Filomat 38:9 (2024), 3001–3010
https://doi.org/10.2298/FIL2409001D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bounds on the zeros of a quaternionic polynomial using matrix methods

Ishfaq Dar[a,*], **N. A. Rather**[b], **Irfan Faiq**[c]

[a]*Department of Applied Sciences, Institute of Technology, University of Kashmir, Srinagar-190006, India*
[b]*Department of Mathematics, University of Kashmir, Srinagar-190006, India*
[c]*Department of Applied Sciences, Institute of Technology, University of Kashmir, Srinagar-190006, India*

Abstract. In this paper, the problem of locating the left eigenvalues of the quaternion matrices and their connection with the zeros of the quaternion polynomials with quaternion coefficients is considered by using various matrix tools. As an application of which, various famous results for locating the zeros of regular polynomials of a quaternionic variable with quaternionic coefficients are obtained, which include the extension of Cauchy's theorem, Parodi's theorem as well.

1. Introduction and statement of results

In an effort to expand complex numbers to greater spatial dimensions, the Irish mathematician Sir William Rowan Hamilton (1805-1865) invented quaternions in 1843. Hamilton became obsessed on quaternions and their uses [4] after inventing them, and he did so for the remainder of his life. However, he probably never imagined that his invention, quaternions, would one day be used to programme video games and steer spacecraft [9]. Quaternions are now widely employed in computer science in addition to being a part of modern mathematics, they are also extensively employed in control theory, physics, mechanics, altitude control, signal processing, and computer graphics (mainly for representing rotations and orientations of objects in three-dimensional space). Quaternions, for instance, are used to control the altitude of spacecraft (See [6] and also references therein). There has been a lot of activity in recent years in the study of the quaternion-related mathematical objects; each year, several research articles are published in a wide range of publications, and various methodologies are used for various objectives.
Quaternion Number: A quaternion is a number of the form $q = \alpha + \beta i + \gamma j + \delta k,\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$ and i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$. The non commutative division ring of quaternions is denoted by $\mathbb{H}$. Every element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$, is composed by the real part $\Re(q) = \alpha$ and the imaginary part $\Im(q) = \beta i + \gamma j + \delta k$. The conjugate of q, denoted by $\bar{q}$ is a quaternion $\bar{q} = \alpha - \beta i - \gamma j - \delta k$ and the norm of q is $|q| = \sqrt{q\bar{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The inverse of each non zero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\bar{q}$.
The study's main challenge is the non commutative multiplication of quaternions, however in this article,

2020 *Mathematics Subject Classification*. Primary 30E10, 30G35; Secondary 16K20.
Keywords. Quaternion polynomials, Matrix methods, Zero-sets, Geršgorin balls, left eigenvalues, Companion matrix.
Received: 02 June 2023; Accepted: 16 October 2023
Communicated by Miodrag Mateljević
* Corresponding author: Ishfaq Dar
Email addresses: ishfaq619@gmail.com (Ishfaq Dar), dr.narather@gmail.com (N. A. Rather), irfanfaiq@uok.edu.in (Irfan Faiq)

we were able to prove various results concerned with (finite) quaternionic matrices, which in turn yield various results concerning the location of zeros of quaternion polynomials.

Quaternionic Polynomials: Unlike the real or complex case, there are several possible ways to define quaternionic polynomials depending on the position of coefficients $a_0, a_1, a_2, ..., a_n \in \mathbb{H}$ with respect to the indeterminate $q \in \mathbb{H}$. The quaternion polynomial of degree n is the expression $f(q) = q^n a_n + q^{n-1}a_{n-1} + ... + qa_1 + a_0$ or $P(q) = a_n q^n + a_{n-1}q^{n-1} + ... + a_1 q + a_0,\ a_n \neq 0$ in the quaternion indeterminate q. This polynomial is called monic quaternion polynomial of degree n if $a_n = 1$.

Quaternion companion matrix: The $n \times n$ companion matrix of a monic quaternion polynomial of the form $f(q) = q^n + q^{n-1}a_{n-1} + ... + qa_1 + a_0$, is given by

$$C_f = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & 0 & ... & 0 & -a_1 \\ 0 & 1 & 0 & ... & 0 & -a_2 \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & -a_{n-1} \end{bmatrix}$$

Whereas, the $n \times n$ companion matrix for a monic quaternion polynomial of the form $P(q) = q^n + a_{n-1}q^{n-1} + ... + a_1 q + a_0$, is given by

$$C_P = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & 1 \\ -a_0 & -a_1 & -a_2 & ... & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

For an $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions, the non commutativity of quaternions result in two different types of eigenvalues (For reference see [5]).

Right eigenvalue: Given an $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions, $\lambda \in \mathbb{H}$ is called the right eigenvalue of A, if $Ax = x\lambda$ for some non-zero eigenvector $x = [x_1, x_2, ..., x_n]^T$ of quaternions.

Left eigenvalue: Given an $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions, $\lambda \in \mathbb{H}$ is called the left eigenvalue of A, if $Ax = \lambda x$ for some non-zero eigenvector $x = [x_1, x_2, ..., x_n]^T$ of quaternions.

For complex case, concerning the location of the eigenvalues, the famous Geršgorin theorem can be stated as;

Theorem 1.1. *All the eigenvalues of a $n \times n$ complex matrix $A = (a_{\mu\nu})$ are contained in the union of n Geršgorin discs defined by $D_\mu = \{z \in \mathbb{C} : |z - a_{\mu\mu}| \leq \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} |a_{\mu\nu}|\}$.*

The quaternion version of Geršgorin theorem for left eigenvalues is mentioned (without proof) in [12], however here we present this theorem along with the proof, more precisely we prove:

Theorem 1.2. *All the left eigenvalues of a $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions lie in the union of the n Geršgorin balls defined by $B_\mu = \{q \in \mathbb{H} : |q - a_{\mu\mu}| \leq \rho_\mu(A)\}$ where $\rho_\mu(A) = \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} |a_{\mu\nu}|$.*

Next, we prove the following result which gives the connection between the left eigenvalues of a companion matrix and the zeros of associated quaternion polynomial.

Theorem 1.3. *Let $P(q) = q^n + a_{n-1}q^{n-1} + ... + a_1 q + a_0$ be a quaternion polynomial with quaternionic coefficients and q be quaternionic variable, then for any diagonal matrix $D = diag(d_1, d_2, ..., d_{n-1}, d_n)$, where $d_1, d_2, ..., d_n$ are positive real numbers, the left eigenvalues of $D^{-1}C_P D$ and the zeros of $P(q)$ are same.*

Note that Theorem 1.3 remains valid if we replace $P(q)$ by $f(q)$.

Remark 1.4. *If we take $d_1 = d_2 = d_3 = ... = d_n = 1$, then Theorem 1.3 reduces to the following result.*

Corollary 1.5. *If λ is the left eigenvalue of the companion matrix C_P associated with the quaternionic polynomial $P(q)$, then λ is a zero of $P(q)$.*

In the complex case, the matrix A and its transpose A^T have same eigenvalues, so one may use the Geršgorin discs of both A and A^T, whichever give the better estimates to locate the regions containing the eigenvalues of A. However, in case of quaternionic matrices, a matrix and its transpose may have different left eigenvalues as shown by following simple example

$$A = \begin{bmatrix} 1 & i \\ j & k \end{bmatrix}$$

On the other hand, very little is known about left eigenvalues of quaternionic matrices. Wood [11] proved that every quaternionic matrix has at least one left eigenvalue. Huang and So [5] completely solved the case of 2×2 matrices. The case $n = 3$ was studied by So [10]. Finally, Zhang [12] and Farid, Wang and Zhang [3] gave several Geršgorin type theorems for quaternionic matrices. Now in view of Corollary 1.5, the left eigenvalues of C_f and the zeros of polynomial $f(q)$ are same, the left eigenvalues of C_P and the zeros of polynomial $P(q)$ are also same, we may therefore use Theorem 1.2 and Theorem 1.3 as a tool in determining the zeros of a given polynomial and vice-versa.
Virtually every discipline of mathematics, from Applied Analysis, Fourier Analysis, and Computer Sciences to Algebraic Number Theory and Algebraic Geometry, possesses its own theory that has been derived from the study of polynomials. Polynomials flourish and much that is beautiful in mathematics is related to polynomials. For a polynomial of degree n with complex coefficients, the exact computation of zeros can be made when n varies from 1 to 4, but there are no general methods to compute the zeros of polynomials of degree $n > 4$ and for this reason, the estimation of regions containing the zeros of polynomials become an interesting area of research. In 1829, concerning the location of zeros of a polynomial with complex coefficients, A. L Cauchy [1] gave a very simple expression of the region containing all the zeros in terms of the coefficients of a polynomial. In fact he proved the following theorem.

Theorem 1.6. *If $P(z) = z^n + z^{n-1}c_{n-1} + z^{n-2}c_{n-2} + ... + zc_1 + c_0$ is a complex polynomial, then all the zeroes of $P(z)$ lie inside the disc $|z| < 1 + \max_{0 \le \nu \le n-1} |c_\nu|$.*

In view of the applications of the zeros of quaternionic polynomials, various authors have shown their interest in this field and were successful in extending various results concerning the location of the zeros of complex polynomials to the quaternion settings. Recently, Carney et al. [2] extended Eneström-Kakeya theorem to quaternion settings by proving following result.

Theorem 1.7. *If $p(q) = q^n a_n + q^{n-1}a_{n-1} + q^{n-2}a_{n-2} + ... + qa_1 + a_0$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying $0 \le a_0 \le a_1 \le ... \le a_n$, then all the zeros of p lie in $|q| \le 1$.*

In the same paper, they generalized of Theorem 1.7 to the polynomials whose coefficients are monotonic but not necessarily non-negative by establishing the following result.

Theorem 1.8. *If $p(q) = q^n a_n + q^{n-1}a_{n-1} + q^{n-2}a_{n-2} + ... + qa_1 + a_0$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying $a_0 \le a_1 \le ... \le a_n$, then all the zeros of p lie in $|q| \le \frac{a_n + |a_0| - a_0}{|a_n|}$.*

Milovanović et al. [7] generalized Theorem 1.7 and Theorem 1.8 by proving the following result.

Theorem 1.9. *If $p(q) = q^n a_n + q^{n-1}a_{n-1} + q^{n-2}a_{n-2} + ... + qa_1 + a_0$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying $a_0 \le a_1 \le ... \le a_{\lambda-1} \le a_\lambda \ge ... \ge a_{n-1} \ge a_n$, where $0 \le \lambda \le n$, then all the zeros of p lie in $|q| \le \frac{2a_\lambda - a_n + |a_0| - a_0}{|a_n|}$.*

Because of the restriction on the coefficients that they should be real and monotonic, the results discussed above are applicable to a small class of polynomials, so its interesting to look for the results without any restriction on the coefficients and applicable to every quaternionic polynomial with quaternion/complex or real coefficients. In this direction, here we present some results concerning the location of the zeros of quaterionic polynomials with quaternionic coefficients by using various quaternion matrix tools including Theorem 1.2 and Theorem 1.3. We begin by proving the following result, which is an extension of Theorem 1.6 to quaternion settings, more precisely we prove:

Theorem 1.10. *If $f(q) = q^n + q^{n-1}a_{n-1} + q^{n-2}a_{n-2} + ... + qa_1 + a_0$ is a quaternion polynomial with quaternion coefficients and q is quaternionic variable, then all the zeroes of $f(q)$ lie inside the ball $|q| < 1 + \max_{0\leq\nu\leq n-1} |a_\nu|$.*

As a generalization of Theorem 1.10, we prove the following result;

Theorem 1.11. *Let $f(q) = q^n + q^{n-1}a_{n-1} + q^{n-2}a_{n-2} + ... + qa_1 + a_0$ be a quaternion polynomial with quaternion coefficients and q be quaternionic variable, then all the zeroes of $f(q)$ lie in the union of balls*

$$\{q \in \mathbb{H} : |q| \leq r\Big(1 + \frac{|a_\nu|}{r^{n-\nu}}\Big), \nu = 0, 1, 2, ..., n-2\} \text{ and } \{q \in \mathbb{H} : |q + a_{n-1}| \leq r\}$$

where r is a positive real number.

Since $|q| = |q + a_{n-1} - a_{n-1}| \leq r + |a_{n-1}| = r\Big(1 + \frac{|a_{n-1}|}{r}\Big)$, the above theorem reduces to the following result.

Corollary 1.12. *Let $f(q) = q^n + q^{n-1}a_{n-1} + q^{n-2}a_{n-2} + ... + qa_1 + a_0$ be a quaternion polynomial with quaternion coefficients and q be quaternionic variable, then all the zeroes of $f(q)$ lie in the union of balls*

$$\{q \in \mathbb{H} : |q| \leq r\Big(1 + \frac{|a_\nu|}{r^{n-\nu}}\Big), \nu = 0, 1, 2, ..., n-1\}$$

where r is a positive real number.

Remark 1.13. *For $r = 1$ Corollary 1.12 reduces to Theorem 1.10.*

Next we prove the following result which is an extension of a theorem due to M. Parodi [8] to quaternion settings, more precisely we prove.

Theorem 1.14. *If $P(q) = q^n + a_{n-1}q^{n-1} + ... + a_1q + a_0$ is a quaternion polynomial with quaternion coefficients and q is quaternionic variable, then all the zeros of $P(q)$ lie in the unions of the balls*

$$\{q \in \mathbb{H} : |q| \leq 1\} \text{ and } \{q \in \mathbb{H} : |q + a_{n-1}| \leq \sum_{\nu=0}^{n-2} |a_\nu|\}.$$

Lastly, we prove the following theorem.

Theorem 1.15. *If $P(q) = q^n + a_{n-1}q^{n-1} + ... + a_1q + a_0$ is a quaternion polynomial with quaternion coefficients and q is quaternionic variable, then all the zeros of $P(q)$ lie in the ball*

$$\{q \in \mathbb{H} : |q| \leq \sum_{\nu=1}^{n} |a_{n-\nu}|^{1/\nu}\}.$$

2. Computations and Analysis

In this section, we present some examples for which existing Eneström-Kakeya type results are applicable to show that the obtained results give better information about the location of the zeros than existing results present in the literature. It is worth mentioning that all existing Eneström-Kakeya type results are applicable to a small class of polynomials with real coefficients satisfying monotonicity condition, whereas the results proved in this paper are applicable to every polynomial with quaternion/complex or real coefficients.

Example 1:
Let $p(q) = q^4 - \frac{q^3}{4} - \frac{q^2}{4} - \frac{q}{4} - \frac{1}{4}$. It is easy to see that Theorem 1.7 is not applicable and on using Theorem 1.8 or Theorem 1.9 (with $\lambda = n = 4$), it follows that all the zeros of $p(q)$ lie in the ball $|q| \leq 1.5$. Whereas, if we use

Theorem 1.10, it follows that all the zeros of $p(q)$ lie in the ball $|q| < 1.25$. Thus, Theorem 1.10 gives better bound with a significant improvement.

Example 2:
Let $p(q) = q^3 + \frac{q^2}{2} + \frac{q}{2} - \frac{1}{2}$. Again it is easy to see that Theorem 1.7 is not applicable and on using Theorem 1.8 or Theorem 1.9 (with $\lambda = n = 3$), it follows that all the zeros of $p(q)$ lie in the ball $|q| \le 2$. Whereas, if we use Theorem 1.10, it follows that all the zeros of $p(q)$ lie in the ball $|q| < 1.5$. Thus, Theorem 1.10 gives better bound with a significant improvement.

Example 3:
Let $p(q) = q^3 + \frac{q^2}{3} + \frac{q}{4} - \frac{1}{2}$. Again it is easy to see that Theorem 1.7 is not applicable and on using Theorem 1.8 or Theorem 1.9 (with $\lambda = n = 3$), it follows that all the zeros of $p(q)$ lie in the ball $|q| \le 2$. Whereas, if we use Theorem 1.10, it follows that all the zeros of $p(q)$ lie in the ball $|q| < 1.5$. Thus, Theorem 1.10 gives better bound with a significant improvement.

3. Lemma

For the proof of Theorem 1.15, we need the following Lemma.

Lemma 3.1. *Let $P(q) = q^n + a_{n-1}q^{n-1} + ... + a_1 q + a_0$ be a quaternion polynomial with quaternion coefficients and q be a quaternionic variable, then for any positive real number r, all the zeros of $P(q)$ lie in the ball*

$$\left\{q \in \mathbb{H} : |q| \le \max\left\{r, \sum_{\nu=0}^{n-1} \frac{|a_\nu|}{r^{n-\nu-1}}\right\}\right\}.$$

Proof. [Proof of Lemma 3.1] Let C_P be the companion matrix of the polynomial $P(q)$ and for any positive real number r define a diagonal matrix $T = diag(1/r^{n-1}, 1/r^{n-2}, ..., 1/r, 1)$. Then

$$T^{-1}C_P T = \begin{bmatrix} r^{n-1} & 0 & ... & 0 & 0 \\ 0 & r^{n-2} & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & r & 0 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & 1 \\ -a_0 & -a_1 & -a_2 & ... & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} 1/r^{n-1} & 0 & ... & 0 & 0 \\ 0 & 1/r^{n-2} & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & 1/r & 0 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & r & 0 & ... & 0 & 0 \\ 0 & 0 & r & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & r \\ -\frac{a_0}{r^{n-1}} & -\frac{a_1}{r^{n-2}} & -\frac{a_2}{r^{n-3}} & ... & -\frac{a_{n-2}}{r} & -a_{n-1} \end{bmatrix}$$

Applying Theorem 1.2 to the matrix $T^{-1}C_P T$, it follows that all the left eigenvalues of the matrix $T^{-1}C_p T$ lie in the union of balls

$$|q| \le r$$

and

$$|q + a_{n-1}| \le \sum_{\nu=0}^{n-2} \frac{|a_\nu|}{r^{n-\nu-1}}.$$

Since $|q| = |q + a_{n-1} - a_{n-1}| \leq \sum_{\nu=0}^{n-2} \frac{|a_\nu|}{r^{n-\nu-1}} + |a_{n-1}| = \sum_{\nu=0}^{n-1} \frac{|a_\nu|}{r^{n-\nu-1}}$, implies, all the left eigenvalues of the matrix $T^{-1}C_PT$ lie in the union of balls

$$|q| \leq r$$

and

$$|q + a_{n-1}| \leq \sum_{\nu=0}^{n-1} \frac{|a_\nu|}{r^{n-\nu-1}}.$$

That is, all the left eigenvalues of the matrix $T^{-1}C_PT$ lie in the ball

$$\left\{q \in \mathbb{H} : |q| \leq \max\left\{r, \sum_{\nu=0}^{n-1} \frac{|a_\nu|}{r^{n-\nu-1}}\right\}\right\}. \tag{1}$$

Since T is a diagonal matrix with real positive entries, by Theorem 1.3, it follows that the left eigenvalues of $T^{-1}C_PT$ are the zeros of $P(q)$. Hence, all the zeros of $P(q)$ lie in the balls given by (1).
That completes the proof. □

4. Proof of the Main Theorems

Proof. [Proof of Theorem 1.2] Let $A = (a_{\mu\nu})$ be $n \times n$ matrix of quaternions and λ be left eigenvalue of A corresponding to the non-zero eigenvector $x = [x_1, x_2, ..., x_n]^T$. Further let x_m be such that $|x_m| \geq |x_\nu|$ for all ν, then $|x_m| > 0$.
Equating m^{th} row of $Ax = \lambda x$, we obtain

$$\sum_{\nu=1}^{n} a_{m\nu}x_\nu = \lambda x_m,$$

that is,

$$\sum_{\substack{\nu=1 \\ \nu\neq m}}^{n} a_{m\nu}x_\nu = (\lambda - a_{mm})x_m,$$

which with the help of triangle inequality yields,

$$\begin{aligned} |\lambda - a_{mm}||x_m| &= \left|\sum_{\substack{\nu=1 \\ \nu\neq m}}^{n} a_{m\nu}x_\nu\right| \\ &\leq \sum_{\substack{\nu=1 \\ \nu\neq m}}^{n} |a_{m\nu}||x_\nu|. \end{aligned}$$

Since $|x_m| \neq 0$, implies

$$|\lambda - a_{mm}| \leq \sum_{\substack{\nu=1 \\ \nu\neq m}}^{n} |a_{m\nu}| = \rho_m(A).$$

Hence, the left eigenvalue λ of A lies in the Geršgorin ball $|q - a_{mm}| \leq \rho_m(A)$. Since λ was chosen arbitrarily, it follows that all the left eigenvalues lie in the union of the Geršgorin balls $B_\mu = \{q \in \mathbb{H} : |q - a_{\mu\mu}| \leq \rho_\mu(A)\}$. where $\rho_\mu(A) = \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} |a_{\mu\nu}|$. □

Proof. [Proof of Theorem 1.3] Let C_P be the companion matrix of the polynomial $P(q)$ and $D = diag(d_1, d_2, ..., d_n)$ be a diagonal matrix, where $d_1, d_2, ..., d_n$ are positive real numbers. Then

$$D^{-1}C_PD = \begin{bmatrix} 0 & d_2/d_1 & 0 & ... & 0 & 0 \\ 0 & 0 & d_3/d_2 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & d_n/d_{n-1} \\ -\frac{a_0d_1}{d_n} & -\frac{a_1d_2}{d_n} & -\frac{a_2d_3}{d_n} & ... & -\frac{a_{n-2}d_{n-1}}{d_n} & -a_{n-1} \end{bmatrix}$$

Now if λ is left eigenvalue of $D^{-1}C_PD$ corresponding to the non-zero eigenvector $v = [v_1, v_2, ..., v_n]^T$, then by definition of left eigenvalues $(D^{-1}C_PD)v = \lambda v$, implies

$$\begin{bmatrix} 0 & d_2/d_1 & 0 & ... & 0 & 0 \\ 0 & 0 & d_3/d_2 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & d_n/d_{n-1} \\ -\frac{a_0d_1}{d_n} & -\frac{a_1d_2}{d_n} & -\frac{a_2d_3}{d_n} & ... & -\frac{a_{n-2}d_{n-1}}{d_n} & -a_{n-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ ... \\ ... \\ v_{n-1} \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ ... \\ ... \\ v_{n-1} \\ v_n \end{bmatrix},$$

which on simplification yields the following system of linear equations

$$\frac{d_2}{d_1}v_2 = \lambda v_1$$
$$\frac{d_3}{d_2}v_3 = \lambda v_2$$
$$.$$
$$.$$
$$\frac{d_n}{d_{n-1}}v_n = \lambda v_{n-1}$$

$$-\frac{a_0d_1v_1}{d_n} - \frac{a_1d_2v_2}{d_n} - \frac{a_2d_3v_3}{d_n} - ... - \frac{a_{n-2}d_{n-1}v_{n-1}}{d_n} - a_{n-1}v_n = \lambda v_n. \tag{2}$$

The first $n-1$ equations on consecutive substitution yield $d_iv_i = \lambda^{i-1}d_1v_1,\ i = 2, 3, ..., n$. In view of this, equation (2) reduces to

$$-a_0d_1v_1 - a_1\lambda d_1v_1 - a_2\lambda^2 d_1v_1 - ... - a_{n-2}\lambda^{n-2}d_1v_1 - a_{n-1}\lambda^{n-1}d_1v_1 = \lambda^n v_1d_1,$$

that is,

$$(-a_0 - a_1\lambda - a_2\lambda^2 - ... - a_{n-2}\lambda^{n-2} - a_{n-1}\lambda^{n-1})v_1 = \lambda^n v_1$$

v being a non zero vector implies that $v_1 \neq 0$, hence from above equation, we conclude

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

This shows that λ is a zero of $P(q)$. Since λ was chosen arbitrarily, it follows that the left eigenvalues of $D^{-1}C_PD$ are the zeros of $P(q)$. □

Proof. [Proof of Theorem 1.10] By definition the companion matrix C_f of $f(q) = q^n + q^{n-1}a_{n-1} + ... + qa_1 + a_0$ is given by

$$C_f = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & 0 & ... & 0 & -a_1 \\ 0 & 1 & 0 & ... & 0 & -a_2 \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & -a_{n-1} \end{bmatrix}$$

where $a_0, a_1, ..., a_{n-1}$ are quaternion coefficients. On applying Theorem 1.2 to C_f, it follows that all the left eigenvalues of C_f lie in the union of the balls;

$$\begin{aligned} &|q| \leq |a_0| < 1 + |a_0|, \\ &|q| \leq 1 + |a_1|, \\ &. \quad . \quad . \quad . \\ &. \quad . \quad . \quad . \\ &|q| \leq 1 + |a_{n-2}| \end{aligned}$$

and

$$|q + a_{n-1}| \leq 1.$$

Since $|q| = |q + a_{n-1} - a_{n-1}| \leq 1 + |a_{n-1}|$, it follows from above n inequalities that all the left eigenvalues of C_f lie in the ball $|q| < 1 + \max\limits_{0 \leq v \leq n-1} |a_v|$. Since the left eigenvalues of C_f are the zeros of $f(q)$, it follows that all the zeros of $f(q)$ lie in the ball $|q| < 1 + \max\limits_{0 \leq v \leq n-1} |a_v|$. □

Proof. [Proof of Theorem 1.11] Let C_f be the companion matrix of the polynomial $f(q)$ and for any positive real number r define a diagonal matrix $T = diag(r^{n-1}, r^{n-2}, ..., r, 1)$. Then

$$T^{-1}C_fT = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & \frac{-a_0}{r^{n-1}} \\ r & 0 & 0 & ... & 0 & \frac{-a_1}{r^{n-2}} \\ 0 & r & 0 & ... & 0 & \frac{-a_2}{r^{n-3}} \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & \frac{-a_{n-2}}{r} \\ 0 & 0 & 0 & ... & r & -a_{n-1} \end{bmatrix}$$

Applying Theorem 1.2 to the matrix $T^{-1}C_fT$, it follows all the left eigenvalues of the matrix $T^{-1}C_fT$ lie in the union of the balls;

$$\begin{aligned} &|q| \leq \frac{|a_0|}{r^{n-1}} < r\left(1 + \frac{|a_0|}{r^n}\right), \\ &\quad |q| \leq r\left(1 + \frac{|a_1|}{r^{n-1}}\right), \\ &\quad . \quad . \quad . \quad . \\ &\quad . \quad . \quad . \quad . \\ &\quad |q| \leq r\left(1 + \frac{|a_{n-2}|}{r^2}\right) \end{aligned}$$

and

$$|q + a_{n-1}| \leq r.$$

That is, all the left eigenvalues of the matrix $T^{-1}C_fT$ lie in the union of the balls

$$\{q \in \mathbb{H} : |q| \leq r\Big(1 + \frac{|a_\nu|}{r^{n-\nu}}\Big), \nu = 0, 1, 2, ..., n-2\} \text{ and } \{q \in \mathbb{H} : |q + a_{n-1}| \leq r\} \tag{3}$$

Since T is a diagonal matrix with real positive entries, by Theorem 1.3, it follows that the left eigenvalues of $T^{-1}C_fT$ are the zeros of $f(q)$. Hence, all the zeros of $f(q)$ lie in the union of the balls given by (3). □

Proof. [Proof of Theorem 1.14] By definition the companion matrix C_P of $P(q) = q^n + a_{n-1}q^{n-1} + ... + a_1q + a_0$ is given by

$$C_P = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & 1 \\ -a_0 & -a_1 & -a_2 & ... & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

where $a_0, a_1, ..., a_{n-1}$ are quaternion coefficients. On applying Theorem 1.2 to C_P, it follows that all the left eigenvalues of C_P lie in the union of the balls $|q| \leq 1$ and $|q + a_{n-1}| \leq \sum_{\nu=0}^{n-2} |a_\nu|$. Since the left eigenvalues of C_P are the zeros of $P(q)$, it follows that all the zeros of $P(q)$ lie in the union of the balls

$$\{q \in \mathbb{H} : |q| \leq 1\} \text{ and } \{q \in \mathbb{H} : |q + a_{n-1}| \leq \sum_{\nu=0}^{n-2} |a_\nu|\}.$$

□

Proof. [Proof of Theorem 1.15] By Lemma 3.1, for any positive real number r, all the zeros of $P(q)$ lie in the ball

$$\Big\{q \in \mathbb{H} : |q| \leq \max\Big\{r, \sum_{\nu=0}^{n-1} \frac{|a_\nu|}{r^{n-\nu-1}}\Big\}\Big\}.$$

Replacing ν by $n - \nu$, it follows that all the zeros of $P(q)$ lie in the ball

$$\Big\{q \in \mathbb{H} : |q| \leq \max\Big\{r, \sum_{\nu=1}^{n} \frac{|a_{n-\nu}|}{r^{\nu-1}}\Big\}\Big\}. \tag{4}$$

Let $r = \max_{1 \leq \nu \leq n} |a_{n-\nu}|^{1/\nu}$, then $r \geq |a_{n-\nu}|^{1/\nu}$ and hence $|a_{n-\nu}|^{\frac{\nu-1}{\nu}} \leq r^{\nu-1}$, $\nu = 1, 2, ..., n$, so that

$$\frac{|a_{n-\nu}|}{r^{\nu-1}} \leq |a_{n-\nu}|^{1/\nu}, \quad 1 \leq \nu \leq n.$$

Therefore from (4), it follows that all the zeros of $P(q)$ lie in the ball

$$\{q \in \mathbb{H} : |q| \leq \sum_{\nu=1}^{n} |a_{n-\nu}|^{1/\nu}\}.$$

□

Declarations

Availability of data and material
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Funding
Not applicable.

Author's contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgement
The author's are highly grateful to the referees for their valuable suggestions and comments which has improved the quality of this work.

References

[1] A. L. Cauchy, *Exercises de mathématique*, in Oeuvres, **9** (1829), 122.
[2] N. Carney, R. Gardner, R. Keaton, A. Powers, *The Eneström-Kakeya theorem for polynomials of a quaternionic variable*, J. Approx. Theory **250** (2020), (Art. 105325), 1-10.
[3] F. O. Farid, Qing-Wen Wang, and Fuzhen Zhang, *On the eigenvalues of quaternion matrices*, Linear Multilinear Algebra, **59** (4):451-473, 2011.
[4] T.L. Hankins, *Sir William Rowan Hamilton*, The Johns Hopkins University Press, Baltimore, 1980.
[5] Liping Huang, Wasin So, *On left eigenvalues of a quaternionic matrix*, Linear Algebra and its Applications **323** (2001) 105-116.
[6] J.B. Kuipers, *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace, and Virtual Reality*, Princeton Univ. Press, Princeton, 2002.
[7] Gradimir V. Milovanović, Abdullah Mir, Abrar Ahmad, *On the zeros of a quaternionic polynomial with restricted coefficients*, Linear Algebra and its Applications, **653** (2022), 231-245, https://doi.org/10.1016/j.laa.2022.08.010.
[8] M. Parodi, *La localisation des valeurs caracteristiques des matrices et ses applications*, Gauthier-Villars, Paris, 1959.
[9] J. Turner, Private communication, January 16, 2006.
[10] Wasin So, *Quaternionic left eigenvalue problem*, Southeast Asian Bull. Math., **29** (3):555-565, 2005.
[11] R. M. W. Wood, *Quaternionic eigenvalues*, Bull. London Math. Soc., **17** (2): 137-138, 1985.
[12] F. Zhang, *Geršgorin type theorems for quaternionic matrices*, Linear Algebra and its Applications, **424** (2007), 139-153.