Filomat 38:9 (2024), 3021–3026
https://doi.org/10.2298/FIL2409021A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Functions simultaneously harmonic and $\mathcal{M}$-harmonic in the unit polydisc

Miloš Arsenović[a], Jelena Gajić[b,*]

[a]*Department of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia*
[b]*Faculty of Natural Sciences and Mathematics, University of Banja Luka, Mladena Stojanovića 2, 78000 Banja Luka, Bosnia and Herzegovina*

Abstract. We give a complete characterization of functions which are at the same time harmonic and $\mathcal{M}$-harmonic in the unit polydisc. These are precisely functions which are harmonic in each of the variables, or equivalently those that can be written as a linear combination of functions which are in each of the variables either holomorphic or conjugate holomorphic. As a consequence we obtain characterization of functions u such that both u and u^s (an integer $s \geq 2$) have this property. If we additionally assume that u is real valued, then u is constant. Our results stand in contrast to results known for such functions in the unit ball in $\mathbb{C}^n$.

1. Introduction

Motivation for this paper is the following result of Walter Rudin:

Theorem 1.1 ([6]). *A function defined on the unit ball in $\mathbb{C}^n$ is pluriharmonic if and only if it is harmonic and $\mathcal{M}$-harmonic (i.e. annihilated by the invariant Laplacian $\tilde{\Delta}$ on $\mathbb{B}^n$).*

We show that this is false if the unit ball is replaced by the unit polydisc $\mathbb{D}^n$. Moreover, we give a description of the space of all functions u which are simultaneously harmonic and $\mathcal{M}$-harmonic in $\mathbb{D}^n$.

We use standard notation and terminology. The open unit ball in $\mathbb{C}^n$ is denoted by $\mathbb{B}^n$, the unit polydisc in $\mathbb{C}^n$ is

$$\mathbb{D}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1, j = 1, \dots, n\}.$$

The space of all holomorphic functions in an open set $\Omega \subset \mathbb{C}^n$ is denoted by $H(\Omega)$. We need also standard first order differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

We sometimes write $\frac{\partial u}{\partial z_j}$ and $\frac{\partial u}{\partial \overline{z_j}}$ as u_{z_j} and $u_{\overline{z_j}}$.

2020 *Mathematics Subject Classification*. Primary 31B35; Secondary 31C10.
Keywords. Harmonic functions, invariant Laplacian, separately harmonic functions, polydisc
Received: 14 July 2023; Accepted: 22 September 2023
Communicated by Miodrag Mateljević
* Corresponding author: Jelena Gajić
Email addresses: milos.arsenovic@matf.bg.ac.rs (Miloš Arsenović), jelena.gajic.mm@gmail.com (Jelena Gajić)

Let us observe that
$$4\frac{\partial^2 u}{\partial z_j \partial \overline{z_j}} = \Delta_j u = \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2}$$
is the usual Laplacian with respect to real variables x_j and y_j.

For the unit ball in $\mathbb{C}^n$ the invariant Laplacian is given by the following formula:

$$(\tilde{\Delta} u)(z) = (1-|z|^2)\left(\Delta u(z) - 4\sum_{i,j=1}^{n} z_i \overline{z_j} \frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}\right).$$

The invariant Laplacian for the unit polydisc is given by the following formula:

$$(\tilde{\Delta} u)(z) = 4\sum_{j=1}^{n}(1-|z_j|^2)^2 \frac{\partial^2 u}{\partial z_j \partial \overline{z_j}} = \sum_{j=1}^{n}(1-|z_j|^2)^2 \Delta_j u(z). \tag{1}$$

A function $u \in C^2(\mathbb{D}^n)$ is said to be $\mathcal{M}$-harmonic if $\tilde{\Delta} u = 0$ in $\mathbb{D}^n$.

We say that a function $u \in C^2(\mathbb{D}^n)$ is separately harmonic if $\Delta_j u = 0$ for all $j = 1, \dots n$, i.e. if u is harmonic in each of the complex variables separately. We alert the reader that in some papers alternative term n-harmonic is used instead of separately harmonic.

A function $u \in C^2(\Omega)$ u is said to be pluriharmonic in a domain $\Omega \subset \mathbb{C}^n$ if it is harmonic on every complex line, see [5] for a discussion of pluriharmonic functions. Equivalently, $u \in C^2(\Omega)$ is pluriharmonic if

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z) = 0 \qquad \text{for all} \quad z \in \Omega, \quad j,k = 1,\dots,n. \tag{2}$$

2. Results

We have several vector spaces of functions defined on $\mathbb{D}^n$: $h(\mathbb{D}^n)$ consisting of harmonic functions, $\mathcal{M}h(\mathbb{D}^n)$ consisting of $\mathcal{M}$-harmonic functions, $sh(\mathbb{D}^n)$ consisting of separately harmonic functions and $ph(\mathbb{D}^n)$ consisting of pluriharmonic functions. The following propositions give some inclusions between these spaces.

Proposition 2.1. *If a function u is separately harmonic in $\mathbb{D}^n$, then u is harmonic and $\mathcal{M}$-harmonic.*

Proof. This follows from $\Delta = \sum_{j=1}^n \Delta_j$ and $\tilde{\Delta} = \sum_{j=1}^n (1-|z_j|^2)^2 \Delta_j$. $\square$

Example 2.2. *For $z = (z_1, z_2) \in \mathbb{D}^2$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ Stoll in his book [4], defines the function*

$$u(z_1, z_2) = \left[\frac{1-|z_1|^2}{|1-z_1\overline{\zeta}_1|^2}\right]^{\lambda_1+\frac{1}{2}} \left[\frac{1-|z_2|^2}{|1-z_2\overline{\zeta}_2|^2}\right]^{\lambda_2+\frac{1}{2}}$$

and proves that, when $\lambda_1, \lambda_2 \neq \frac{1}{2}$ and $\lambda_1^2 + \lambda_2^2 = \frac{1}{2}$, that this function is $\mathcal{M}$-harmonic but not separately harmonic in $\mathbb{D}^2$.

Proposition 2.3. *If a function u is pluriharmonic in $\mathbb{D}^n$, then u is separately harmonic. In particular it is harmonic and $\mathcal{M}$-harmonic.*

Proof. Every real valued pluriharmonc function u on $\mathbb{D}^n$ is locally the real part of a holomorphic function. Therefore every pluriharmonic function on $\mathbb{D}^n$ is locally represented as a sum $f + \bar{g}$, where f and g are analytic. This implies u is separately harmonic. $\square$

It is natural to ask if u is necessarily pluriharmonic under these conditions? The answer is no, as is seen from the following example.

Example 2.4. *The function $u(z_1, z_2) = z_1\bar{z}_2$ is harmonic and $\mathcal{M}$-harmonic in $\mathbb{D}^2$, but it is not pluriharmonic in $\mathbb{D}^2$.*

Indeed, for a fixed z_1 this function is conjugate analytic in z_2 and for a fixed z_2 it is analytic in z_1. Hence it is separately harmonic and therefore $\mathcal{M}$-harmonic. However, the restriction of u to the unit disc in the plane $\{(\lambda, \lambda) : \lambda \in \mathbb{C}\}$ produces a function $\phi(\lambda) = |\lambda|^2$, which is subharmonic but not harmonic. Alternatively:

$$\frac{\partial^2 u}{\partial z_1 \partial \overline{z_2}} = 1 \neq 0.$$

Clearly, this example can be extended to any complex dimension $n \geq 3$ by setting $u(z_1, z_2, \ldots z_n) = z_1\bar{z}_2 z_3 \cdots z_n$. This function is separately harmonic and therefore both harmonic and $\mathcal{M}$ - harmonic, but it is not pluriharmonic. Therefore the analogue of Theorem 1.1 for the polydisc fails in any dimension $n \geq 2$.

In order to present another class of functions, we need some notation. For $J \subset \{1, \ldots, n\} = I_n$ we define a mapping $C_J : \mathbb{D}^n \to \mathbb{D}^n$ by setting $C_J(z) = w$, where $w_j = z_j$ if $j \notin J$ and $w_j = \overline{z_j}$ if $j \in J$. If $\psi : \mathbb{D}^n \to \mathbb{C}$ is holomorphic then $\psi \circ C_J$ is holomorphic in variables z_j for $j \notin J$ and conjugate holomorphic in variables z_j for $j \in J$. Hence, for holomorphic ψ, $\psi \circ C_J$ is always separately harmonic. The set of all such functions is a vector space which we denote by $h_J(\mathbb{D}^n)$, that is:

$$h_J(\mathbb{D}^n) = \{\psi \circ C_J : \psi \in H(\mathbb{D}^n)\}. \tag{3}$$

The functions in $h_J(\mathbb{D}^n)$ are conjugate analytic in variables z_j with $j \in J$ and analytic in the remaining variables.

The vector sum of these function spaces is the following vector space:

$$\Sigma h(\mathbb{D}^n) = \left\{ \sum_{J \subset I_n} \varphi_J : \varphi_J \in h_J(\mathbb{D}^n) \text{ for all } J \subset I_n \right\}. \tag{4}$$

It contains all holomorphic and all conjugate holomorphic functions. Using Propositions 2.1, 2.3 and the above remarks we obtain the following inclusions:

$$ph(\mathbb{D}^n) \cup \Sigma h(\mathbb{D}^n) \subset sh(\mathbb{D}^n) \subset h(\mathbb{D}^n) \cap \mathcal{M}h(\mathbb{D}^n). \tag{5}$$

From the main result of this paper, Theorem 2.6 below, we can deduce that these inclusions are in fact equalities.

Every subspace $h_J(\mathbb{D}^n)$ of $\Sigma h(\mathbb{D}^n)$ is a function algebra, which means that it is closed under multiplication. However, the space $\Sigma h(\mathbb{D}^n)$ is not an algebra: for example $f(z_1, z_2) = z_1 + \bar{z}_1 z_2$ is in $\Sigma h(\mathbb{D}^2)$ but f^2 is not in $\Sigma h(\mathbb{D}^2)$. Proposition 2.9 below gives more information on this failure of $\Sigma h(\mathbb{D}^n)$ to be an algebra. Aditionaly, the vector sum

$$\Sigma h(\mathbb{D}^n) = \sum_{J \subset I_n} h_J(\mathbb{D}^n)$$

is not direct, even if we factor out constants. For example, if $K = J_1 \cap J_2 \neq \emptyset$, where $J_1, J_2 \subset I_n$ are distinct, then any function f in $h_K(\mathbb{D}^n)$ which depends only on variables z_j with $j \in K$ belongs to both spaces $h_{J_1}(\mathbb{D}^n)$ and $h_{J_2}(\mathbb{D}^n)$.

Let P_k stand for the space of all homogeneous polynomials in $x_1, y_1, \ldots, x_n, y_n$ (or, equivalently in $z_1, \overline{z_1}, \ldots, z_n, \overline{z_n}$), of degree k, $k \geq 0$ and set $P_{-1} = P_{-2} = 0$.

Let u be a real analytic function on $\mathbb{D}^n$. We have the following expansion:

$$u(z) = \sum_{k=0}^{\infty} u_k(z), \qquad z \in \mathbb{D}^n, \tag{6}$$

where $u_k \in P_k$, $k \geq 0$. The series above converges locally uniformly and can be differentiated term by term repeatedly. It is well known that harmonic functions are real analytic, the same is true for $\mathcal{M}$ -harmonic functions by theory of elliptic equations.

It is convenient to use multi-index notation adapted for our purposes. Namely, for non negative integers $p_1, q_1, \ldots, p_n, q_n$ we set $p = (p_1, \ldots, p_n)$,and $q = (q_1, \ldots, q_n)$. We interpret $p \cdot q$ as the usual dot product, i.e. as $\sum_j p_j q_j$. We call (p, q) *pure* if $p \cdot q = 0$, otherwise we call it *mixed*.
The order of multi-index (p, q) is $|(p, q)| = |p| + |q| = \sum_{j=1}^n p_j + q_j$. Let us set $p + 1_j = (p_1, \ldots, p_j + 1, \ldots, p_n)$, $q + 1_j = (q_1, \ldots, q_j + 1, \ldots, q_n)$, and $p - 1_j = (p_1, \ldots, p_j - 1, \ldots, p_n)$, $q - 1_j = (q_1, \ldots, q_j - 1, \ldots, q_n)$. Here we interpret $p_j - 1$ as zero, in the case $p_j = 0$, similarly for q_j. We define monomials $z^p \bar{z}^q$ of degree $|(p, q)|$ by

$$z^p \bar{z}^q = z_1^{p_1} \cdots z_n^{p_n} \bar{z}_1^{q_1} \cdots \bar{z}_n^{q_n}.$$

Note that $z^p \overline{z}^q$ is analytic (resp. belongs to $\Sigma h(\mathbb{D}^n)$) if and only if $q = 0$ (resp. (p, q) is pure). Using this notation in (6) we can write

$$u_k(z) = \sum_{|(p,q)|=k} c_{p,q} z^p \bar{z}^q, \qquad u(z) = \sum_{(p,q)} c_{p,q} z^p \bar{z}^q. \tag{7}$$

Now assume (p, q) is pure and set $J = \{j : q_j \neq 0\}$. Then the monomial $z^p \bar{z}^q$ belongs to $h_J(\mathbb{D}^n)$. Hence $u = \sum_{(p,q)} c_{p,q} z^p \bar{z}^q$ belongs to $\Sigma h(\mathbb{D}^n)$ if and only if $c_{p,q} = 0$ for all mixed (p, q).

Let us introduce the following second order linear partial differential operators acting on the space of real analytic functions. Namely, we set

$$\Lambda u(z) = \sum_{j=1}^n z_j \bar{z}_j \Delta_j u(z) = 4 \sum_{(p,q)} (p_1 q_1 + \cdots p_n q_n) c_{p,q} z^p \bar{z}^q \tag{8}$$

and

$$\Psi u(z) = \sum_{j=1}^n z_j^2 \bar{z}_j^2 \Delta_j u(z) = 4 \sum_{(p,q)} \sum_{j=1}^n p_j q_j c_{p,q} z^{p+1_j} \bar{z}^{q+1_j}. \tag{9}$$

Note that Λ maps the space P_k of all homogeneous polynomials of order k into itself, Δ maps P_k into P_{k-2} while Ψ maps P_k into P_{k+2}. We clearly have

$$\tilde{\Delta} u = \Delta u - 2\Lambda u + \Psi u. \tag{10}$$

Lemma 2.5. *Let u be a harmonic function in the unit polydisc and let*

$$u(z) = \sum_{(p,q)} c_{p,q} z^p \bar{z}^q \tag{11}$$

be its expansion into monomials in z_j and $\bar{z}_j$. Then u is $\mathcal{M}$-harmonic if and only if the coefficients $c_{p,q}$ satisfy the following recurrent formula:

$$(p \cdot q) c_{p,q} = \frac{1}{2} \sum_{j=1}^n (p_j - 1)(q_j - 1) c_{p-1_j, q-1_j}. \tag{12}$$

Proof. Since $\Delta u = 0$ we see, using (10), that $\tilde{\Delta} u = 0$ if and only if $\Psi u = 2\Lambda u$ in $\mathbb{D}^n$. However, we can rewrite (9) in the following form:

$$\Psi u(z) = 4 \sum_{j=1}^n \sum_{(p,q)} (p_j - 1)(q_j - 1) c_{p-1_j, q-1_j} z^p \bar{z}^q. \tag{13}$$

Uniqueness of expansion of real analytic functions combined with (13) and (8) gives equivalence of $\Psi u = 2\Lambda u$ with (12). $\square$

Theorem 2.6. $\Sigma h(\mathbb{D}^n) = sh(\mathbb{D}^n) = h(\mathbb{D}^n) \cap \mathcal{M}h(\mathbb{D}^n)$.

Proof. In view of (5) it suffices to prove that $h(\mathbb{D}^n) \cap \mathcal{M}h(\mathbb{D}^n) \subset \Sigma h(\mathbb{D}^n)$. Let us choose $u \in h(\mathbb{D}^n) \cap \mathcal{M}h(\mathbb{D}^n)$. Then we have

$$u(z) = \sum_{(p,q)} a_{p,q} z^p \bar{z}^q = \sum_{(p,q)\text{ pure}} a_{p,q} z^p \bar{z}^q + \sum_{(p,q)\text{ mixed}} a_{p,q} z^p \bar{z}^q = u_p(z) + u_m(z). \tag{14}$$

Clearly u_p belongs to $\Sigma h(\mathbb{D}^n)$ and u_m belongs to $h(\mathbb{D}^n) \cap \mathcal{M}h(\mathbb{D}^n)$. Let us prove that $u_m = 0$, which completes the proof. We have an expansion

$$u_m(z) = \sum_{(p,q)} c_{p,q} z^p \bar{z}^q, \qquad z \in \mathbb{D}^n, \tag{15}$$

where $c_{p,q} = 0$ for pure (p,q) and $c_{p,q} = a_{p,q}$ for mixed (p,q). Clearly $c_{p,q} = 0$ for $|(p,q)| \leq 1$. The coefficients $c_{p,q}$ in (15) satisfy relation (12). Therefore, if $c_{p,q} = 0$ for all $|(p,q)| = k$, then $c_{p,q} = 0$ for all mixed (p,q) such that $|(p,q)| = k+2$. Since $c_{p,q} = 0$ for all pure (p,q), this is precisely inductive step which completes the proof. $\square$

This theorem gives two descriptions of the space of functions which are simultaneously harmonic and $\mathcal{M}$-harmonic in $\mathbb{D}^n$. Hence, it can be seen as an analogue of Theorem 1.1 for the polydisc.

Now we turn to applications of Theorem 2.6. Related results, in the case of the unit ball, can be found in [1] and [2].

Note that if $u \in C^2(\mathbb{D}^n)$ is a real valued function, then for an integer $s \geq 2$, we have

$$\Delta_j u^s = s u^{s-1} \Delta_j u + s(s-1) u^{s-2} |\nabla_j u|^2 \tag{16}$$

for $1 \leq j \leq n$.

Proposition 2.7. *Let u be a real valued harmonic and $\mathcal{M}$ - harmonic function in $\mathbb{D}^n$ and suppose $u^s \in \mathcal{M}h(\mathbb{D}^n)$ for some integer $s \geq 2$. Then u is a constant function.*

Proof. By Theorem 2.6, u is separately harmonic, therefore by (16) $\Delta_j u^s = s(s-1)u^{s-2}|\nabla_j u|^2$. Since $\tilde{\Delta} u^s = 0$ we obtain

$$0 = s(s-1)u^{s-2} \sum_{j=1}^{n} (1 - |z_j|^2)^2 |\nabla_j u(z)|^2.$$

from this we get $\nabla_j u = 0$ for all $j = 1, 2, \ldots, n$ and this implies u is constant. $\square$

Proposition 2.8. *If u is a real $\mathcal{M}$-harmonic function in the unit polydisc such that $u^s \in h(\mathbb{D}^n) \cap \mathcal{M}h(\mathbb{D}^n)$ for some integer $s \geq 2$, then u is a constant function.*

Proof. By Theorem 2.6 the function u^s is separately harmonic and using formula (16), we get

$$u \Delta_j u + (s-1)|\nabla_j u|^2 = 0 \quad \text{for all} \quad 1 \leq j \leq n.$$

Since $\sum_{j=1}^{n} (1 - |z_j|^2)^2 \Delta_j u(z) = 0$, we have $\sum_{j=1}^{n} (1 - |z_j|^2)^2 |\nabla_j u(z)|^2 = 0$ and this gives again $\nabla_j u = 0$ for all $j = 1, 2, \ldots, n$, which suffices. $\square$

Proposition 2.9. *Let $u \in \Sigma h(\mathbb{D}^n)$ and let $s \geq 2$ be an integer. Then u^s is in $\Sigma h(\mathbb{D}^n)$ if and only if u belongs to $h_J(\mathbb{D}^n)$ for some $J \subset I_n$.*

Proof. We already noted that $h_J(\mathbb{D}^n)$ is an algebra, so the "if" part is trivial. Now we assume u and u^s are in $\Sigma h(\mathbb{D}^n)$ for some integer $s \geq 2$. By Theorem 2.6 u and u^s are separately harmonic, hence for each $1 \leq j \leq n$ we have

$$\frac{1}{4}\Delta_j(u^s) = su^{s-1}u_{z_j\overline{z_j}} + s(s-1)u^{s-2}u_{z_j}u_{\overline{z_j}} = 0.$$

Since $u_{z_j\overline{z_j}} = 0$, this gives $u^{s-2}u_{z_j}u_{\overline{z_j}} = 0$. This is a product of three real analytic functions, hence at least one of the factors is identically equal to zero. Excluding the trivial case when u is zero function, this means that u does not depend on z_j and $\overline{z_j}$ simultaneously. This condition is equivalent to $u \in \cup_{J\subset I_n} h_J(\mathbb{D}^n)$. □

It is an interesting problem to determine whether in the above propostion one can replace condition $u^s \in \Sigma h(\mathbb{D}^n)$ with a weaker condition $\tilde{\Delta} u^s = 0$?

Note that an $\mathcal{M}$ - harmonic function is separately harmonic if and only if it is harmonic, this is a part of Theorem 2.6. However, the following proposition is proved in [3].

Proposition 2.10. *([3], Corollary 1.2.) If $u \in C(\overline{\mathbb{D}^n})$ is $\mathcal{M}$-harmonic function then u is separately harmonic.*

Proposition 2.11. *Let $u \in C(\overline{\mathbb{D}^n})$, $\tilde{\Delta} u = 0$ and suppose $\tilde{\Delta} u^s = 0$ for some integer $s \geq 2$. Then u belongs to $h_J(\mathbb{D}^n)$ for some $J \subset I_n$.*

Proof. The proposition follows from Proposition 2.10, Theorem 2.6 and Proposition 2.9. □

Acknowledgements. The authors would like to thank the anonymous reviewers for their valuable comments.

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