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Hankel determinant and related problems for q-analogue of convex functions

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Abstract. In this article, by using the idea of q-analogue of the hyperbolic tangent functions we define a new class of q-convex functions. This study's main contribution is the development of sharp coefficient bounds in open unit disc, particularly the first five bounds, the Fekete-Szegö type functional, and the upper bounds of the third-order Hankel determinant. We also consider the Zalcman and generalized Zalcman conjectures for our newly defined class.

1. Introduction and Preliminaries

Let ξ represents an analytic function in the open unit disc $U = \{\tau : \tau \in \mathbb{C} \text{ and } |\tau| < 1\}$ is affirmatively true under the conditions $\xi(0) = 0$ and $\xi^{'}(0) = 1$, as well as for all similar types of functions occurring in class $\mathcal{A}$ and every $\xi \in \mathcal{A}$ has the following series of the form:

$$\xi(\tau) = \tau + \sum_{n=2}^{\infty} a_n \tau^n. \tag{1}$$

The analytic function ξ is called univalent in U, if there exists one to one correspondance between U and its image under ξ. The set of all such normalized univalent functions is denoted by $\mathcal{S}$. An analytic function w along with the conditions $w(0) = 0$,and $|w(\tau)| < 1$,is called Schwarz function. Let us suppose that two functions ξ and ξ_1 are analytic in U and ξ is subordinate to ξ_1 (denoted by $\xi \prec \xi_1$), if there exists a Schwarz function w such that

$$\xi(\tau) = \xi_1(w(\tau)). \tag{2}$$

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In particular if ξ_1 is univalent in U then $\xi \prec \xi_1$ if and only if $\xi(0) = \xi_1(0)$, and $\xi(U) \subset \xi_1(U)$.

Those functions that satisfy the requirements $p(0) = 1$ and $\Re(p(\tau)) > 0$, are called Caratheodory functions and are represented by the class $\mathcal{P}$. For every $p \in \mathcal{P}$, there is a series expansion of the form:

$$p(\tau) = 1 + \sum_{n=1}^{\infty} b_n \tau^n. \tag{3}$$

In 1992, Ma and Minda [11] made an interesting contribution and defined a general form of the family of univalent functions as follows:

$$\mathcal{S}^*(\varphi) = \left\{ \xi \in \mathcal{A} : \frac{\tau \xi'(\tau)}{\xi(\tau)} \prec \varphi(\tau) \right\} \tag{4}$$

and

$$C(\varphi) = \left\{ \xi \in \mathcal{A} : \frac{\left(\tau \xi'(\tau)\right)'}{\xi'(\tau)} \prec \varphi(\tau) \right\}. \tag{5}$$

In recent years, a number of sub-families of the normalized analytic functions have been studied as a special case of $\mathcal{S}^*(\varphi)$ and $C(\varphi)$. For example, Janowski [10] investigated the class of Janowski starlike $\mathcal{S}^*(L, M)$ and convex functions $C(L, M)$ for $-1 \leq M < L \leq 1$. Recently, Cho et al. [4] choose $\varphi(\tau) = 1 + \sin \tau$ and defined a class ($\mathcal{S}^*_{\sin}$) of starlike functions:

$$\mathcal{S}^*_{\sin} = \left\{ \xi \in \mathcal{A} : \frac{\tau \xi'(\tau)}{\xi(\tau)} \prec 1 + \sin \tau \right\}.$$

Mendiratta et al. [14] studied the function class $\mathcal{S}^*_e \equiv \mathcal{S}^*(e^\tau)$ of strongly starlike functions using the technique of subordination. Recently, Bano and Raza [3] chose $\varphi(\tau) = \cos \tau$.

The Hankel determinant is a similar coefficient problem to Fekete and Szegö. In the study of singularities and power series with integral coefficients, Hankel determinants are highly helpful.

For $\xi \in \mathcal{A}$, $n, j \in \mathbb{N}$, and $a_1 = 1$ the j^{th} Hankel determinant $\mathcal{H}_{j,n}(\xi)$ is defined by

$$\mathcal{H}_{j,n}(\xi) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \end{vmatrix}.$$

In 1966, Pommerenke [15] investigated the Hankel determinants for univalent starlike functions. Babalola [2] examined the third Hankel determinant for particular kinds of univalent functions in 2010. The authors in [21] investigated the second Hankel determinant for bi-univalent functions associated with nephroid domain. Using the Gegenbauer polynomials Srivastava et al. [22] find the Fekete Szegö functional for analytic functions satisfying a certain subordination condition. Recently Srivastava et al. [23] computed the best possible upper and the best possible lower estimates for the Hermitian-Toeplitz of the third order for a class of starlike functions associated with cardioid shape region in the right half plane. In [24] authors investigated third Hankel determinant involving Hohlov operator. The estimates of the fourth Hankel determinant for a class of analytic functions involving cardioid domain was investigated in [25]. In 2022 Shi et al. [19] investigated Hankel determinant for inverse functions subordinated to the exponential functions.

Jackson [8, 9] used the concept of fundamental q-calculus and defined the q-analogues of derivatives, and further this operator was utilized by Ismail et al. [7] to define a q-analogue of starlike functions. Historically, however, Srivastava initially provided a systematic usage of the $q-$calculus in the context of Geometric Function Theory and also used the basic (or $q-$) hypergeometric functions in a book chapter (see

[20, pp. 347 et seq.]). Several new subclasses of starlike and convex functions, as well as sharp bounds for second and third-order Hankel determinants, have been investigated recently by researchers using the operator D_q. Srivastava et al. for example, [26] determine the Hankel determinant for bi-univalent functions by using symmetric q-derivative operator. In [27], the authors achieved the same work for close-to-convex functions and different mathematicians applied the q-derivative operator and investigated some interesting properties for different classes of analytic functions (see, for example [13, 17, 33]) Concurrently, Srivastava et al. [29] explored Hankel and Toeplitz determinants connected to the generalized conic domain and identified a new subclass of q-starlike functions. Authors in [28] investigated Fekete Szegö inequalities for starlike functions by using symmetrical points in 2020. An upper bound of the third Hankel determinant by using k-Fibonacci numbers for a subclass of q-starlike functions is considered in [18]. In 2021, Srivastava et al. [30] generalized the class of q-starlike functions associated with exponential functions and determined the third order Hankel determinant. In this article, we tackle the sharp third order Hankel determinant and related problems for a subclass of q-convex functions. Now we discuss some basic definitions related to q-calculus.

Definition 1.1. *([8]). The q-derivative operator or (q-difference operator) for $\xi \in \mathcal{A}$ is defined by*

$$\begin{aligned} D_q\xi(\tau) &= \frac{\xi(\tau)-\xi(q\tau)}{(1-q)\tau}, \quad \tau \neq 0, q \neq 1, \\ &= 1+\sum_{n=2}^{\infty}[n]_q a_n \tau^{n-1}, \end{aligned} \tag{6}$$

where,

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{v=0}^{n-1} q^v.$$

Inspired fundamentally by the aforementioned works (see, [17, 26, 29]) on coefficient estimate problems, we establish a new class $C_{\tanh}(q)$ of q-convex functions connected to the q-analogue of the hyperbolic tangent function using the idea of subordination.

Definition 1.2. *A function $\xi \in \mathcal{S}$ belonging to the class $C_{\tanh}(q)$, if it satisfies the following subordination condition*

$$\frac{D_q\left(\tau D_q\xi(\tau)\right)}{D_q\xi(\tau)} \prec 1+\tanh(q\tau), \; \tau \in U. \tag{7}$$

The image of the function $1 + \tanh(q\tau)$ under U is symmetric about real axis. It turns from circular disk to oval shaped and then to eight-shaped region as the parameter $q \to 1-$, as shown in the Figure 1.

Remark 1.3. *When $q \to 1-$, then $C_{\tanh}(q) = C_{\tanh}$.*

2. A Set of Lemmas

Following lemmas will be used to investigate the sharp coefficient problems for the class $C_{\tanh}(q)$.

Lemma 2.1. *([6]). Let the function $p(\tau)$ of the form (3), then*

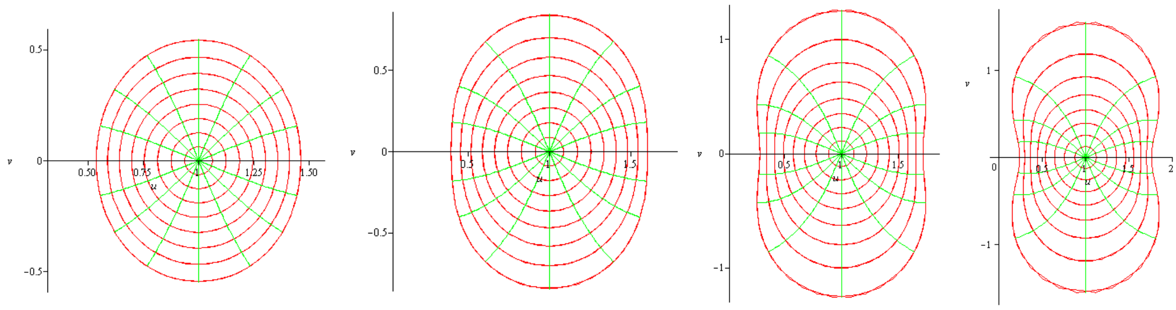
$$|b_n| \leq 2, \; n \in \mathbb{N}. \tag{8}$$

Also,

$$\left|b_n - \mu b_i b_{n-i}\right| \leq 2, \; n > i, \; \mu \in [0,1]. \tag{9}$$

The equality holds for

$$\xi(\tau) = \frac{(1+\tau)}{(1-\tau)}.$$



(a) For $q = 0.5$ (b) For $q = 0.7$ (c) For $q = 0.9$ (d) For $q = 0.9999$

Figure 1: The images of $1 + \tanh(qU)$ for different values of q.

Lemma 2.2. *([1]). Let the function $p \in \mathcal{P}$, be given by (3), then*

$$\left|b_3 - 2Qb_1b_2 + Wb_1^3\right| \leq 2,$$

if

$$0 \leq Q \leq 1, \text{ and } Q(2Q-1) \leq W \leq Q.$$

Lemma 2.3. *([6]). Let an analytic function $p(\tau)$ of the form (3), then*

$$2b_2 = b_1^2 + x(4 - b_1^2)$$

and

$$4b_3 = b_1^3 + 2(4 - b_1^2)b_1x - (4 - b_1^2)b_1x^2 + 2(4 - b_1^2)(1 - |x|^2)\tau,$$

where, $x, \tau \in \mathbb{C}$, with $|\tau| \leq 1$ and $|x| \leq 1$.

Lemma 2.4. *([16]). Consider the function $p \in \mathcal{P}$ of the form (3), $0 < Q_1 < 1$, $0 < Q_2 < 1$ and*

$$\begin{aligned} & 8Q_1(1-Q_2)\left\{(Q_2Q_3 - 2Q_4)^2 + (Q_2(Q_1+Q_2) - Q_3)^2\right\} + Q_2(1-Q_2)(Q_3 - 2Q_1Q_2)^2 \\ \leq\ & 4Q_2^2Q_1(1-Q_2)^2(1-Q_1). \end{aligned} \tag{10}$$

Then

$$\left|Q_4b_1^4 + Q_1b_2^2 + 2Q_2b_1b_3 - \frac{3}{2}Q_3b_1^2b_2 - b_4\right| \leq 2. \tag{11}$$

Lemma 2.5. *[5]. Let $\overline{E} = \{\tau \in \mathbb{C} : \tau \leq 1\}$, and for real numbers X, Y, Z, let*

$$\Upsilon(X, Y, Z) := \max\left\{\left|X + Yx + Zx^2\right| + 1 - |x|^2,\ \ x \in \overline{E}\right\}.$$

If $XZ \geq 0$, then

$$\Upsilon(X, Y, Z) = \begin{cases} |X| + |Y| + |Z|, & \text{if } |Y| \geq 2(1 - |Z|) \\ 1 + |X| + \frac{Y^2}{4(1-|Z|)}, & \text{if } |Y| < 2(1 - |Z|). \end{cases}$$

3. Main Results

Theorem 3.1. *If ξ be an analytic function of the form (1) belongs to $C_{\tanh}(q)$, then*

$$\begin{aligned}|a_2| &\leq \frac{1}{[2]_q},\\ |a_3| &\leq \frac{1}{[3]_q[2]_q},\\ |a_4| &\leq \frac{1}{[4]_q[3]_q},\\ |a_5| &\leq \frac{1}{[5]_q[4]_q}.\end{aligned}$$

All bounds of Theorem 3.1 are sharp for the functions given in (21)-(24).

Proof. Let $\xi \in C_{\tanh}(q)$, and satisfies (7), then from (2) we have

$$\frac{D_q\left(\tau D_q\xi(\tau)\right)}{D_q\xi(\tau)} = 1+\tanh(q(w(\tau))). \tag{12}$$

Let

$$\begin{aligned}w(\tau) &= \frac{p(\tau)-1}{p(\tau)+1}\\ &= \frac{1}{2}b_1\tau+\frac{1}{2}\left(b_2-\frac{1}{2}b_1^2\right)\tau^2+\frac{1}{2}\left(b_3-b_1b_2+\frac{1}{4}b_1^3\right)\tau^3+\cdots.\end{aligned} \tag{13}$$

In view of (12) and (13), we obtain the following series

$$\begin{aligned}&1+\tanh(q(w(\tau)))\\ =\ &1+\frac{q}{2}b_1\tau+q\left(\frac{1}{2}b_2-\frac{1}{4}b_1^2\right)\tau^2\\ &+q\left(\frac{1}{2}b_3-\frac{1}{2}b_1b_2+\frac{\left(3-q^2\right)}{24}b_1^3\right)\tau^3+\\ &+\left(q\left(\frac{1}{2}b_4-\frac{1}{2}b_1b_3-\frac{1}{4}b_1^2-\frac{1}{16}b_1^4+\frac{3}{8}b_1^2b_2\right)-\frac{q^2}{4}\left(\frac{b_1^2b_2}{2}-\frac{b_1^4}{4}\right)\right)\tau^4+....\end{aligned} \tag{14}$$

Similarly

$$\begin{aligned}&\frac{D_q\left(\tau D_q\xi(\tau)\right)}{D_q\xi(\tau)}\\ =\ &1+[2]_q\left([2]_q-1\right)a_2\tau+\left\{[3]_q\left([3]_q-1\right)a_3-[2]_q^2\left([2]_q-1\right)a_2^2\right\}\tau^2\\ &+\left\{[4]_q\left([4]_q-1\right)a_4-[3]_q[2]_q\left\{[2]_q+[3]_q-2\right\}a_2a_3+[2]_q^3\left([2]_q-1\right)a_2^3\right\}\tau^3\\ &+\left\{[5]_q\left([5]_q-1\right)a_4-[4]_q[2]_q\left\{[4]_q+[2]_q-2\right\}a_2a_4-[3]_q^2\left([3]_q-1\right)a_3^2\right.\\ &\left.+[2]_q^2[3]_q\left\{2[2]_q+[3]_q-3\right\}a_3a_2^2-[2]_q^4\left([2]_q-1\right)a_2^4\right\}\tau^4+....\end{aligned} \tag{15}$$

Equating the corresponding coefficients of (14) and (15), we obtain the following series

$$a_2 = \frac{b_1}{2\,[2]_q}, \tag{16}$$

$$a_3 = \frac{b_2}{2\,[3]_q\,[2]_q}, \tag{17}$$

$$a_4 = \frac{1}{2\,[4]_q\,[3]_q}\left\{b_3 - \frac{q}{2[2]_q}b_1b_2 - \frac{q^2}{12}b_1^3\right\}, \tag{18}$$

$$a_5 = \frac{1}{2\,[5]_q\,[4]_q}\left\{\begin{array}{c} \frac{-q[2]_q}{2[3]_q}b_1b_3 - \frac{q\left(q^3-2q^2-q-3\right)}{24[3]_q}b_1^4 - \frac{q}{2[2]_q}b_2^2 \\ +b_4 - \frac{\left(q^2+1\right)}{4[3]_q}b_1^2b_2 \end{array}\right\}. \tag{19}$$

Applying the Lemma 2.1 on (16), yields

$$|a_2| \le \frac{1}{[2]_q}.$$

Applying the Lemma 2.1 on (17), yields

$$|a_3| \le \frac{1}{[2]_q\,[3]_q}.$$

Now consider (18), as

$$|a_4| = \frac{q}{2\,[4]_q\,[3]_q}\left|b_3 - \frac{q}{2[2]_q}b_1b_2 - \frac{q^2}{12}b_1^3\right|,$$

Assuming that, $Q_q = \frac{q}{4[2]_q}$ and $W_q = -\frac{q^2}{12}$, then

$$Q_q(2Q_q-1) - W_q = \frac{q\left(2q^3+4q^2-q-6\right)}{24\,[2]_q^2} < 0, \text{ for } q \in (0,1)$$

and

$$Q_q - W_q = \frac{q\left(q^2+q+3\right)}{12\,[2]_q} > 0, \text{ for } q \in (0,1).$$

Clearly, $Q_q(2Q_q-1) \le W_q \le Q_q$, when $q \in (0,1)$, so by the applications of Lemma 2.2, we obtain

$$|a_4| \le \frac{1}{[4]_q\,[3]_q}.$$

Now from (19), consider

$$|a_5| = \frac{1}{2\,[5]_q\,[4]_q}\left|Q_4b_1^4 + Q_1b_2^2 + 2Q_2b_1b_3 - \frac{3}{2}Q_3b_1^2b_2 - b_4\right|,$$

where

$$Q_1 = \frac{q}{2\,[2]_q}, \quad Q_2 = \frac{q(1+q)}{2\,[3]_q}$$

$$Q_3 = \frac{-\left(q^2+1\right)}{6\,[3]_q}.$$

$$Q_4 = \frac{q\left(q^3-2q^2-q-3\right)}{24\,[3]_q}.$$

Clearly, $0 < Q_1 < 1$, $0 < Q_2 < 1$, for $q \in (0,1)$ and after simple calculation, we have

$$\begin{aligned} & 8Q_1(1-Q_2)\left\{(Q_2Q_3-2Q_4)^2+(Q_2(Q_1+Q_2)-Q_3)^2\right\} \\ & Q_2(1-Q_2)(Q_3-2Q_1Q_2)^2-4Q_2^2Q_1(1-Q_2)^2(1-Q_1) \\ = & \frac{1}{2[5]_q[4]_q}\{\varphi_1(q)-\varphi_2(q)\} \\ = & \Psi(q,Q_1,Q_2,Q_3), \end{aligned}$$

where

$$\begin{aligned} \varphi_1(q) &= \frac{q}{144[3]_q^4[2]_q^2}\begin{pmatrix} 100q^{15}+784q^{14}+3022q^{13}+7634q^{12}+13940q^{11} \\ +19342q^{10}+20864q^9+17666q^8+11714q^7+6024q^6 \\ +2349q^5+680q^4+156q^3+34q^2+7q+2 \end{pmatrix} \\ \varphi_2(q) &= \frac{q(1+q)\left(q^2+q+2\right)^3}{16[3]_q^5} > 0, \text{ for } q \in (0,1). \end{aligned}$$

Clearly $0 < Q_1 < 1$, $0 < Q_2 < 1$ and $\varphi_1(q) \le \varphi_2(q)$ for $q \in (0,1)$. Therefore we have $\Psi(q,Q_1,Q_2,Q_3) \le 0$, when $0 < q < 1$.
Now by using the Lemma 2.4, we have

$$|a_5| \le \frac{1}{[5]_q[4]_q}.$$

For sharpness, consider the function $\xi_n : U \longrightarrow \mathbb{C}$, defined by

$$\frac{D_q\left(\tau D_q\xi_n(\tau)\right)}{D_q\xi_n(\tau)} = 1+\tanh(q\tau^n),\ n = 2,3,4,5. \tag{20}$$

The results are sharp, as shown by the following functions

$$\xi_2(\tau) = \tau + \frac{q}{[2]_q\left([2]_q-1\right)}\tau^2+..., \tag{21}$$

$$\xi_3(\tau) = \tau + \frac{q}{[3]_q\left([3]_q-1\right)}\tau^3+..., \tag{22}$$

$$\xi_4(\tau) = \tau + \frac{q}{[4]_q\left([4]_q-1\right)}\tau^4+..., \tag{23}$$

$$\xi_5(\tau) = \tau + \frac{q}{[5]_q\left([5]_q-1\right)}\tau^5+.... \tag{24}$$

□

Zalcman and Generalized Zalcman Conjecture

In 1960, Zalcman defined the conjecture for univalent functions. He stated that every $\xi \in \mathcal{S}$ of the form (1) satisfies the inequality:

$$\left|a_n^2-a_{2n-1}\right| \le (n-1)^2,\ \ n \ge 2. \tag{25}$$

In 1999, Ma [12] proved generalized version of Zalcman conjecture and stated that every univalent function $\xi \in \mathcal{S}$ satisfies the following inequality:

$$|a_na_i-a_{n+i-1}| \le (n-1)(i-1),\ \ \forall\, i,n \in \mathbb{N},\ n \ge 2,\ i \ge 2. \tag{26}$$

Furthermore, we have estimated the bounds of the third-order Hankel determinant for the class $C_{\tanh}(q)$ for different values of n and i. For $n = 2$, the inequality (25) has the form

$$\left|a_2^2 - a_3\right| \leq 1.$$

Theorem 3.2. *If $\xi \in C_{\tanh}(q)$, where ξ is of the form (1), then*

$$\left|a_2^2 - a_3\right| \leq \frac{1}{[2]_q [3]_q}. \tag{27}$$

The inequality (27) is sharp for the function ξ_3 given in (22).

Proof. From (16) and (17), consider

$$\left|a_3 - a_2^2\right| = \frac{1}{2[3]_q [2]_q} \left|b_2 - vb_1^2\right|$$

where

$$v = \frac{1 + q + q^2}{2[2]_q}.$$

Since, $0 < q < 1$, therefore $v \in (0, 1)$. Now, using the Lemma 9, for $n = 2$, $i = 1$, we obtain (27).

For sharpness, consider the function ξ_3 given in (22) such that

$$a_2 = 0, \text{ and } a_3 = \frac{q}{[3]_q\left([3]_q - 1\right)}.$$

□

Take $n = 3$, $i = 2$, in the inequality (26), then we have $|a_4 - a_2a_3| \leq 2$. Now we discuss it as follows:

Theorem 3.3. *If $\xi \in C_{\tanh}(q)$, where ξ is the form (1), then*

$$|a_4 - a_2a_3| \leq \frac{1}{[3]_q [4]_q}, \quad 0 < q < 1.$$

The inequality is sharp, for the function ξ_4 defined in (23).

Proof. From (16), (17), and (18), we have

$$|a_4 - a_2a_3| = \frac{1}{2[4]_q [3]_q} \left|b_3 - 2\beta b_1 b_2 + \delta b_1^3\right|.$$

Assuming the values

$$\beta = \frac{q^3 + 2q^2 + 2q + 1}{4[2]_q},$$

$$\delta = -\frac{q^2}{12}.$$

By using these values, we get

$$\beta(2\beta - 1) - \delta = \frac{\left(5q^4 + 4q^3 - q^2 - 6q - 3\right)}{24[2]_q^2} < 0, \text{ when } 0 < q < 1,$$

and

$$\beta-\delta=\frac{\left(q^5+3q^4+3q^3+4q^2+3q+3\right)}{12\,[2]_q^3}>0,\ \text{when}\ \ 0<q<1,$$

which shows that

$$\beta\,(2\beta-1)<\delta<\beta.$$

Thus, using Lemma 2.2, we get

$$|a_4-a_2a_3|\leq\frac{1}{[3]_q\,[4]_q}.$$

The equality holds for the extremal function

$$\xi_4(\tau)=\tau+\frac{q}{[4]_q\left([4]_q-1\right)}\tau^4+....$$

□

Theorem 3.4. *If $\xi\in\mathcal{C}_{\tanh}(q)$, where ξ is the form (1), then*

$$\left|a_3^2-a_5\right|\leq\frac{1}{[5]_q\,[4]_q},\quad 0<q<1.$$

The inequality is sharp, for the function ξ_5 given in (24).

Proof. From (17) and (19), consider

$$\left|a_3^2-a_5\right|=\frac{1}{2\,[5]_q\,[4]_q}\left|Q_4b_1^4+Q_1b_2^2+2Q_2b_1b_3-\frac{3}{2}Q_3b_1^2b_2-b_4\right|,$$

where

$$\begin{aligned}Q_1&=\frac{q^6+2q^5+4q^4+5q^3+2q^2+2q+1}{2\,[3]_q^2\,[2]_q},\\ Q_2&=\frac{q(1+q)}{[3]_q},\quad Q_3=\frac{-q\left(q^2+1\right)}{6\,[3]_q}.\\ Q_4&=\frac{q^4-2q^3-q^2-3q}{24\,[3]_q},\end{aligned}$$

Hence, $0<Q_1<1$, $0<Q_2<1$, for $q\in(0,1)$. Consider

$$\begin{aligned}&8Q_1(1-Q_2)\left\{(Q_2Q_3-2Q_4)^2+(Q_2\,(Q_1+Q_2)-Q_3)^2\right\}\\ &+Q_2\,(1-Q_2)\,(Q_3-2Q_1Q_2)^2-4Q_2^2Q_1\,(1-Q_2)^2\,(1-Q_1).\\ =\;&\Psi(q),\end{aligned}$$

where

$$\Psi(q)=\frac{-1}{72\,[3]_q^{10}\,[2]_q^2}\left\{\Psi_1(q)+\Psi_2(q)+\Psi_3(q)+\Psi_4(q)+\Psi_5(q)\right\}$$

$$\begin{aligned}
\Psi_1(q) &= q^{26}+6q^{25}+82q^{24}+598q^{23}+2860q^{22}+9322q^{21}+20852q^{20}+26330q^{19},\\
\Psi_2(q) &= -15588q^{18}-191242q^{17}-617479q^{16}-1379276q^{15}-2436965q^{14}-3571256q^{13},\\
\Psi_3(q) &= -4436315q^{12}-4722224q^{11}-4327491q^{10}-3416210q^{9}-2316068q^{8}-1339176q^{7},\\
\Psi_4(q) &= -652417q^{6}-262586q^{5}-84549q^{4}-20610q^{3}-3415q^{2}-294q-1.
\end{aligned}$$

This calculation shows that $\Psi(q)<0$, for $q\in(0,1)$. Now by using Lemma 2.4, we obtain

$$\left|a_3^2-a_5\right| \le \frac{1}{[5]_q\,[4]_q},\quad 0<q<1.$$

The equality holds for the extremal function

$$\xi_5(\tau)=\tau+\frac{q}{[5]_q\left([5]_q-1\right)}\tau^4+\dots.$$

□

Theorem 3.5. *If $\xi\in C_{\tanh}(q)$, where ξ is the form (1), then*

$$|a_2a_4-a_5| \le \frac{1}{[5]_q\,[4]_q},\quad 0<q<1.$$

The inequality is sharp, for the function ξ_5 given in (24).

Proof. From (16)(18) and (19), consider

$$|a_2a_4-a_5| = \frac{1}{2\,[5]_q\,[4]_q}\left|Q_4b_1^4+Q_1b_2^2+2Q_2b_1b_3-\frac{3}{2}Q_3b_1^2b_2-b_4\right|,$$

where

$$\begin{aligned}
Q_1 &= \frac{q}{2\,[2]_q},\quad Q_2=\frac{1+q+q^2}{4\,(q+1)}\\
Q_3 &= \frac{q^2-1}{6\,(q+1)},\\
Q_4 &= \frac{-q\left(q^3-q^2+2q+3\right)}{24\,[2]_q}.
\end{aligned}$$

It is clear that, $0<Q_1<1$, $0<Q_2<1$, for $q\in(0,1)$. By taking

$$\begin{aligned}
8Q_1(1-Q_1) &= \frac{2q\,(q+2)}{[2]_q^2},\\
(Q_2Q_3-2Q_4)^2 &= \frac{q^2\left(q^5+q^4+3q^3+8q^2+10q+3\right)^2}{144\,[2]_q^6},\\
(Q_2\,(Q_1+Q_2)-Q_3)^2 &= \frac{\left(3q^4+4q^3+7q^2+12q+3\right)^2}{2304\,[2]_q^4},\\
Q_2\,(1-Q_2)\,(Q_3-2Q_1Q_2)^2 &= \frac{-q^4\left(6q^2+5q+6\right)^2(1+q+q^2)}{36\,[2]_q^6},\\
4Q_2^2Q_1\,(1-Q_2)^2\,(1-Q_1) &= \frac{[3]_q^2\,q^5\,(q+2)\,(1+q+q^2)^2}{[2]_q^6}.
\end{aligned}$$

By simple calculation it is clear that

$$\begin{aligned} & 8Q_1(1-Q_2)\left\{(Q_2Q_3-2Q_4)^2+(Q_2(Q_1+Q_2)-Q_3)^2\right\}+Q_2(1-Q_2)(Q_3-2Q_1Q_2)^2 \\ \leq\ & 4Q_2^2Q_1(1-Q_2)^2(1-Q_1). \end{aligned}$$

Now by using Lemma 2.4, we obtain the required result

$$|a_2a_4-a_5| \leq \frac{1}{[5]_q[4]_q}, \quad 0<q<1.$$

□

In the following result, the second Hankel determinant $H_{2,2}(\xi)$ will be proved.

Theorem 3.6. *If $\xi \in C_{\tanh}(q)$, where ξ is of the form (1), then*

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{[3]_q^2[2]_q^2}.$$

Proof. By using of (16), (17), and (18), to get

$$\left|a_2a_4-a_3^2\right| = \frac{1}{4[4]_q[3]_q[2]_q}\left|-\frac{q^2}{12}b_1^4-\frac{q}{2[2]_q}b_1^2b_2+b_1b_3-\frac{(1+q+q^2+q^3)}{[3]_q[2]_q}b_2^2\right|,$$

By using the Lemma 2.3, and assume that $b=b_1$, $(0\leq b\leq 2)$, so that

$$\left|a_2a_4-a_3^2\right| = \frac{1}{16[4]_q[3]_q[2]_q}\left|\begin{array}{c} \frac{-q^2\left(q^3+2q^2+5q+1\right)}{3[3]_q[2]_q}b^4-\frac{q\left(q^2-q-1\right)\left(4-b^2\right)b^2x}{[3]_q[2]_q} \\ -\frac{\left(b^2q+4q^2+4\right)\left(4-b^2\right)x^2}{[3]_q}+2b\left(4-b^2\right)\left(1-|x|^2\right)\tau \end{array}\right|, \tag{28}$$

Using $|\tau|\leq 1$ and $|x|\leq 1$, and applying the triangle inequality and take $b=2$, we obtain

$$\left|a_2a_4-a_3^2\right| = \frac{q^2\left(q^3+2q^2+5q+1\right)}{3[4]_q[3]_q^2[2]_q^2} < \frac{1}{[4]_q[3]_q[2]_q^2}$$

By taking $b=0$, then

$$\left|a_2a_4-a_3^2\right| = \frac{4\left(4q^2+4q\right)|x|^2}{[3]_q} \leq \frac{1}{[4]_q[3]_q[2]_q^2}.$$

Suppose that $b\in(0,1)$, then by applying the triangle inequality, to get

$$\left|a_2a_4-a_3^2\right| \leq \frac{b\left(4-b^2\right)}{16[4]_q[3]_q[2]_q}\Psi\left(X_q,Y_q,Z_q\right)$$

where

$$\begin{aligned} X_q &= \frac{-q^2\left(q^3+2q^2+5q+1\right)}{3[3]_q[2]_q} \\ Y_q &= \frac{-q\left(q^2-q-1\right)b}{[3]_q[2]_q}, \\ Z_q &= \frac{-\left(b^2q+4q^2+4\right)}{[3]_q b}. \end{aligned}$$

Clearly, $X_qZ_q > 0$ for $q \in (0,1)$, next to show is $\left|Y_q\right| \geq 2\left(1-\left|Z_q\right|\right)$, or $\left|Y_q\right| - 2\left(1-\left|Z_q\right|\right) \geq 0$.

For this consider the function

$$\phi(b) = b^2q^3 + b^2q^2 - 2bq^3 + b^2q - 4bq^2 + 8q^3 - 4bq + 8q^2 - 2b + 8q + 8$$

From above we see that $\phi^{'}$ is increasing and

$$\max \phi^{'}(b) = \phi^{'}(2) = 2q^3 - 2 < 0, \text{ for } q \in (0,1).$$

Hence ϕ is decreasing function for $b \in (0,2)$ and

$$\min \phi(b) = \phi(2) = 8q^3 + 4[3]_q > 0 \text{ for } q \in (0,1)$$

Thus $|B| - 2(1-|C|) > 0$, and by using the Lemma 2.5, we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \frac{b\left(4-b^2\right)}{16[4]_q[3]_q[2]_q}\left\{\left|X_q\right| + \left|Y_q\right| + \left|Z_q\right|\right\}.$$

Or

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &\leq \frac{1}{48[4]_q[3]_q^2[2]_q^2}\left\{\begin{array}{c}\left(q^5+2q^4+8q^3-5q^2-6q\right)b^4 \\ +\left(-24q^3+12q^2+12q-12\right)b^2 + 48[4]_q\end{array}\right\}, \\ &= g(b)\end{aligned}$$

and

$$g^{'}(b) = \frac{1}{48[4]_q[3]_q^2[2]_q^2}\left(\begin{array}{c}4\left(q^5+2q^4+8q^3-5q^2-6q\right)b^3 \\ +2\left(-24q^3+12q^2+12q-12\right)b\end{array}\right).$$

Clearly, $g^{'}(b) < 0$ for $q \in (0,1)$, Therefore $g(b)$ is decreasing and hence

$$g(b) < g(0).$$

Thus

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{[3]_q^2[2]_q^2}.$$

The result is sharp for the function ξ_3 given in (20). □

Theorem 3.7. *If $\xi \in C_{\tanh}(q)$, where ξ is of the form (1), and $0 < q < 1$, then*

$$\left|H_{3,1}(\xi)\right| \leq \frac{1}{[2]_q[3]_q[5]_q[4]_q} + \frac{1}{[3]_q^2[4]_q^2} + \frac{1}{[3]_q^3[2]_q^3}.$$

Proof. We know that

$$\left|H_{3,1}(\xi)\right| \leq |a_5|\left|a_3 - a_2^2\right| + |a_4||a_4 - a_2a_3| + |a_3|\left|a_2a_4 - a_3^2\right|.$$

Using the Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.6, we have required result when $0 < q < 1$,

$$\left|H_{3,1}(\xi)\right| \leq \frac{1}{[2]_q[3]_q[5]_q[4]_q} + \frac{1}{[3]_q^2[4]_q^2} + \frac{1}{[3]_q^3[2]_q^3}.$$

□

4. Conclusion

In the current paper, a subclass of q-convex functions connected to q–analogous of tangent hyperbolic functions was taken into consideration. For this class, in Theorem 3.1, first five sharps coefficients bounds are investigated. We used Zalcman and Generalized Zalcman conjecture with Lemma 2.1 and Lemma 2.2 and then investigated Theorem 3.2 and Theorem 3.3. By using the Lemma 2.4, we determined the sharp results in Theorem 3.4 and Theorem 3.5. In Theorem 3.6, the second Hankel determinant $H_{2,2}(\xi)$ is proved and in Theorem 3.7, third-order Hankel determinant bounds are established. All of the estimates which we have been proved in this study are sharp.

In this article as well as in a remarkably large number of other earlier q-investigations on the subject for $0 < q < 1$ can easily (and possibly trivially) be translated into the corresponding (p, q)-analogues (with $0 < q < p \leq 1$) by applying some obvious parametric and argument variations of the types indicated above, the additional parameter p being redundant, see for example, ([31], pp 340) and ([32]), section 3, pp 1505 – 1506).

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