that $u_j v_1, \cdots, u_j v_p(v_q, v_m), u_j v'_q, u_j v'_m, \cdots, u_j v_t$ in t distinct edges incident to u_j. Then the edges containing $u_j v_1, \cdots, u_j v_p(v_q, v_m), u_j v'_q, u_j v'_m \cdots, u_j v_t$ and $u_k v_1, \cdots, u_j v_p(v_q, v_m), u_k v'_q, u_j v'_M, \cdots, u_k v_t$ form a Berge-$K_{2,t}$ in H, a contradiction.

$\vdots$

(1.t). If there are t pairs $u_j v_1, \cdots, u_j v_t$ contained in one edge incident to u_j, there must exist $t-1$ vertices $v'_2, \cdots, v'_t \in N_{u_j u_k} \setminus \{h_i^u\}$ such that $u_j v_1, u_j v'_2, \cdots, \cdots, u_j v_t$ in t distinct edges incident to u_j. Then the edges containing $u_j v_1(v_2, \cdots, v_t), u_j v'_2, \cdots, \cdots, u_j v'_t$ and $u_k v_1(v_2, \cdots, v_t), u_k v'_2, \cdots, \cdots, u_k v'_t$ form a Berge-$K_{2,t}$ in H, a contradiction.

Otherwise the pairs $u_j v_1, \cdots, u_j v_t$ are contained in t different edges incident to u_j, then the $2t$ edges containing the pairs $u_k v_1, \cdots, u_k v_t, u_j v_1, \cdots, u_j v_t$ form a Berge-$K_{2,t}$ in H, a contradiction. So, for the case that $d(u_k) \geq t+1$ and $d(u_j) \geq t+1$,we have $|N_{u_j u_k}| \leq t(r-1) - 1$.

From the above discussion, we know that $|N_{u_{j,i}u_{k,i}}| \leq t(r-1) - 1$. □

Lemma 3.3. *For $r \geq 3$, let H be a r-uniform linear hypergraph without Berge-$K_{2,t}$ and $U_{u_{j,i}}$ be defined as (1). For $i \in [d(u)]$, we have*

$$\sum_{i=1}^{d(u)} \sum_{j=1}^{r-1} |U_{u_{j,i}}| \leq (t-1)(n-1-|N_u|) + \frac{(r-1)^2(r-2)(t-1)}{2} d(u).$$

Proof. Let $h_i^u = \{u\} \cup \{u_{j,i} | j \in [r-1]\}$ be a hyperdege in H, and $u_{j,i}, u_{k,i} \in h_i^u$ be any two distinct vertices. By (1) and (2), it is easy to see that

$$|U_i| = |\bigcup_{j=1}^{r-1} U_{u_{j,i}}| \geq \sum_{j=1}^{r-1} |U_{u_{j,i}}| - \sum_{1\leq j<k\leq r-1} |U_{u_{j,i}} \bigcap U_{u_{k,i}}|. \tag{5}$$

Further by Lemma 3.2, we get

$$\begin{aligned} |U_{u_{j,i}} \bigcap U_{u_{k,i}}| &\leq |Nu_j u_k \setminus (N_u \cup \{u\})| \\ &\leq |N_{u_j u_k} \setminus (h_i^u \setminus \{u_{j,i}, u_{k,i}\})| \\ &\leq t(r-1) - 1 - (r-2) = (t-1)(r-1). \end{aligned}$$

By (5), we have

$$|U_i| \geq \sum_{j=1}^{r-1} |U_{u_{j,i}}| - \binom{r-1}{2}(t-1)(r-1) = \sum_{j=1}^{r-1} |U_{u_{j,i}}| - \frac{(r-1)^2(r-2)(t-1)}{2}.$$

Then

$$\begin{aligned} \sum_{j=1}^{r-1} |U_{u_{j,i}}| &\leq |U_i| + \frac{(r-1)^2(r-2)(t-1)}{2}, \\ \sum_{i=1}^{d(u)} \sum_{j=1}^{r-1} |U_{u_{j,i}}| &\leq \sum_{i=1}^{d(u)} |U_i| + \sum_{i=1}^{d(u)} \frac{(r-1)^2(r-2)(t-1)}{2} \\ &= \sum_{i=1}^{d(u)} |U_i| + \frac{(r-1)^2(r-2)(t-1)}{2} d(u). \end{aligned} \tag{6}$$

In order to attain our desirable result, now we only want to prove the following inequality.

$$\sum_{i=1}^{d(u)} |U_i| \leq (t-1)(n-1-|N_u|). \tag{7}$$

Note that $h_i^u = \{u\} \cup \{u_{j,i} | j \in [r-1]\}$ be the hyperedge associated to U_i for $i \in [d(u)]$.

If $t > d(u)$, we have $d(u) \leq t-1$, and for any $v \in \overline{N_u}$ it belongs to at most $d(u)$ sets $U_i (i \in [d(u)])$. Then

$$\sum_{i=1}^{d(u)} |U_i| \leq d(u)(n-1-|N_u|) \leq (t-1)(n-1-|N_u|).$$

If $t \leq d(u)$, we can claim that for any $v \in \overline{N_u}$ it belongs to at most $t-1$ sets $U_i (i \in [d(u)])$. Otherwise, there exists a vertex $v_0 \in \overline{N_u}$, which is contained in t sets, without loss of generality, say $U_1, U_2, \ldots, U_t$. For each U_j, $j \in [t]$, we can select a hyperedge $l_j^{v_0}$ containing v_0. Then the $2t$ hyperedges $l_1^v, l_2^v, \cdots, l_t^v$ and $h_1^u, h_2^u, \cdots, h_t^u$ form a Berge-$K_{2,t}$ in H, a contradiction. Further it is easy to see that (7) holds.

By (6) and (7), we have

$$\sum_{i=1}^{d(u)} \sum_{j=1}^{r-1} |U_{u_{j,i}}| \leq (t-1)(n-1-|N_u|) + \frac{(r-1)^2(r-2)(t-1)}{2} d(u).$$

□

Lemma 3.4. *For $r \geq 3$, let H be a r-uniform linear hypergraph without Berge-$K_{2,t}$. Then for any vertex $u \in V(H)$,*

$$\sum_{u_{j,i} \in N_u} (r-1)d(u_{j,i}) \quad \leq \quad (t-1)(n-1) + \frac{(n-1)[3(t-1)r^2 - (7t-9)r + 2(t-2)]}{2}.$$

Proof. From the linearity of H, it has $d(u) \leq \frac{n-1}{r-1}$. Note that $|N_u| = (r-1)d(u)$, by (4) and Lemma 3.3, we have

$$\begin{aligned}
& \sum_{u_{j,i} \in N_u} (r-1)d(u_{j,i}) \\
\leq \ & \sum_{u_{j,i} \in N_u} |U_{u_{j,i}}| + (r-1)^3(t-1)d(u) + (r-1)^2 d(u) \\
= \ & \sum_{i=1}^{d(u)} \sum_{j=1}^{r-1} |U_{u_{j,i}}| + (r-1)^3(t-1)d(u) + (r-1)^2 d(u) \\
\leq \ & (t-1)(n-1-|N_u|) + \frac{(r-1)^2(r-2)(t-1)}{2} d(u) + (r-1)^3(t-1)d(u) + (r-1)^2 d(u) \\
= \ & (t-1)(n-1) + \frac{(r-1)^2(r-2)(t-1)}{2} d(u) + (r-t)(r-1)d(u) + (r-1)^3(t-1)d(u) \\
= \ & (t-1)(n-1) + \frac{(r-1)[3(t-1)r^2 - (7t-9)r + 2(t-2)]}{2} d(u) \\
\leq \ & (t-1)(n-1) + \frac{(n-1)[3(t-1)r^2 - (7t-9)r + 2(t-2)]}{2}.
\end{aligned}$$

□

Theorem 3.5. *Let $\mathcal{H}$ be the set of r-uniform linear hypergraphs without Berge-$K_{2,t}$, and ρ be the maximum spectral radius in $\mathcal{H}$. Then $\rho^2 \leq (n-1)[\frac{3(t-1)}{2} - \frac{t-3}{2(r-1)}]$.*

Proof. By Lemma 2.1 and Lemma 3.4, for any vertex $v \in N_u$, we have

$$\begin{aligned}
\rho^2 & \leq \frac{1}{r-1} \sum_{v \in N_u} d(v) \\
& = \frac{\sum_{v \in N_u}(r-1)d(v)}{(r-1)^2} \\
& \leq \frac{(t-1)(n-1)}{(r-1)^2} + \frac{(n-1)[3(t-1)r^2 - (7t-9)r + 2(t-2)]}{2(r-1)^2} \\
& = (n-1)\frac{[3(t-1)r^2 - (7t-9)r + 2(2t-3)]}{2(r-1)^2} \\
& = (n-1)\frac{(r-1)[3(t-1)r - 2(2t-3)]}{2(r-1)^2} \\
& = (n-1)\frac{3(t-1)r - 2(2t-3)}{2(r-1)} \\
& = (n-1)[\frac{3(t-1)}{2} - \frac{t-3}{2(r-1)}].
\end{aligned}$$

□

Note that $K_{2,t} = C_4$ for $t = 2$, then we have the following corollary. Obviously, this corollary improve the result of Theorem 6 in [13].

Corollary 3.6. *Let $\mathcal{H}$ be the set of linear r-uniform hypergraphs of order n without Berge-C_4, and ρ be the maximum spectral radius in $\mathcal{H}$. Then $\rho^2 \leq (n-1)\left(\frac{3}{2} + \frac{1}{2(r-1)}\right)$.*

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