Filomat 38:9 (2024), 3275–3294
https://doi.org/10.2298/FIL2409275H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inequalities with parameters for twice-differentiable functions involving Riemann–Liouville fractional integrals

Fatih Hezenci[a]

[a] *Department of Mathematics, Faculty of Science and Arts, Duzce University, Turkiye*

Abstract. In this paper, it is given an equality for twice-differentiable functions whose second derivatives in absolute value are convex. By using this equality, it is established several left and right Hermite–Hadamard type inequalities and Simpson type inequalities for the case of Riemann–Liouville fractional integral. Namely, midpoint, trapezoid and also Simpson type inequalities are obtained for Riemann–Liouville fractional integral by using special cases of main results.

1. Introduction

The theory of inequalities has an important place in nonlinear analysis. Over the last two decade, one of the most famous inequalities for convex functions is Hermite–Hadamard inequality because of its rich geometrical significance and applications (see Ref. [11] and p. 137 of Ref. [28]). Thus, remarkable number of mathematicians have considered the Hermite–Hadamard-type inequalities and related these inequalities such as trapezoid, midpoint, and Simpson type inequalities.

The Hermite–Hadamard-type inequalities are investigated firstly by C. Hermite and J. Hadamard for convex functions. Let $f : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then, the following inequalities

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

are valid for the case of all convex functions, is well-known in the literature as the Hermite–Hadamard inequality. If f is concave, then both inequalities in (1) hold to the reverse direction.

In recent years, a great number of studies have been written about trapezoid and midpoint type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1), respectively. Dragomir and Agarwal first obtained trapezoid inequalities for convex functions in [10] and Kirmacı first established midpoint inequalities for convex functions in the paper [21]. In the paper [34], it was generalized the inequalities (1) for fractional integrals and it was also presented some corresponding

2020 *Mathematics Subject Classification*. Primary 26B25, 26D10; Secondary 26D15.
Keywords. Hermite–Hadamard inequality, Simpson inequality, Fractional integral operators, Convex function, Twice differentiable function.
Received: 27 May 2023; Accepted: 27 September 2023
Communicated by Miodrag Spalević
Email address: fatihezenci@gmail.com (Fatih Hezenci)

trapezoid type inequalities. In addition to this, some fractional midpoint type inequalities for the case of convex functions were investigated in the paper [18]. On the other hand, Dragomir established the Hermite–Hadamard inequalities for co–ordinated convex functions in the paper [12]. Moreover, the midpoint and trapezoid type inequalities for co–ordinated convex functions were established in the papers [22] and [32], respectively. Furthermore, Tunç et al. proved some fractional midpoint type inequalities for co–ordinated convex functions in [40]. In [38], Sarikaya and Ertuğral first introduced new fractional integrals, are called generalized fractional integrals. The authors also proved several midpoint and trapezoid type inequalities for generalized fractional integrals. For results connected with these type of inequalities one can see Refs. [6, 8, 9, 20, 25] and the references therein.

In the literature, considerable number of researchers have focused on twice differentiable functions to obtain many important inequalities. For instance, Barani et al. established inequalities for twice-differentiable convex functions, which are connected with the Hermite–Hadamard inequalities in [4, 5]. In the paper [26], it was proved the Hermite-Hadamard type inequalities for functions whose second derivatives absolute values are $P-$convex. Some new generalized fractional integral inequalities of midpoint and trapezoid type for twice differentiable convex functions are obtained in [24]. Moreover, the authors established some new inequalities of the Simpson and the Hermite–Hadamard-type for functions whose absolute values of derivatives are convex in [30]. In addition, J. Park investigated new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for functions whose second derivatives in absolute value at certain powers are convex and quasi–convex functions in [27]. In addition, it is proved some midpoint and trapezoid type inequalities for functions whose second derivatives in absolute value are convex in [7]. For more information about these type of inequalities involving twice-differentiable functions, we refer to[2, 16, 31, 35, 41, 42].

We will present mathematical preliminaries of fractional calculus theory which are used further in the following of this paper.

Definition 1.1. *Let us consider $f \in L_1[a,b]$. The Riemann–Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are described as*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and its defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du.$$

Let us note that $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

Remark 1.2. *For $\alpha = 1$ in Definition 1.1, the fractional integral becomes to the classical integral.*

The fractional integral inequalities and applications by using Riemann–Liouville fractional integral have been investigated by many authors. For example, Sarıkaya and Yıldırım investigated the the Hermite–Hadamard-type inequality for the Riemann–Liouville fractional integrals in [36]. In the paper [29], some new integral inequalities of Hermite–Hadamard and Simpson type using $s-$ $(\alpha, m)-$convex function by Riemann–Liouville fractional integrals in order to generalize Hermite–Hadamard-type inequalities. Furthermore, Tomar et al. proved for several new Hermite–Hadamard-type of inequalities for Riemann–Liouville fractional integrals on twice-differentiable functions in [39]. The reader is referred to [13, 14, 23, 33] and the references therein for more information and unexplained subjects about several properties of Riemann–Liouville fractional integrals. While many mathematicians have studied the

Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals, some authors have also investigated the Hermite–Hadamard inequalities for other type of fractional integrals such as Hadamard fractional integrals, Conformable fractional integrals, k–fractional integral, etc. There have been a great number of research papers written on these subjects, (see, [1, 3, 19, 37]) and the references therein.

It is remarkable that Sarikaya et al. [34] first give the following interesting integral inequalities of the Hermite–Hadamard-type involving Riemann–Liouville fractional integrals.

Theorem 1.3 (See [34]). *Let $f : [a, b] \to \mathbb{R}$ denote a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. Let f be a convex function on $[a, b]$. Then, the following inequalities for fractional integrals*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{2}$$

are valid with $\alpha > 0$.

Remark 1.4. *If we choose $\alpha = 1$, then inequality (2) reduces to inequality (1).*

Iqbal et. al. [18] establish new upper bound for the left-hand side of (2) for convex functions is proposed in the following theorem.

Theorem 1.5 (See [18]). *Let us consider that $f : [a, b] \to \mathbb{R}$ is differentiable function on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following midpoint type inequality for Riemann–Liouville fractional integrals*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]\right| \le \frac{b-a}{2^{\alpha+1}(\alpha+1)}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]$$

is valid with $0 < \alpha \le 1$.

Theorem 1.6 (See [34]). *Let us note that $f : [a, b] \to \mathbb{R}$ is differentiable function on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following trapezoid type inequality for Riemann–Liouville fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]\right| \le \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

The main purpose of this paper is to establish left and right Hermite–Hadamard-type inequalities and also Simpson type inequalities for Riemann–Liouville fractional integral. The entire structure of the study takes the form of four sections including introduction. In Sect. 2, an identity for twice-differentiable functions is investigated. By utilizing this equality, parameterized Hermite–Hadamard-types inequalities and Simpson type inequalities for Riemann–Liouville fractional integral are proved. With the help of special choices, midpoint, trapezoid and simpson type inequalities are given in subsections of Sect. 3, respectively. Moreover, some remarks and corollaries are presented. Some conclusions and further directions of research are discussed in Sect. 4.

2. Main results

Let's start with the following lemma, which will form the basic structure of our article to obtain our main results.

Lemma 2.1. *Let us note that $f : [a, b] \to \mathbb{R}$ is an absolutely continuous mapping (a, b) such that $f'' \in L_1([a, b])$. Then, the following equality holds:*

$$\frac{(2^{\alpha}(\alpha+1) + \beta_1 - \lambda_1)}{2^{\alpha}(\alpha+1)}\left(\frac{f(a)+f(b)}{2}\right) + \frac{\lambda_1 - \beta_1}{2^{\alpha}(\alpha+1)} f\left(\frac{a+b}{2}\right) \tag{3}$$

$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]$$

$$=\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left[I_2+I_3-I_1-I_4\right],$$

where

$$\begin{cases} I_1=\int\limits_0^1\left((1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right)f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt, \\ I_2=\int\limits_0^1\left(\lambda_1 t+\lambda_2-(1-t)^{\alpha+1}\right)f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt, \\ I_3=\int\limits_0^1\left(\beta_1 t+\beta_2-(2-t)^{\alpha+1}\right)f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt, \\ I_4=\int\limits_0^1\left((2-t)^{\alpha+1}-\beta_1 t-\beta_2\right)f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \end{cases}$$

and $\lambda_1+\lambda_2+\beta_2=2^{\alpha+1}$ with $\alpha>0$.

Proof. By using integration by parts, we obtain

(4)

$$I_1=\int\limits_0^1\left((1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right)f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$=\frac{2(\lambda_1+\lambda_2)}{b-a}f'(a)+\frac{2(1-\lambda_2)}{b-a}f'\left(\frac{a+b}{2}\right)$$

$$-\frac{2}{(b-a)}\int\limits_0^1\left(\lambda_1+(\alpha+1)(1-t)^{\alpha}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$=\frac{2(\lambda_1+\lambda_2)}{b-a}f'(a)+\frac{2(1-\lambda_2)}{b-a}f'\left(\frac{a+b}{2}\right)+\frac{4\lambda_1}{(b-a)^2}f(a)$$

$$-\frac{4(\alpha+1+\lambda_1)}{(b-a)^2}f\left(\frac{a+b}{2}\right)+\frac{4\alpha(\alpha+1)}{(b-a)^2}\int\limits_0^1(1-t)^{\alpha-1}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt.$$

By using the equation (4) and with the help of the change of the variable $x=\frac{1+t}{2}a+\frac{1-t}{2}b$ for $t\in[0,1]$, it can be rewritten as follows

(5)

$$I_1 = \frac{2(\lambda_1+\lambda_2)}{b-a} f'(a) + \frac{2(1-\lambda_2)}{b-a} f'\left(\frac{a+b}{2}\right) + \frac{4\lambda_1}{(b-a)^2} f(a)$$

$$- \frac{4(\alpha+1+\lambda_1)}{(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} (x-a)^{\alpha-1} f(x)\,dx.$$

Similarly, we have

$$I_2 = \frac{2(\lambda_1+\lambda_2)}{b-a} f'(b) + \frac{2(1-\lambda_2)}{b-a} f'\left(\frac{a+b}{2}\right) - \frac{4\lambda_1}{(b-a)^2} f(b)$$

$$+ \frac{4(\alpha+1+\lambda_1)}{(b-a)^2} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} (b-x)^{\alpha-1} f(x)\,dx, \tag{6}$$

$$I_3 = \frac{2(\beta_1+\beta_2-1)}{b-a} f'\left(\frac{a+b}{2}\right) + \frac{2\left(2^{\alpha+1}-\beta_2\right)}{b-a} f'(a) - \frac{4(\alpha+1+\beta_1)}{(b-a)^2} f\left(\frac{a+b}{2}\right)$$

$$+ \frac{4(2^{\alpha}(\alpha+1)+\beta_1)}{(b-a)^2} f(a) - \frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} (b-x)^{\alpha-1} f(x)\,dx, \tag{7}$$

and

$$I_4 = \frac{2(\beta_1+\beta_2-1)}{b-a} f'\left(\frac{a+b}{2}\right) + \frac{2\left(2^{\alpha+1}-\beta_2\right)}{b-a} f'(b) + \frac{4(\alpha+1+\beta_1)}{(b-a)^2} f\left(\frac{a+b}{2}\right)$$

$$- \frac{4(2^{\alpha}(\alpha+1)+\beta_1)}{(b-a)^2} f(b) + \frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} (x-a)^{\alpha-1} f(x)\,dx. \tag{8}$$

From equations (5) and (8), we get

$$I_2 + I_3 - I_1 - I_4 \tag{9}$$

$$= \frac{2\left(2^{\alpha+1}-\beta_2-\lambda_1-\lambda_2\right)}{b-a}\left(f'(a)-f'(b)\right) + \frac{4(2^{\alpha}(\alpha+1)+\beta_1-\lambda_1)}{(b-a)^2}\left(f(a)+f(b)\right)$$

$$+ \frac{8(\lambda_1-\beta_1)}{(b-a)^2} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right].$$

Choosing $\lambda_1 + \lambda_2 + \beta_2 = 2^{\alpha+1}$ and multiplying the both sides of (9) by $\frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}}$, we obtain equation (3). This completes the proof of Lemma 2.1. □

Theorem 2.2. *Let us consider that the assumptions of Lemma 2.1 hold. Let us also consider that the mapping* $|f''|$ *is convex on* $[a, b]$ *with* $\alpha > 0$. *Then, we have the following inequality*

$$\begin{aligned}
&\left|\frac{(2^{\alpha}(\alpha+1)+\beta_1-\lambda_1)}{2^{\alpha}(\alpha+1)}\left(\frac{f(a)+f(b)}{2}\right)+\frac{\lambda_1-\beta_1}{2^{\alpha}(\alpha+1)}f\left(\frac{a+b}{2}\right)\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\
&\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}}\left[\Omega_1(\alpha;\lambda_1,\lambda_2)+\Omega_2(\alpha;\beta_1,\beta_2)\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right].
\end{aligned}$$

Here,

$$\begin{cases}
\Omega_1(\alpha;\lambda_1,\lambda_2)=\int_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|dt,\\
\Omega_2(\alpha;\beta_1,\beta_2)=\int_0^1\left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|dt.
\end{cases}\tag{10}$$

Proof. By taking modulus in Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{(2^{\alpha}(\alpha+1)+\beta_1-\lambda_1)}{2^{\alpha}(\alpha+1)}\left(\frac{f(a)+f(b)}{2}\right)+\frac{\lambda_1-\beta_1}{2^{\alpha}(\alpha+1)}f\left(\frac{a+b}{2}\right)\right. \qquad (11)\\
&\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\
&\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}}\left[\int_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.\\
&+\int_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\\
&+\int_0^1\left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\\
&\left.+\int_0^1\left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right].
\end{aligned}$$

By using convexity of $\left|f''\right|$, we obtain

$$\left|\frac{\left(2^{\alpha}\left(\alpha+1\right)+\beta_1-\lambda_1\right)}{2^{\alpha}\left(\alpha+1\right)}\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right)+\frac{\lambda_1-\beta_1}{2^{\alpha}\left(\alpha+1\right)}f\left(\frac{a+b}{2}\right)\right.$$

$$\left.-\frac{\Gamma\left(\alpha+1\right)}{2\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f\left(a\right)\right]\right|$$

$$\leq\frac{\left(b-a\right)^{2}}{\left(\alpha+1\right)2^{\alpha+3}}\left[\int_{0}^{1}\left|\left(1-t\right)^{\alpha+1}-\lambda_1t-\lambda_2\right|\left[\frac{1+t}{2}\left|f''\left(a\right)\right|+\frac{1-t}{2}\left|f''\left(b\right)\right|\right]dt\right.$$

$$+\int_{0}^{1}\left|\left(1-t\right)^{\alpha+1}-\lambda_1t-\lambda_2\right|\left[\frac{1-t}{2}\left|f''\left(a\right)\right|+\frac{1+t}{2}\left|f''\left(b\right)\right|\right]dt$$

$$+\int_{0}^{1}\left|\left(2-t\right)^{\alpha+1}-\beta_1t-\beta_2\right|\left[\frac{2-t}{2}\left|f''\left(a\right)\right|+\frac{t}{2}\left|f''\left(b\right)\right|\right]dt$$

$$\left.+\int_{0}^{1}\left|\left(2-t\right)^{\alpha+1}-\beta_1t-\beta_2\right|\left[\frac{t}{2}\left|f''\left(a\right)\right|+\frac{2-t}{2}\left|f''\left(b\right)\right|\right]dt\right]$$

$$=\frac{\left(b-a\right)^{2}}{\left(\alpha+1\right)2^{\alpha+3}}\left[\int_{0}^{1}\left|\left(1-t\right)^{\alpha+1}-\lambda_1t-\lambda_2\right|dt+\int_{0}^{1}\left|\left(2-t\right)^{\alpha+1}-\beta_1t-\beta_2\right|dt\right]\left[\left|f''\left(a\right)\right|+\left|f''\left(b\right)\right|\right].$$

This ends the proof of Theorem 2.2. □

Theorem 2.3. *Assume that the assumptions of Lemma 2.1 is valid. Assume also that the function* $\left|f''\right|^{q}$, $q>1$ *is convex on* $[a,b]$ *with* $\alpha>0$. *Then, the following inequalities hold:*

$$\left|\frac{\left(2^{\alpha}\left(\alpha+1\right)+\beta_1-\lambda_1\right)}{2^{\alpha}\left(\alpha+1\right)}\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right)+\frac{\lambda_1-\beta_1}{2^{\alpha}\left(\alpha+1\right)}f\left(\frac{a+b}{2}\right)\right.$$

$$\left.-\frac{\Gamma\left(\alpha+1\right)}{2\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f\left(a\right)\right]\right|$$

$$\leq\frac{\left(b-a\right)^{2}}{\left(\alpha+1\right)2^{\alpha+3}}\left(\varphi_1\left(\alpha,p;\lambda_1,\lambda_2\right)+\varphi_2\left(\alpha,p;\beta_1,\beta_2\right)\right)$$

$$\times\left[\left(\frac{3\left|f''\left(a\right)\right|^{q}+\left|f''\left(b\right)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''\left(a\right)\right|^{q}+3\left|f''\left(b\right)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)^2}{(\alpha+1)\,2^{\alpha+1+\frac{2}{q}}}\left(\varphi_1(\alpha,p;\lambda_1,\lambda_2)+\varphi_2(\alpha,p;\beta_1,\beta_2)\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{cases} \varphi_1(\alpha,p;\lambda_1,\lambda_2)=\left(\int_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|^p dt\right)^{\frac{1}{p}}, \\ \varphi_2(\alpha,p;\beta_1,\beta_2)=\left(\int_0^1\left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|^p dt\right)^{\frac{1}{p}}. \end{cases}$$

Proof. By applying Hölder inequality in inequality (11), it follows

$$\left|\frac{(2^\alpha(\alpha+1)+\beta_1-\lambda_1)}{2^\alpha(\alpha+1)}\left(\frac{f(a)+f(b)}{2}\right)+\frac{\lambda_1-\beta_1}{2^\alpha(\alpha+1)}f\left(\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{(\alpha+1)\,2^{\alpha+3}}\left[\int_0^1\left(\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\int_0^1\left(\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By using convexity of $\left|f''\right|^q$, we obtain

$$\left|\frac{(2^\alpha(\alpha+1)+\beta_1-\lambda_1)}{2^\alpha(\alpha+1)}\left(\frac{f(a)+f(b)}{2}\right)+\frac{\lambda_1-\beta_1}{2^\alpha(\alpha+1)}f\left(\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(b-a)^2}{(\alpha+1)\,2^{\alpha+3}} \left[\left(\int_0^1 \left| (1-t)^{\alpha+1} - \lambda_1 t - \lambda_2 \right|^p dt \right)^{\frac{1}{p}} \right.$$

$$\times \left[\left(\int_0^1 \left[\frac{1+t}{2} \left| f''(a) \right|^q + \frac{1-t}{2} \left| f''(b) \right|^q \right] dt \right)^{\frac{1}{q}} + \left(\int_0^1 \left[\frac{1-t}{2} \left| f''(a) \right|^q + \frac{1+t}{2} \left| f''(b) \right|^q \right] dt \right)^{\frac{1}{q}} \right]$$

$$+ \left(\int_0^1 \left| (2-t)^{\alpha+1} - \beta_1 t - \beta_2 \right|^p dt \right)^{\frac{1}{p}}$$

$$\left. \times \left[\left(\int_0^1 \left[\frac{2-t}{2} \left| f''(a) \right|^q + \frac{t}{2} \left| f''(b) \right|^q \right] dt \right)^{\frac{1}{q}} + \left(\int_0^1 \left[\frac{t}{2} \left| f''(a) \right|^q + \frac{2-t}{2} \left| f''(b) \right|^q \right] dt \right)^{\frac{1}{q}} \right] \right]$$

$$= \frac{(b-a)^2}{(\alpha+1)\,2^{\alpha+3}} \left[\left(\int_0^1 \left| (1-t)^{\alpha+1} - \lambda_1 t - \lambda_2 \right|^p dt \right)^{\frac{1}{p}} + \left(\int_0^1 \left| (2-t)^{\alpha+1} - \beta_1 t - \beta_2 \right|^p dt \right)^{\frac{1}{p}} \right]$$

$$\times \left[\left(\frac{3 \left| f''(a) \right|^q + \left| f''(b) \right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left| f''(a) \right|^q + 3 \left| f''(b) \right|^q}{4} \right)^{\frac{1}{q}} \right].$$

Let us consider $a_1 = 3\left|f''(a)\right|^q$, $b_1 = \left|f''(b)\right|^q$, $a_2 = \left|f''(a)\right|^q$ and $b_2 = 3\left|f''(b)\right|^q$. Using the facts that,

$$\sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s, \ 0 \leq s < 1$$

and $1 + 3^{\frac{1}{q}} \leq 4$. The desired result can be obtained straightforwardly. This finishes the proof of Theorem 2.3. □

Theorem 2.4. *Suppose that the assumptions of Lemma 2.1 hold. Suppose also that the function* $\left|f''\right|^q$, $q \geq 1$ *is convex on* $[a, b]$ *with* $\alpha > 0$. *Then, we have the following inequality*

$$\left| \frac{(2^\alpha(\alpha+1) + \beta_1 - \lambda_1)}{2^\alpha(\alpha+1)} \left(\frac{f(a) + f(b)}{2} \right) + \frac{\lambda_1 - \beta_1}{2^\alpha(\alpha+1)} f\left(\frac{a+b}{2} \right) \right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right|$$

$$\leq \frac{(b-a)^2}{(\alpha+1)\,2^{\alpha+3}} \left[(\Omega_1(\alpha; \lambda_1, \lambda_2))^{1-\frac{1}{q}} \right.$$

$$\times\left[\left(\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)+\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(a)\right|^q+\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)-\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)-\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(a)\right|^q+\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)+\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+\left(\Omega_2(\alpha;\beta_1,\beta_2)\right)^{1-\frac{1}{q}}\left[\left(\frac{2\Omega_2(\alpha;\beta_1,\beta_2)-\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(a)\right|^q+\frac{\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(a)\right|^q+\frac{2\Omega_2(\alpha;\beta_1,\beta_2)-\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]\right].$$

Here, $\Omega_1(\alpha;\lambda_1,\lambda_2)$ and $\Omega_2(\alpha;\beta_1,\beta_2)$ are defined in (10) *and*

$$\begin{cases}\Omega_3(\alpha,\lambda_1,\lambda_2)=\int\limits_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|t\,dt,\\ \Omega_4(\alpha;\beta_1,\beta_2)=\int\limits_0^1\left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|t\,dt.\end{cases}\tag{12}$$

Proof. By applying power-mean inequality in inequality (11), we have

$$\left|\frac{(2^\alpha(\alpha+1)+\beta_1-\lambda_1)}{2^\alpha(\alpha+1)}\left(\frac{f(a)+f(b)}{2}\right)+\frac{\lambda_1-\beta_1}{2^\alpha(\alpha+1)}f\left(\frac{a+b}{2}\right)\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}}\left[\left(\int\limits_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int\limits_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int\limits_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right| dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right| \left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right| dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1 \left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right| \left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $\left|f''\right|^q$ is convex, we obtain

$$\left|\frac{(2^\alpha(\alpha+1)+\beta_1-\lambda_1)}{2^\alpha(\alpha+1)}\left(\frac{f(a)+f(b)}{2}\right)+\frac{\lambda_1-\beta_1}{2^\alpha(\alpha+1)}f\left(\frac{a+b}{2}\right)\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}}\left[\left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right| dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right| \left[\frac{1+t}{2}\left|f''(a)\right|^q+\frac{1-t}{2}\left|f''(b)\right|^q\right] dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right| \left[\frac{1-t}{2}\left|f''(a)\right|^q+\frac{1+t}{2}\left|f''(b)\right|^q\right] dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_0^1 \left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right| dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 \left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|\left[\frac{2-t}{2}\left|f''(a)\right|^q+\frac{t}{2}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1 \left|(2-t)^{\alpha+1}-\beta_1 t-\beta_2\right|\left[\frac{t}{2}\left|f''(a)\right|^q+\frac{2-t}{2}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]\right]$$

$$=\frac{(b-a)^2}{(\alpha+1)\,2^{\alpha+3}}\left[(\Omega_1(\alpha;\lambda_1,\lambda_2))^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)+\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(a)\right|^q+\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)-\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)-\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(a)\right|^q+\frac{\Omega_1(\alpha;\lambda_1,\lambda_2)+\Omega_3(\alpha,\lambda_1,\lambda_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+(\Omega_2(\alpha;\beta_1,\beta_2))^{1-\frac{1}{q}}\left[\left(\frac{2\Omega_2(\alpha;\beta_1,\beta_2)-\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(a)\right|^q+\frac{\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(a)\right|^q+\frac{2\Omega_2(\alpha;\beta_1,\beta_2)-\Omega_4(\alpha;\beta_1,\beta_2)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]\right].$$

Then, we obtain the desired result of Theorem 2.4. □

3. Special cases

3.1. Special choices of main results to obtain fractional midpoint type inequalities

In this subsection, let us consider $\lambda_1=\lambda_2=0$, $\beta_1=-(\alpha+1)\,2^{\alpha}$ and $\beta_2=2^{\alpha+1}$. Then, main results will be reduced midpoint type inequalities for the case of Riemann–Liouville fractional integral forms.

Remark 3.1. *Assume that the assumptions of Theorem 2.2 hold. Then, the following midpoint type inequality for Riemann–Liouville fractional integral holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left[\frac{1}{\alpha+2}+\frac{\alpha-3}{8}\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which is proved by [17].

Remark 3.2. *Let us consider $\alpha = 1$ in Remark 3.1. Then, the following midpoint type inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \frac{(b-a)^2}{48} \left[\left| f''(a) \right| + \left| f''(b) \right| \right],$$

which is given by [31, Theorem 5].

Corollary 3.3. *Suppose that the assumptions of Theorem 2.3 hold. Then, the following inequality*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
&\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}} \left(\varphi_1(\alpha, p; 0, 0) + \varphi_2\left(\alpha, p; -(\alpha+1)2^\alpha, 2^{\alpha+1}\right) \right) \\
&\quad \times \left[\left(\frac{3\left| f''(a) \right|^q + \left| f''(b) \right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left| f''(a) \right|^q + 3\left| f''(b) \right|^q}{4} \right)^{\frac{1}{q}} \right] \\
&\leq \frac{(b-a)^2}{(\alpha+1)2^{1+\alpha+\frac{2}{q}}} \left(\varphi_1(\alpha, p; 0, 0) + \varphi_2\left(\alpha, p; -(\alpha+1)2^\alpha, 2^{\alpha+1}\right) \right) \left[\left| f''(a) \right| + \left| f''(b) \right| \right]
\end{aligned}$$

is valid. Here, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\varphi_1(\alpha, p; 0, 0) = \left(\frac{1}{(\alpha+1)p+1} \right)^{\frac{1}{p}}$.

Remark 3.4. *In Corollary 3.3, let us note that $\alpha = 1$. Then, the following inequalities hold:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \frac{(b-a)^2}{16} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\frac{3\left| f''(b) \right|^q + \left| f''(a) \right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3\left| f''(a) \right|^q + \left| f''(b) \right|^q}{4} \right)^{\frac{1}{q}} \right] \\
&\leq \frac{(b-a)^2}{16} \left(\frac{4}{2p+1} \right)^{\frac{1}{p}} \left[\left| f''(a) \right| + \left| f''(b) \right| \right],
\end{aligned}$$

which are established in the paper [7, Corollary 4.8].

Corollary 3.5. *If the assumptions of Theorem 2.4 hold, then the following inequality*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
&\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}} \left[\frac{1}{\alpha+2} \left[\left(\frac{(\alpha+4)\left| f''(a) \right|^q + (\alpha+2)\left| f''(b) \right|^q}{2(\alpha+3)} \right)^{\frac{1}{q}} \right. \right.
\end{aligned}$$

$$+\left(\frac{(\alpha+2)\left|f''(a)\right|^{q}+(\alpha+4)\left|f''(b)\right|^{q}}{2(\alpha+3)}\right)^{\frac{1}{q}}\Bigg]+\left(\frac{2^{\alpha+2}-1}{\alpha+2}+2^{\alpha-1}(\alpha-3)\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\left(\left(\frac{2^{\alpha+3}-1}{2(\alpha+3)}+2^{\alpha-1}\left(\frac{2\alpha-7}{3}\right)\right)\left|f''(a)\right|^{q}+\left(\frac{2^{\alpha+3}-(\alpha+4)}{2(\alpha+2)(\alpha+3)}+\frac{2^{\alpha-1}(\alpha-2)}{3}\right)\left|f''(b)\right|^{q}\right)^{\frac{1}{q}}$$

$$+\left(\left(\frac{2^{\alpha+3}-(\alpha+4)}{2(\alpha+2)(\alpha+3)}+\frac{2^{\alpha-1}(\alpha-2)}{3}\right)\left|f''(a)\right|^{q}+\left(\frac{2^{\alpha+3}-1}{2(\alpha+3)}+2^{\alpha-1}\left(\frac{2\alpha-7}{3}\right)\right)\left|f''(b)\right|^{q}\right)^{\frac{1}{q}}\Bigg]\Bigg]$$

is valid.

Proof. Let us consider the following notations

$$\begin{cases}\Omega_1(\alpha;0,0)=\int_0^1\left|(1-t)^{\alpha+1}\right|dt=\frac{1}{\alpha+2},\\[2ex] \Omega_2\left(\alpha;-(\alpha+1)2^{\alpha},2^{\alpha+1}\right)=\int_0^1\left|(2-t)^{\alpha+1}+(\alpha+1)2^{\alpha}t-2^{\alpha+1}\right|dt=\frac{2^{\alpha+2}-1}{\alpha+2}+2^{\alpha-1}(\alpha-3),\\[2ex] \Omega_3(\alpha;0,0)=\int_0^1\left|(1-t)^{\alpha+1}\right|t\,dt=\frac{1}{(\alpha+2)(\alpha+3)},\\[2ex] \Omega_4\left(\alpha;-(\alpha+1)2^{\alpha},2^{\alpha+1}\right)=\int_0^1\left|(2-t)^{\alpha+1}+(\alpha+1)2^{\alpha}t-2^{\alpha+1}\right|t\,dt=\frac{2^{\alpha+3}-(\alpha+4)}{(\alpha+2)(\alpha+3)}+\frac{2^{\alpha}(\alpha-2)}{3}.\end{cases}\tag{13}$$

If we substitute equalities (13) in Theorem 2.4, then the desired result of Corollary 3.5 is obtained. □

Remark 3.6. *If we choose $\alpha=1$ in Corollary 3.5, then the following midpoint type inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(t)\,dt\right|$$

$$\leq\frac{(b-a)^2}{48}\left[\left(\frac{3\left|f''(b)\right|^{q}+5\left|f''(a)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''(a)\right|^{q}+5\left|f''(b)\right|^{q}}{8}\right)^{\frac{1}{q}}\right],$$

which is presented in [30, Proposition 5].

3.2. Special cases of main results to obtain fractional trapezoid type inequalities

In this subsection, let us consider $\lambda_1=\beta_1=-2^{\alpha}$, $\lambda_2=2^{\alpha}$ and $\beta_2=2^{\alpha+1}$. Then, main results will be reduced trapezoid type inequalities for Riemann–Liouville fractional integral forms.

Corollary 3.7. *Suppose that the assumptions of Theorem 2.2 hold. Then, the following trapezoid type inequality for Riemann–Liouville fractional integral holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq \frac{(b-a)^2 \alpha}{4(\alpha+1)(\alpha+2)} \left[\left| f''(a) \right| + \left| f''(b) \right| \right].$$

Remark 3.8. *Let us now note that for $\alpha = 1$ in Corollary 3.7, then the following trapezoid type inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \frac{(b-a)^2}{24} \left[\left| f''(a) \right| + \left| f''(b) \right| \right],$$

which is given in [30, Proposition 2].

Corollary 3.9. *Assume that the assumptions of Theorem 2.3 hold. Then, the following trapezoid type inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right|$$

$$\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}} \left(\varphi_1(\alpha, p; -2^\alpha, 2^\alpha) + \varphi_2\left(\alpha, p; -2^\alpha, 2^{\alpha+1}\right) \right)$$

$$\times \left[\left(\frac{3\left| f''(a) \right|^q + \left| f''(b) \right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left| f''(a) \right|^q + 3\left| f''(b) \right|^q}{4} \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+1+\frac{2}{q}}} \left(\varphi_1(\alpha, p; -2^\alpha, 2^\alpha) + \varphi_2\left(\alpha, p; -2^\alpha, 2^{\alpha+1}\right) \right) \left[\left| f''(a) \right| + \left| f''(b) \right| \right]$$

is valid. Here, $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 3.10. *If the assumptions of Theorem 2.4 hold, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right|$$

$$\leq \frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}} \left[\left(2^{\alpha-1} - \frac{1}{\alpha+2} \right)^{1-\frac{1}{q}} \right.$$

$$\times \left[\left(\left(\frac{2^\alpha}{3} - \frac{\alpha+4}{2(\alpha+2)(\alpha+3)} \right) \left| f''(a) \right|^q + \left(\frac{2^{\alpha-1}}{3} - \frac{1}{2(\alpha+3)} \right) \left| f''(b) \right|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\left(\frac{2^{\alpha-1}}{3} - \frac{1}{2(\alpha+3)} \right) \left| f''(a) \right|^q + \left(\frac{2^\alpha}{3} - \frac{\alpha+4}{2(\alpha+2)(\alpha+3)} \right) \left| f''(b) \right|^q \right)^{\frac{1}{q}} \right]$$

$$+ \left(\frac{1-2^{\alpha+2}}{\alpha+2} + 3 \cdot 2^{\alpha-1} \right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\left(7\frac{2^{\alpha-1}}{3}+\frac{1-2^{\alpha+3}}{2(\alpha+3)}\right)\left|f''(a)\right|^q+\left(\frac{2^\alpha}{3}+\frac{\alpha+4-2^{\alpha+3}}{2(\alpha+2)(\alpha+3)}\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{2^\alpha}{3}+\frac{\alpha+4-2^{\alpha+3}}{2(\alpha+2)(\alpha+3)}\right)\left|f''(a)\right|^q+\left(7\frac{2^{\alpha-1}}{3}+\frac{1-2^{\alpha+3}}{2(\alpha+3)}\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].$$

Proof. Consider the following notations

$$\begin{cases} \Omega_1(\alpha;-2^\alpha,2^\alpha)=\int_0^1\left|(1-t)^{\alpha+1}-2^\alpha(1-t)\right|dt=2^{\alpha-1}-\frac{1}{\alpha+2}, \\ \Omega_2\left(\alpha;-2^\alpha,2^{\alpha+1}\right)=\int_0^1\left|(2-t)^{\alpha+1}-2^\alpha(2-t)\right|dt=\frac{1-2^{\alpha+2}}{\alpha+2}+3\cdot 2^{\alpha-1}, \\ \Omega_3(\alpha;-2^\alpha,2^\alpha)=\int_0^1\left|(1-t)^{\alpha+1}-2^\alpha(1-t)\right|t\,dt=\frac{2^{\alpha-1}}{3}-\frac{1}{(\alpha+2)(\alpha+3)}, \\ \Omega_4\left(\alpha;-2^\alpha,2^{\alpha+1}\right)=\int_0^1\left|(2-t)^{\alpha+1}-2^\alpha(2-t)\right|t\,dt=\frac{2^{\alpha+1}}{3}+\frac{\alpha+4-2^{\alpha+3}}{(\alpha+2)(\alpha+3)}. \end{cases} \tag{14}$$

If we substitute equalities (14) in Theorem 2.4, then the desired result of Corollary 3.10 is obtained. □

Remark 3.11. *Let us now note that for $\alpha=1$ in Corollary 3.10, the following trapezoid type inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(t)\,dt\right|$$

$$\leq\frac{(b-a)^2}{24}\left[\left(\frac{11\left|f''(a)\right|^q+5\left|f''(b)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{5\left|f''(a)\right|^q+11\left|f''(b)\right|^q}{16}\right)^{\frac{1}{q}}\right],$$

which is established by [30, Proposition 6].

3.3. Special choices of main results to obtain fractional Simpson type inequalities

In this subsection, let us consider $\lambda_1=-\frac{2^{\alpha-1}(\alpha+1)}{3}$, $\lambda_2=\frac{2^{\alpha-1}(\alpha+1)}{3}$, $\beta_1=-\frac{5(\alpha+1)2^{\alpha-1}}{3}$ and $\beta_2=2^{\alpha+1}$. Then, main results will be reduced Simpson type inequalities for the case of Riemann–Liouville fractional integral forms.

Remark 3.12. *Assume that the assumptions of Theorem 2.2 hold. Then, the following midpoint type inequality for Riemann–Liouville fractional integral holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}}\left[\Omega_1\left(\alpha;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)\right.$$

$$+\Omega_2\left(\alpha;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)\Bigg]\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which is proved by [15]. Here,

$$\begin{cases} \Omega_1\left(\alpha;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)=\int\limits_0^1\left|(1-t)^{\alpha+1}-\lambda_1 t-\lambda_2\right|dt \\ =\frac{1}{\alpha+2}\left(1+\alpha\left(\frac{2^{\alpha-1}(\alpha+1)}{3}\right)^{1+\frac{2}{\alpha}}\right)-\frac{2^{\alpha-1}(\alpha+1)}{6}, \\ \Omega_2\left(\alpha;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)=\int\limits_0^1\left|(2-t)^{\alpha+1}+\frac{5(\alpha+1)2^{\alpha-1}}{3}t-2^{\alpha+1}\right|dt. \end{cases}$$

Remark 3.13. *Let us consider $\alpha=1$ in Remark 3.1. Then, the following Simpson type inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int\limits_a^b f(t)\,dt\right|\leq\frac{(b-a)^2}{162}\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which is given in [30, Proposition 3].

Remark 3.14. *Suppose that the assumptions of Theorem 2.3 hold. Then, the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{(\alpha+1)2^{\alpha+3}}\left(\varphi_1\left(\alpha,p;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)\right.$$

$$\left.+\varphi_2\left(\alpha,p;-\frac{5(\alpha+1)2^{\alpha-1}}{3},2^{\alpha+1}\right)\right)$$

$$\times\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(b-a)^2}{(\alpha+1)2^{1+\alpha+\frac{2}{q}}}\left(\varphi_1\left(\alpha,p;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)\right.$$

$$\left.+\varphi_2\left(\alpha,p;-\frac{5(\alpha+1)2^{\alpha-1}}{3},2^{\alpha+1}\right)\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which are established by [15, Theorem 3].

Corollary 3.15. *If the assumptions of Theorem 2.4 hold, then the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(b-a)^2}{(\alpha+1)\,2^{\alpha+3}}\left[(\psi_1(\alpha))^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\frac{\psi_1(\alpha)+\psi_3(\alpha)}{2}\left|f''(a)\right|^q+\frac{\psi_1(\alpha)-\psi_3(\alpha)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\psi_1(\alpha)-x_3(\alpha)}{2}\left|f''(a)\right|^q+\frac{\psi_1(\alpha)+\psi_3(\alpha)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+(\psi_2(\alpha))^{1-\frac{1}{q}}\left[\left(\frac{2\psi_2(\alpha)-\psi_4(\alpha)}{2}\left|f''(a)\right|^q+\frac{\psi_4(\alpha)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{\psi_4(\alpha)}{2}\left|f''(a)\right|^q+\frac{2\psi_2(\alpha)-\psi_4(\alpha)}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]\right],$$

is valid. Here, $\psi_1(\alpha)=\Omega_1\left(\alpha;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)$ *and* $\psi_2(\alpha)=\Omega_2\left(\alpha;-\frac{5(\alpha+1)2^{\alpha-1}}{3},2^{\alpha+1}\right)$ *are defined in* (10) *and* $\psi_3(\alpha)=\Omega_3\left(\alpha;-\frac{2^{\alpha-1}(\alpha+1)}{3},\frac{2^{\alpha-1}(\alpha+1)}{3}\right)$ *and* $\psi_4(\alpha)=\Omega_4\left(\alpha;-\frac{5(\alpha+1)2^{\alpha-1}}{3},2^{\alpha+1}\right)$ *are given in* (12).

Remark 3.16. *If we choose* $\alpha=1$ *in Corollary 3.15, then the following midpoint type inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{(b-a)^2}{162}\left[\left(\frac{59\left|f''(a)\right|^q+133\left|f''(b)\right|^q}{3\times 2^6}\right)^{\frac{1}{q}}+\left(\frac{133\left|f''(a)\right|^q+59\left|f''(b)\right|^q}{3\times 2^6}\right)^{\frac{1}{q}}\right],$$

which is given in [30, Proposition 7].

4. Conclusion

In this paper, an identity is presented for the case of twice-differentiable functions whose second derivatives are convex. With the help of this equality, it is given midpoint, trapezoid and simpson type inequalities for Riemann–Liouville fractional integral. Furthermore, it is provided the our results by using special cases of obtained theorems.

In future studies of the mathematicians, improvement or generalization of our results can be investigated by using different kind of convex function classes or other type fractional integral operators. Moreover, the field of mathematical inequalities remains ripe for exploration and innovation. Future researchers may continue to explore new inequalities with diverse fractional types, building upon the foundation we have laid with these type of inequalities.

References

[1] P. Agarwal, J. Tariboon, and S. K. Ntouyas, Some generalized Riemann–Liouville *k*-fractional integral inequalities, Journal of Inequalities and Applications, 2016(122) (2016).
[2] M.A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak, and F. Hezenci, Some new Simpson's-Formula-Type inequalities for twice–differentiable convex functions via generalized fractional operators, Symmetry, 13(12) (2021) 2249.
[3] G.A. Anastassiou, General fractional Hermite–Hadamard inequalities using $m-$convexity and $(s,m)-$convexity, Frontiers in Time Scales and Inequalities, (2016) 237–255.
[4] A. Barani, S. Barani, S.S. Dragomir, Refinements of Hermite-Hadamard type inequality for functions whose second derivatives absolute values are quasi convex, RGMIA Res. Rep. Coll, 14 (2011).
[5] A. Barani, S. Barani, S.S. Dragomir, Refinements of Hermite–Hadamard inequalities for functions when a power of the absolute value of the second derivative is P–convex, Journal of Applied Mathematics, 2012 (2012).
[6] H. Budak, F. Hezenci, and H. Kara, On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co–ordinated convex functions via generalized fractional integrals, Advances in Difference Equations, 2021 (2021) 1–32.
[7] H. Budak, F. Ertugral, and E. Pehlivan, Hermite-Hadamard type inequalities for twice differantiable functions via generalized fractional integrals, Filomat, 33(15) (2019) 4967–4979.
[8] H. Budak, and E. Pehlivan, Weighted Ostrowski, trapezoid and midpoint type inequalities for Riemann–Liouville fractional integrals, AIMS Mathematics, 5(3) (2020) 1960–1984.
[9] F. Chen, A note on the Hermite–Hadamard inequality for convex functions on the co–ordinates, Journal of Mathematical Inequalities, 8(4) (2014) 915–923.
[10] S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Applied Mathematics Letters, 11(5) (1998) 91–95.
[11] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, (2000).
[12] S.S. Dragomir, On Hadamard's inequality for convex functions on the co–ordinates in a rectangle from the plane, Taiwanese Journal of Mathematics, 4 (2001) 775–788.
[13] S.S. Dragomir, M.I. Bhatti, M. Iqbal, M. Muddassar, Some new Hermite-Hadamard's type fractional integral inequalities, Journal of Computational Analysis and Application, 18 (2015) 655–661.
[14] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, Wien: Springer-Verlag, 1997, 223–276.
[15] F. Hezenci, A note on Fractional Simpson type inequalities for twice differentiable functions, Mathematica Slovaca, 73(3) (2023) 675–686.
[16] F. Hezenci, H. Budak, and H. Kara, New version of Fractional Simpson type inequalities for twice differentiable functions, Advances in Difference Equations, 2021(460) (2021).
[17] F. Hezenci, M. Bohner, and H. Budak, Fractional midpoint type inequalities for twice differentiable functions, Filomat, 37(24) (2023) 8131–8144.
[18] M. Iqbal, M.I. Bhatti, K. Nazeer, Generalization of inequalities analogous to Hermite–Hadamard inequality via fractional integrals, Bulletin of the Korean Mathematical Society, 52(3) (2015) 707–716.
[19] A. Iqbal, M.A. Khan, S. Ullah, and Y. Chu, Some new Hermite–Hadamard-type inequalities associated with conformable fractional integrals and their applications, Journal of Function Spaces, 2020 (2020).
[20] H. Kavurmaci, M. Avci, and M.E. Ozdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, Journal of Inequalities and Applications, 2011(1) 1–11.
[21] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, Applied Mathematics and Computation, 147(5) (2004) 137–146.
[22] M.A. Latif and S.S. Dragomir, On some new inequalities for differentiable co–ordinated convex functions, Journal of Inequalities and Applications, 2012(1) (2012) 1–13.
[23] S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, NewYork:Wiley, 1993.
[24] P.O. Mohammed and M.Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, Journal of Computational and Applied Mathematics, 372 (2020) 112740.
[25] M.E. Ozdemir, C. Yildiz, and A.O. Akdemir, On some new Hadamard-type inequalities for co-ordinated quasi-convex functions, Hacettepe Journal of Mathematics and Statistics, 41(5) (2012) 697–707.
[26] M.E. Ozdemir, and C. Yildiz, New inequalities for Hermite-Hadamard and Simpson type with applications, Tamkang Journal of Mathematics, 44(2) (2013) 209–216.
[27] J. Park, On Some Integral Inequalities for Twice Differentiable Quasi–Convex and Convex Functions via Fractional Integrals, Applied Mathematical Sciences, 9(62) (2015) 3057–3069.
[28] J.E. Pecaric and Y.L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press; 1st edition, Boston, 1992.
[29] X. Li, S. Qaisar, J. Nasir et al., Some results on integral inequalities via Riemann–Liouville fractional integrals, Journal of Inequalities and Applications, 2019(214)(2019) 1–11.
[30] M.Z. Sarikaya, N. Aktan, On the generalization of some integral inequalities and their applications, Mathematical and computer Modelling, 54(9-10) (2011) 2175–2182.
[31] M.Z. Sarikaya, A. Saglam, and H. Yildirim, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi–convex, International Journal of Open Problems in Computer Science and Mathematics, 5(3) (2012) 2074–2827.

[32] M.Z. Sarikaya, E. Set, M.E. Ozdemir, and S.S. Dragomir, New some Hadamard's type inequalities for coordinated convex functions, Tamsui Oxford Journal of Information and Mathematical Sciences, 28(2) (2012) 137–152.

[33] M.Z. Sarikaya, E. Set, and M.E. Ozdemir, On new inequalities of Simpson's type for $s-$convex functions, Computers & Mathematics with Applications, 60(8) (2010) 2191–2199.

[34] M.Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57(9–10) (2013) 2403–2407.

[35] M.Z. Sarikaya and H. Budak, Some Hermite -Hadamard type integral inequalities for twice differentiable mappings via fractional integrals, Facta Universitatis. Series: Mathematics and Informatics, 29(4) (2014) 371–384.

[36] M.Z. Sarikaya and H. Yildirim, On Hermite–Hadamard type inequalities for Riemann-Liouville fractional integrals, Miskolc Mathematical Notes, 17(2) (2016) 1049–1059.

[37] M.Z. Sarikaya, A. Akkurt, H. Budak, M.E. Yildirim, and H. Yildirim, Hermite-Hadamard's inequalities for conformable fractional integrals, An International Journal of Optimization and Control: Theories & Applications, 9(1) (2019) 49–59.

[38] M.Z. Sarikaya and F. Ertugral, On the generalized Hermite–Hadamard inequalities, Annals of the University of Craiova, Mathematics and Computer Science Series, 47(1) (2020) 193–213.

[39] M. Tomar, E. Set, and M.Z. Sarıkaya, Hermite-Hadamard type Riemann-Liouville fractional integral inequalities for convex functions, AIP Conference Proceedings, 1726, 2016, 020035.

[40] T. Tunç, M.Z. Sarikaya, and H. Yaldiz, Fractional Hermite-Hadamard's type inequality for co–ordinated convex functions, TWMS Journal of Pure and Applied Mathematics, 11(1) (2020) 3–29.

[41] M. Vivas-Cortez, T. Abdeljawad, P.O. Mohammed, and Y. Rangel–Oliveros, Simpson's integral inequalities for twice differentiable convex functions, Mathematical Problems in Engineering, 2020.

[42] X. You, F. Hezenci, H. Budak, and H. Kara, New Simpson type inequalities for twice differentiable functions via generalized fractional integrals, AIMS Mathematics, 7(3) (2021) 3959–3971.