Filomat 38:9 (2024), 3055–3067
https://doi.org/10.2298/FIL2409055B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Well-posedness and stability result for a swelling porous elastic system with neutral delay and porous damping

Marwa Boudeliou[a], Brahim Kilani[a], Abdelhak Djebabla[a,*]

[a]*Numerical Analysis, Optimization and Statistics Laboratory, Badji Mokhtar-Annaba University, Algeria*

Abstract. This study takes into account a one-dimensional swelling porous elastic system with neutral delay and porous damping acting on the second equation. We verify the existence and uniqueness of the solution using Faedo-Galerkin approach, and we prove that the porous damping dissipation is powerful enough to stabilize the system exponentially even in the presence of neutral delay using the multiplier method.

1. Introduction

In recent decades, the theory of mixtures of solids has received a lot of attention by researchers and an increasing interest has been oriented to the study of the qualitative properties of solutions related to mixtures composed by two interacting continua, see [1]-[2]-[3]. Note that one of the first works in continuity theory applied to mixtures were the contributions of [4]-[5]. Eringen in [6] developed the first mathematical model consisting of three partial differential equations that give form to the problem of saturation of porous solids by the action of a gas or fluids. This mathematical model represents, in fact, the theory of mixtures for the saturation of porous solids by the action of a gas or fluid. Then, several mathematical results on the existence, uniqueness and asymptotic behaviour for this theory have been developed by many researchers see [7]-[8]. Alves et al. in [9], considered the one-dimensional system composed of a mixture of two thermoelastic solids. By using the semi-group method, they established a necessary and sufficient condition over the coefficients of the system to get the exponential stability of the corresponding semigroup. As established by Ieşan [10], towards the end of the 19th century and simplified by Quintanilla in [11], the basic field equations for the theory of swelling of one-dimensional porous elastic soils are given by

$$\begin{aligned} \rho_z z_{tt} &= P_{1x} - G_1 + H_1, \\ \rho_u u_{tt} &= P_{2x} + G_2 + H_2, \end{aligned} \tag{1}$$

here P_i denotes the partial tensions, H_i are the external forces and G_i are internal body forces associated with the dependent variables z and u, respectively. And we assume that the constitutive equations of partial

2020 *Mathematics Subject Classification*. 93D20, 35B40.
Keywords. Swelling; neutral delay; porous damping; Faedo-Galerkin method; Lyapunov functional; exponential stability.
Received: 06 June 2023; Revised: 07 September 2023; Accepted: 25 September 2023
Communicated by Marko Nedeljkov
* Corresponding author: Abdelhak Djebabla
Email addresses: marwa.boudeliou@univ-annaba.org (Marwa Boudeliou), kilbra2000@yahoo.fr (Brahim Kilani), adjebabla@yahoo.com (Abdelhak Djebabla)

tensions are given by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}}_{=A} \begin{pmatrix} z_x \\ u_x \end{pmatrix}, \tag{2}$$

where $a_1,\ a_3 > 0$ and $a_2 \neq 0$ is a real number. The matrix A is positive definite such that $a_1 a_3 > a_2^2$.
Ramos et al. [12] studied the case $H_1 = 0$ and $H_2 = -\gamma(t)\, g(u_t)$ where $\gamma(t)\, g(u_t)$ is a nonlinear damping term, which acts only in the second equation. Using the multiplier method and some properties of convex functions, they established an exponential decay rate provided that the wave speeds of the system are equal. Similarly, in [13] Wang and Guo considered (1) with initial and some mixed boundary conditions and took

$$G_1 = G_2 = 0, \quad H_1 = -\rho_u \gamma(x) u_t, \quad H_2 = 0, \tag{3}$$

where $\gamma(x)$ is an internal viscous damping function with positive mean. Using the Riesz basis approach, they proved that the whole system can be exponentially stabilized by a single internal viscous damping. For more interesting results on swelling porous elastic soils, we refer the reader to [14]-[15].

Now, on the other hand, the scientific community is observing a considerable growth interest in problems involving time delays, because most phenomena naturally depend not only on the current state but also on some past events see [16]-[17]. Tatar in [18], considered the following damped wave equation with a neutral delay.

$$u_{tt} = u_{xx} - u_t - \int_0^t h(t-s)\, u_{tt}(s)\, ds,\ x \in (0,1),\ t > 0,$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,1), \\ u(0,t) = u(1,t) = 0,\ t \geq 0, \end{cases}$$

he demonstrated that the solution decays exponentially under certain conditions on the kernel h. In many cases, it has been shown that delay is a source of instability unless additional conditions or control terms are added, as in the work of Kerbal and Tatar [19], where they investigated the following neutrally delayed viscoelastic Timoshenko beam system

$$\begin{cases} \varphi_{tt} = (\varphi_x + \psi)_x, \\ \left(\psi_t + \displaystyle\int_0^t k(t-s)\, \psi_t(s)\, ds\right)_t = \psi_{xx} - \displaystyle\int_0^t g(t-s)\, \psi_{xx}(s)\, ds - (\varphi_x + \psi), \end{cases}$$

for $x \in (0,1)$, $t > 0$ with initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,1), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \quad x \in (0,1). \end{cases}$$

$$\left\{ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad t \geq 0, \right.$$

they obtained an exponential stability result.

Our present work focuses on the study of system (1) with internel and eternal body forces which act only on the elastic solid present the porous dissipation and the neutral delay term respectively.

$$G_1 = 0,\ G_2 = -\beta u_t,\ H_1 = 0, \quad H_2 = -\left[\int_0^t h(t-r) u_t(r) dr\right]_t, \tag{4}$$

where β is a positive constant and h is a given kernel.

Thus, when we substitute (4) into (1), our system becomes

$$\begin{array}{ll} \rho_z z_{tt} = a_1 z_{xx} + a_2 u_{xx}, & \text{in } \Omega, \\ \rho_u \left[u_t + \int_0^t h(t-r) u_t(x,r) dr \right]_t = a_3 u_{xx} + a_2 z_{xx} - \beta u_t, & \text{in } \Omega, \end{array} \tag{5}$$

where $\Omega = (0,1) \times (0,\infty)$, with initial conditions

$$\begin{array}{l} z(x,0) = z_0(x), z_t(x,0) = z_1(x), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ x \in [0,1] \end{array} \tag{6}$$

and boundary conditions given by

$$z(0,t) = z(1,t) = u(0,t) = u(1,t) = 0, \ t \geq 0. \tag{7}$$

In what follows, we consider (z,u) to be a solution of system (5)-(7) with some assumptions on the kernel h needed to justify the calculations. In section 2, we present preliminary materials which will be helpful in obtaining our results. In section 3, we prove the existence and uniqueness of the solution by using Faedo-Galerkin method. In section 4, we establish some useful lemmas to study the exponential stability result of our system.

2. Preliminaries

As a starting point, we make certain assumptions about the kernel h, and then we introduce the energy functional, which is nonnegative; at the very least, we provide a lemma and a definition that will be employed later.

(**H**) The kernel h is a nonnegative continuously differentiable and summable function satisfying

$$h'(t) \leq -\eta h(t), \ \int_0^{+\infty} e^{\tau r} |h(t)| dr < \infty, \ t \geq 0, \tag{8}$$

for some positive constants η and τ. The associated energy $E(t)$ is a nonnegative functional defined as

$$E(t) = \frac{1}{2} \int_0^1 \left(\rho_z z_t^2 + a_1 z_x^2 + 2a_2 z_x u_x + \rho_u u_t^2 + a_3 u_x^2 \right) dx + \frac{\rho_u}{2} \int_0^1 \left(\int_0^t h(t-r) u_t^2(x,r) dr \right) dx.$$

Observing that

$$a_3 u_x^2 + 2a_2 u_x z_x + a_1 z_x^2 = \frac{1}{2} \left[a_3 \left(u_x + \frac{a_2}{a_3} z_x \right)^2 + a_1 \left(z_x + \frac{a_2}{a_1} u_x \right)^2 + \left(a_3 - \frac{a_2^2}{a_1} \right) u_x^2 + \left(a_1 - \frac{a_2^2}{a_3} \right) z_x^2 \right],$$

and using the assumption $a_1 a_3 > a_2^2$, we get

$$a_3 u_x^2 + 2a_2 u_x z_x + a_1 z_x^2 > \frac{1}{2} \left[\left(a_3 - \frac{a_2^2}{a_1} \right) u_x^2 + \left(a_1 - \frac{a_2^2}{a_3} \right) z_x^2 \right].$$

Now, we conclude that

$$E(t) > \frac{1}{2} \int_0^1 [\rho_z z_t^2 + a_1' z_x^2 + \rho_u u_t^2 + a_3' u_x^2 + \rho_u \int_0^t h(t-r) u_t^2(x,r) dr] dx,$$

where $2a_1' = a_1 - \frac{a_2^2}{a_3}$ and $2a_3' = a_3 - \frac{a_2^2}{a_1}$.

Lemma 2.1. *We have for $t \geq 0$*

$$\int_0^1 u_t(t)\int_0^t h(t-r)u_{tt}(x,r)drdx$$
$$= -\frac{1}{2}\int_0^1 h'\Box u_t dx + \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^t h(t-r)u_t^2(x,r)drdx + \frac{h(t)}{2}\int_0^1 u_t^2 dx - h(t)\int_0^1 u_t(t)u_t(0)dx, \quad (9)$$

for all $u_t \in C^1\left([0,\infty);L^2(0,1)\right)$ *and* $h \in C^1([0,\infty))$.

Definition 2.2. *We define the binary operators $\Box$ and $\circ$ respectively by*

$$h\Box u_t = \int_0^t |h(t-r)|\,|u_t(t)-u_t(s)|\,dr,$$
$$h\circ u_t = \int_0^t |h(t-r)|\,|u_t(t)-u_t(s)|^2\,dr.$$

In this study, we shall utilize the regular Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their typical scalar products and norms .

The space H is defined as

$$H = \left[H_0^1(0,1)\times L^2(0,1)\right]^2. \quad (10)$$

3. The well-posedness result

In this part, we will use Faedo-Galerkin technique to demonstrate the global existence and uniqueness of system's solutions (5)-(7). The following theorem is used as the first outcome.

Theorem 3.1. *For all $(z_0,z_1,u_0,u_1)\in H$,* (**H**) *is verified, and $T>0$, there existes a unique weak solution of problem* (5)-(7) *on $(0,T)$, such that*

$$(z,u)\in C([0,T],H_0^1(0,1))\cap C^1([0,T],L^2(0,1)),$$
$$(z_t,u_t)\in L^2([0,T],(0,1))\cap L^2([0,T],H_0^1(0,1)).$$

Proof. **Existence :** The main tool of our proof is the use of Faedo-Galerkin's method, which base on the construction of approximations of the solutions, then we obtain an energy estimates proving that $t_n = T$ for $n\in\mathbb{N}$. Finally, we pass to the limit of the approximations, for more details see [20]-[21].

Step 1 : Faedo-Galerkin approximations.

For every $n\geq 1$, let $V_n = Span\{w_1,...,w_n\}$, $1\leq i\leq n$, be an Hilbert basis of the space $H_0^1(0,1)$ and $L^2(0,1)$. As Hilbert space is a separable space, we can choose z_0^n, z_1^n, u_0^n and $u_1^n\in[w_1,...,w_n]$ such that

$$z_0^n = \sum_{k=1}^n \alpha_k^n w_k \to z_0 \text{ in } H_0^1(0,1),$$
$$z_1^n = \sum_{k=1}^n \beta_k^n w_k \to z_1 \text{ in } H_0^1(0,1),$$
$$u_0^n = \sum_{k=1}^n \bar{\alpha}_k^n w_k \to u_0 \text{ in } H_0^1(0,1),$$
$$u_1^n = \sum_{k=1}^n \bar{\beta}_k^n w_k \to u_1 \text{ in } H_0^1(0,1). \quad (11)$$

Now, we search for solution having the following form

$$z^n = \sum_{k=1}^{n} g_k^n(t)\, w_k(x),$$
$$u^n = \sum_{k=1}^{n} \bar{g}_k^n(t)\, w_k(x),$$

of the following approximate system, for $k = 1, ..., n$

$$\begin{cases} \rho_z \int_0^1 z_{tt}^n w_k dx + a_1 \int_0^1 z_x^n w_{kx} dx + a_2 \int_0^1 u_x^n w_{kx} dx = 0, \\ \rho_u \int_0^1 \left[u_t^n + \int_0^t h(t-r) u_t^n(x,r) dr\right]_t w_k dx + a_3 \int_0^1 u_x^n w_{kx} dx \\ +a_2 \int_0^1 z_x^n w_{kx} dx + \beta \int_0^1 u_t^n w_k dx = 0, \end{cases} \tag{12}$$

with initial data

$$z^n(x,0) = z_0^n(x), z_t^n(x,0) = z_1^n(x), u^n(x,0) = u_0^n(x), u_t^n(x,0) = u_1^n(x). \tag{13}$$

By using the Caratheodory theorem for an ordinary differential equation, we derive that the aforementioned Cauchy problem (12) – (13) has a unique global solution $\left(g_k^n(t), \bar{g}_k^n(t)\right)_{k=1,...,n}$ defined on $[0, t_n]$.

Step 2 : Energy estimates.

The main purpose of this step is to prove that $t_n = T$, we obtain this result by multiplying in L^2 the first and the second equation of system (12) by $\left(\left(g_k^n(t)\right)', \left(\bar{g}_k^n(t)\right)'\right)$ respectively, and by using integration by parts, boundary-initial conditions and lemma 1, we find for all $t > 0$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_z (z_t^n)^2 + a_1 (z_x^n)^2 + 2a_2 u_x^n z_x^n + \rho_u (u_t^n)^2 + a_3 (u_x^n)^2 + \rho_u \int_0^t h(t-r)(u_t^n(r))^2 dr\right] dx \\ &= -\left(\frac{\rho_u}{2} h(t) + \beta\right) \int_0^1 (u_t^n)^2 dx + \frac{\rho_u}{2} \int_0^1 h' \square u_t^n dx, \end{aligned} \tag{14}$$

for every $n \geq 1$. From the hypotheses on the function h, and by integrating (14) over $(0, t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left[\rho_z (z_t^n)^2 + a_1 (z_x^n)^2 + 2a_2 u_x^n z_x^n + \rho_u (u_t^n)^2 + a_3 (u_x^n)^2 + \rho_u \int_0^t h(t-r)(u_t^n(r))^2 dr\right] dx \\ &\leq \frac{1}{2}\int_0^1 \left[\rho_z \left(z_1^n\right)^2 + a_1 \left(z_0^n\right)^2 + 2a_2 u_0^n z n_0^n + \rho_u \left(u_1^n\right)^2 + a_3 \left(u_0^n\right)^2 + \rho_u \int_0^t h(t-r)\left(u_1^n(r)\right)^2 dr\right] dx. \end{aligned} \tag{15}$$

(15) may be noted as

$$E^n(t) \leq E^n(0). \tag{16}$$

Now, since the energy is nonnegative, from (11) and the hypotheses on the function h, we can write (16) as

$$E^n(t) \leq E(0) \leq C, \tag{17}$$

where C is a positive constant independent of n and t. From (17), we deduce that

$$\begin{aligned} &(z^n, u^n) \text{ are bounded in } L^\infty\left([0,T]; H_0^1(0,1)\right), \\ &(z_t^n, u_t^n) \text{ are bounded in } L^\infty\left([0,T]; H_0^1(0,1)\right). \end{aligned} \tag{18}$$

Step 3 : The limit process.

Now, by using Aubin-Lions Theorem [22] and up to the subsequence, we can observe from (18) that there exists a subsequences (z^m, u^m) of (z^n, u^n), such that

$$\begin{aligned} &(z^m, u^m) \to (z, u) \quad \text{weak * in } L^\infty\left([0,T]; H_0^1(0,1)\right), \\ &(z_t^m, u_t^m) \to (z_t, u_t) \ \text{ weak * in } L^\infty\left([0,T]; H_0^1(0,1)\right), \end{aligned} \tag{19}$$

and by using the fact that

$$L^\infty\left([0,T]; H_0^1(0,1)\right) \hookrightarrow L^2\left([0,T]; H_0^1(0,1)\right).$$

We get

$$\begin{aligned} &(z^n, u^n) \ \text{are bounded in } L^2\left([0,T]; H_0^1(0,1)\right), \\ &(z_t^n, u_t^n) \ \text{are bounded in } L^2\left([0,T]; H_0^1(0,1)\right). \end{aligned} \tag{20}$$

Therefore,

$$(z^n, u^n) \ \text{are bounded in } H^1\left([0,T]; H^1(0,1)\right). \tag{21}$$

Since the embedding $H^1\left([0,T]; H^1(0,1)\right) \hookrightarrow L^2\left([0,T]; L^2(0,1)\right)$ is compact, then the subsequences

$$\begin{aligned} &(z^m, u^m) \to (z, u) \quad \text{strongly in } L^2\left([0,T]; L^2(0,1)\right), \\ &(z_t^m, u_t^m) \to (z_t, u_t) \ \text{ strongly in } L^2\left([0,T]; L^2(0,1)\right). \end{aligned} \tag{22}$$

Uniqueness : Let $\left(z^1, u^1\right)$ and $\left(z^2, u^2\right)$ two solutions of (5)-(7), and let
$\tilde{z} = z^1 - z^2$ and $\tilde{u} = u^1 - u^2$ satisfy

$$\begin{cases} \rho_z \tilde{z}_{tt} - a_1 \tilde{z}_{xx} - a_2 \tilde{u}_{xx} = 0, \\ \rho_u \left[\tilde{u}_t + \int_0^t h(t-r)\tilde{u}_t(x,r)dr\right]_t - a_3 \tilde{u}_{xx} - a_2 \tilde{z}_{xx} + \beta \tilde{u}_t = 0. \end{cases} \tag{23}$$

Multiplying (23) by $\tilde{z}_t$, $\tilde{u}_t$ respectively, then, integrating over $(0,1)$ and by using lemma 1, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_z \tilde{z}_t^2 + a_1 \tilde{z}_x^2 + 2a_2 \tilde{u}_x \tilde{z}_x + \rho_u \tilde{u}_t^2 + a_3 \tilde{u}_x^2 + \rho_u \int_0^t h(t-r)\tilde{u}_t^2(r)dr\right] dx \\ &= -\left(\frac{\rho_u}{2} h(t) + \beta\right)\int_0^1 \tilde{u}_t^2 dx + \frac{\rho_u}{2}\int_0^1 h' \Box \tilde{u}_t dx. \end{aligned} \tag{24}$$

Now, we integrate (24) over $(0,t)$, we obtain :

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left[\rho_z \tilde{z}_t^2 + a_1 \tilde{z}_x^2 + 2a_2 \tilde{u}_x \tilde{z}_x + \rho_u \tilde{u}_t^2 + a_3 \tilde{u}_x^2 + \rho_u \int_0^t h(t-r)\tilde{u}_t^2(r)dr\right] dx \\ &= -\left(\frac{\rho_u}{2}\int_0^t h(\tau)d\tau + \beta t\right)\int_0^1 \tilde{u}_t^2 dx ds - \frac{\rho_u \eta}{2}\int_0^t \int_0^1 h \Box \tilde{u}_t dx ds \le 0. \end{aligned} \tag{25}$$

From estimate (25), we deduce that $(\tilde{z}, \tilde{u}) = (0,0)$, which implies that problem (5)-(7) has a unique solution.
Continuous dependence : Multiplying the first and the second equation of system (5) by z_t, u_t respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_z z_t^2 + a_1 z_x^2 + 2a_2 u_x z_x + \rho_u u_t^2 + a_3 u_x^2 + \rho_u \int_0^t h(t-r)u_t^2(r)dr\right] \le 0, \tag{26}$$

integrating (26) over $(0, t)$, and from the positivity of the energy, we can get the following estimate :

$$E(t) \leq E(0) + \frac{1}{2}\int_0^t \left[\int_0^1 \left(\rho_z z_t^2 + a_1 z_x^2 + \underbrace{2a_2 u_x z_x}_{f_0} + \rho_u u_t^2 + \rho_u \int_0^t h(t-r)u_t^2(r)dr + a_3 u_x^2\right) dx\right] d\tau. \tag{27}$$

Now, applying young's inequality to f_0 with the fact that $a_1 a_3 > a_2^2$, gives

$$\begin{aligned} E(t) &\leq E(0) + \frac{1}{2}\int_0^t \left[\int_0^1 \left(\rho_z z_t^2 + 2a_1 z_x^2 + +\rho_u u_t^2 + 2a_3 u_x^2 + \rho_u \int_0^t h(t-r)u_t^2(r)dr\right) dx\right] d\tau \\ &\leq E(0) + \rho_1 \int_0^t \left[\int_0^1 \left(z_t^2 + z_x^2 + +u_t^2 + u_x^2 + \int_0^t h(t-r)u_t^2(r)dr\right) dx\right] d\tau, \end{aligned} \tag{28}$$

where $2\rho_1 = \max(\rho_z, 2a_1, \rho_u, 2a_3)$. On the other side, we know that

$$\begin{aligned} E(t) &> \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + a_1' z_x^2 + \rho_u u_t^2 + a_3' u_x^2 + \rho_u \int_0^t h(t-r)u_t^2(x,r)dr\right] dx, \\ &> \rho_2 \int_0^1 \left[z_t^2 + z_x^2 + u_t^2 + \int_0^t h(t-r)u_t^2(x,r)dr + u_x^2\right] dx, \end{aligned} \tag{29}$$

where $2\rho_2 = \max\left(\rho_z, a_1', \rho_u, a_3'\right)$, $2a_1' = a_1 - \frac{a_2^2}{a_3}$ and $2a_3' = a_3 - \frac{a_2^2}{a_1}$. From (28) − (29), we get

$$\begin{aligned} &\rho_2 \int_0^1 \left[z_t^2 + z_x^2 + u_t^2 + \int_0^t h(t-r)u_t^2(x,r)dr + u_x^2\right] dx \\ &< E(0) + \rho_1 \int_0^t \left[\int_0^1 \left(z_t^2 + z_x^2 + +u_t^2 + u_x^2 + \int_0^t h(t-r)u_t^2(r)dr\right) dx\right] d\tau. \end{aligned} \tag{30}$$

Applying Gronwall's inequality on (30), we obtain

$$\int_0^1 \left[z_t^2 + z_x^2 + u_t^2 + \int_0^t h(t-r)u_t^2(x,r)dr + u_x^2\right] dx \leq E(0)e^{\rho_3 t}, \tag{31}$$

where ρ_3 is a positive constant. From (31), we deduce that the solution of problem (5)-(7) continuously depends on initial data. □

4. Stability Result

In this section, we state and prove some technical lemmas. Subsequently, we end the section with the statement and proof of our stability result.

Lemma 4.1. *The energy functional E, defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_z z_t^2 + a_1 z_x^2 + 2a_2 z_x u_x + \rho_u u_t^2 + a_3 u_x^2\right) dx + \frac{\rho_u}{2}\int_0^1 \left(\int_0^t h(t-r)u_t^2(x,r)dr\right) dx, \ t > 0, \tag{32}$$

satisfies

$$E'(t) \leq -\beta \int_0^1 u_t^2 dx + \frac{\rho_u}{2}\int_0^1 h' \Box u_t dx, \ t > 0. \tag{33}$$

Proof. Multiplying the first equation of (5) by z_t, and integrating by parts over $(0,1)$, we obtain

$$\frac{\rho_z}{2}\frac{d}{dt}\int_0^1 z_t^2dx + \frac{a_1}{2}\frac{d}{dt}\int_0^1 z_x^2dx + a_2\int_0^1 u_x z_{tx}dx = 0,\ t>0. \tag{34}$$

Multiplying the second equation of (5) by u_t, and integrating by parts over $(0,1)$, we obtain

$$\frac{\rho_u}{2}\frac{d}{dt}\int_0^1 u_t^2dx + \rho_u\int_0^1\left[\int_0^t h(t-r)u_t(r)dr\right]_t u_tdx + \frac{a_3}{2}\frac{d}{dt}\int_0^1 u_x^2dx + a_2\int_0^1 z_xu_{tx}dx + \beta\int_0^1 u_t^2dx = 0. \tag{35}$$

We have

$$\left[\int_0^t h(t-r)u_t(r)dr\right]_t = \int_0^t h(t-r)u_{tt}(r)dr + h(t)u_t(0). \tag{36}$$

By using (9) and (36), we get

$$\begin{aligned}&\frac{\rho_u}{2}\frac{d}{dt}\int_0^1 u_t^2dx - \frac{\rho_u}{2}\int_0^1 h'\Box u_tdx + \frac{\rho_u}{2}h(t)\int_0^1 u_t^2dx + \frac{\rho_u}{2}\frac{d}{dt}\int_0^1\int_0^t h(t-r)u_t^2(r)drdx + \frac{a_3}{2}\frac{d}{dt}\int_0^1 u_x^2dx\\ &+ a_2\int_0^1 z_xu_{tx}dx + \beta\int_0^1 u_t^2dx = 0.\end{aligned} \tag{37}$$

Now by adding (34) to (37) and by using the fact that the kernel h is a nonnegative function, we get, for any $t>0$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1[\rho_z z_t^2 + a_1z_x^2 + 2a_2z_xu_x + \rho_uu_t^2 + a_3u_x^2 + \rho_u\int_0^t h(t-r)u_t^2(x,r)dr]dx\\ &\le -\beta\int_0^1 u_t^2dx + \frac{\rho_u}{2}\int_0^1 h'\Box u_tdx.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.2. *The functional*

$$I_1(t) = \rho_u\int_0^1\left(u_t + \int_0^t h(t-r)u_t(r)dr\right)udx + \frac{\beta}{2}\int_0^1 u^2dx - \frac{a_2}{a_1}\rho_z\int_0^1 z_tudx,\quad t>0, \tag{38}$$

satisfies, for any $\varepsilon_1>0$,

$$I_1'(t) \le -\left(a_3 - \frac{a_2^2}{a_1}\right)\int_0^1 u_x^2dx + \varepsilon_1\int_0^1 z_t^2dx + C_1\int_0^1 u_t^2dx + \frac{1}{2}\left(\int_0^t h(r)dr\right)\int_0^1 h\circ u_tdx,\ t>0, \tag{39}$$

where $C_1 = \frac{a_2^2\rho_z^2}{4\varepsilon_1a_1^2} + \rho_u\left(1 + \int_0^t h(r)dr + \frac{\rho_u}{2}\right)$.

Proof. A simple differentiation of (38) and by using the first and the second equation of (5), we find

$$I_1'(t) = -\left(a_3 - \frac{a_2^2}{a_1}\right)\int_0^1 u_x^2dx + \rho_u\int_0^1 u_t^2dx \underbrace{- \frac{a_2}{a_1}\rho_z\int_0^1 z_tu_tdx}_{f_1} + \underbrace{\rho_u\int_0^1 u_t\int_0^t h(t-r)u_t(r)drdx}_{f_2},\ t>0. \tag{40}$$

Applying Young's inequality to f_1 and f_2, gives

$$f_1 \leq \varepsilon_1 \int_0^1 z_t^2 dx + \frac{a_2^2 \rho_z^2}{4\varepsilon_1 a_1^2} \int_0^1 u_t^2 dx. \tag{41}$$

$$\begin{aligned} f_2 &= \rho_u \int_0^t h(r)dr \int_0^1 u_t^2 dx - \rho_u \int_0^1 u_t \int_0^t h(t-r)\left(u_t(t) - u_t(r)\right) drdx, \\ &\leq \rho_u \left(\int_0^t h(r)dr + \frac{\rho_u}{2} \right) \int_0^1 u_t^2 dx + \frac{1}{2} \int_0^t h(r)dr \int_0^1 h \circ u_t dx. \end{aligned} \tag{42}$$

Substituting (41) and (42) into (40), we find (39). □

Lemma 4.3. *The functional*

$$I_2(t) = a_2 \left[\int_0^1 \left(u_t + \int_0^t h(t-r) u_t(r) dr \right) z dx - \int_0^1 z_t u dx + \frac{\beta}{\rho_u} \int_0^1 z u dx \right], \ t > 0, \tag{43}$$

satisfies, for any $\varepsilon_2 > 0$,

$$\begin{aligned} I_2'(t) \leq & -\frac{a_2^2}{2\rho_u} \int_0^1 z_x^2 dx + a_2^2 \varepsilon_2 \int_0^1 z_t^2 dx + \frac{1}{\varepsilon_2} \left(\int_0^t h(r)dr \right)^2 \int_0^1 u_t^2 dx + \frac{1}{\varepsilon_2} \left(\int_0^t h(r)dr \right)^2 \int_0^1 h \circ u_t dx \\ & + C_2(\varepsilon_2) \int_0^1 u_x^2 dx, \ t > 0, \end{aligned} \tag{44}$$

where $C_2(\varepsilon_2) = \frac{a_2^2}{\rho_z} + \frac{\rho_u}{2} \left(\frac{a_1}{\rho_z} - \frac{a_2}{\rho_u} \right)^2 + \frac{\beta^2}{2\varepsilon_2 \rho_u^2}$.

Proof. A simple differentiation of (43) and using the first and the second equation of (5), we obtain

$$\begin{aligned} I_2'(t) = & -\frac{a_2^2}{\rho_u} \int_0^1 z_x^2 dx + \underbrace{a_2 \left(\frac{a_1}{\rho_z} - \frac{a_3}{\rho_u} \right) \int_0^1 z_x u_x dx}_{f_3} + \frac{a_2^2}{\rho_z} \int_0^1 u_x^2 dx + \underbrace{a_2 \int_0^1 z_t \int_0^t h(t-r) u_t(r) drdx}_{f_4} \\ & + \underbrace{\frac{a_2 \beta}{\rho_u} \int_0^1 z_t u dx}_{f_5}. \end{aligned} \tag{45}$$

Now, by using Poincaré's and Young's inequalities, we get

$$f_3 \leq \frac{a_2^2 \varepsilon_3}{2} \int_0^1 z_x^2 dx + \frac{1}{2\varepsilon_3} \left(\frac{a_1}{\rho_z} - \frac{a_2}{\rho_u} \right)^2 \int_0^1 u_x^2 dx, \tag{46}$$

$$\begin{aligned} f_4 &\leq \frac{a_2^2 \varepsilon_2}{2} \int_0^1 z_t^2 dx + \frac{1}{2\varepsilon_2} \int_0^1 \left(\int_0^t h(t-r) u_t(r) dr \right)^2 dx \\ &\leq \frac{a_2^2 \varepsilon_2}{2} \int_0^1 z_t^2 dx + \frac{1}{2\varepsilon_2} \int_0^1 \left(- \int_0^t h(t-r) \left[u_t(t) - u_t(r) - u_t(t) \right] dr \right)^2 dx \\ &\leq \frac{a_2^2 \varepsilon_2}{2} \int_0^1 z_t^2 dx + \frac{1}{\varepsilon_2} \int_0^1 \left(\int_0^t h(t-r) \left(u_t(t) - u_t(r) \right) dr \right)^2 dx + \frac{1}{\varepsilon_2} \left(\int_0^t h(r)dr \right)^2 \int_0^1 u_t^2 dx \\ &\leq \frac{a_2^2 \varepsilon_2}{2} \int_0^1 z_t^2 dx + \frac{1}{\varepsilon_2} \left(\int_0^t h(r)dr \right) \int_0^1 h \circ u_t dx + \frac{1}{\varepsilon_3} \left(\int_0^t h(r)dr \right)^2 \int_0^1 u_t^2 dx, \end{aligned} \tag{47}$$

$$f_5 \leq \frac{\varepsilon_2 a_2^2}{2} \int_0^1 z_t^2 dx + \frac{\beta^2}{2\varepsilon_2 \rho_u^2} \int_0^1 u_x^2 dx. \tag{48}$$

We end up with (44), by substituting (46), (47) and (48) into (45) and taking $\varepsilon_3 = \frac{1}{\rho_u}$. □

Lemma 4.4. *The functional*

$$I_3(t) = -\rho_z \int_0^1 z_t z dx, \ t > 0, \tag{49}$$

satisfies

$$I_3'(t) \leq -\rho_z \int_0^1 z_t^2 dx + 2a_1 \int_0^1 z_x^2 dx + \frac{a_3}{4} \int_0^1 u_x^2 dx, \ t > 0. \tag{50}$$

Proof. A simple differentiation of (49) and using the second equation of (5), we find

$$I_3'(t) = -\rho_z \int_0^1 z_t^2 dx + a_1 \int_0^1 z_x^2 dx + \underbrace{a_2 \int_0^1 u_x z_x dx}_{f_5}. \tag{51}$$

Since $a_1 a_3 > a_2^2$ and thanks to Young's inequality, we find

$$f_5 \leq a_1 \int_0^1 z_x^2 dx + \frac{a_3}{4} \int_0^1 u_x^2 dx. \tag{52}$$

Substituting (52) into (51), we end up with (50). □

Lemma 4.5. *The functional*

$$I_4(t) = e^{-\delta t} \int_0^1 \int_0^t e^{\delta r} K(t-r) u_t^2(r) dr dx, \ t > 0, \tag{53}$$

where

$$K(t) = \int_t^{+\infty} e^{\tau r} |h(t)| \, dr,$$

satisfies

$$I_4'(t) = -\delta I_4(t) + \int_0^1 K(0) u_t^2 dx - \int_0^1 \int_0^t h(t-r) u_t^2(r) dr dx, \ t > 0. \tag{54}$$

Now, by using the previous lemmas and the Lyapunov functional $F(t)$ defined by

$$F(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + I_3(t) + I_4(t), \ t > 0, \tag{55}$$

where N, N_1 and N_2 are positive constants. Then, our stability result reads.

Lemma 4.6. *For some positive constants α_1 and α_2, the Lyapunov functional $F(t)$ introduced by* (55) *is equivalent to $NE(t) + I_4(t)$, such that*

$$\alpha_1 E(t) \leq F(t) \leq \alpha_2 \left[E(t) + I_4(t) \right], \ t > 0. \tag{56}$$

Proof. By exploiting relation (55) with the use of (38), (43) and (49), we get

$$\begin{aligned}
&|F(t) - NE(t) - I_4(t)| \\
&= |N_1 I_1(t) + N_2 I_2(t) + I_3(t)| \\
&\leq \rho_u N_1 \int_0^1 \left| u_t + \int_0^t h(t-r) u_t(r) dr \right| |u|\, dx + \frac{\beta N_1}{2} \int_0^1 u^2 dx + N_2 a_2 \int_0^1 \left| u_t + \int_0^t h(t-r) u_t(r) dr \right| |z|\, dx \\
&+ \frac{N_2 \beta a_2}{\rho_u} \int_0^1 |z|\,|u|\, dx - a_2 \left(\frac{N_1 \rho_z}{a_1} + N_2 \right) \int_0^1 |z_t|\,|u|\, dx - \rho_z \int_0^1 |z_t|\,|z|\, dx,
\end{aligned}$$

by using Poincaré's, Cauchy-Schwarz and Young's inequalities, we obtain

$$|F(t) - NE(t) - I_4(t)| \leq \lambda E(t), \quad \lambda > 0. \tag{57}$$

Thus, (57) leads to

$$(N - \lambda)\, E(t) + I_4(t) \leq F(t) \leq (N + \lambda)\, E(t) + I_4(t),$$

we conclude (56), when we set $\alpha_1 = \min\{N - \lambda, 1\}$ and $\alpha_2 = \max\{N + \lambda, 1\}$ such that N is sufficiently large. □

Theorem 4.7. *Under the above assumptions* (**H**), *we have*

$$E(t) \leq M e^{-\delta t}, \quad t > 0, \tag{58}$$

for some positive constants M *and* δ.

Proof. Recalling (33), (39), (44), (50) and (54), we obtain

$$\begin{aligned}
F'(t) \leq &- \left[N\beta - N_1 C_1 - \frac{N_2}{\varepsilon_2} \left(\int_0^t h(r) dr \right)^2 - K(0) \right] \int_0^1 u_t^2 dx \\
&- \left[N_1 \left(a_3 - \frac{a_2^2}{a_1} \right) - N_2 C_2\left(\varepsilon_2\right) - \frac{a_3}{4} \right] \int_0^1 u_x^2 dx - \left[\rho_z - N_1 \varepsilon_1 - N_2 a_2^2 \varepsilon_2 \right] \int_0^1 z_t^2 dx \\
&- \left[\frac{N_2 a_2^2}{2\rho_u} - 2a_1 \right] \int_0^1 z_x^2 dx + \left[\frac{N_1}{2} \int_0^t h(r) dr + \frac{N_2}{\varepsilon_2} \left(\int_0^t h(r) dr \right)^2 \right] \int_0^1 h \circ u_t dx \\
&+ \frac{N\rho_u}{2} \int_0^1 h' \Box u_t dx - \delta I_4(t) - \int_0^1 \int_0^t h(t-r) u_t^2(r) dr dx,
\end{aligned}$$

by taking $\varepsilon_1 = \frac{\rho_z}{4N_1}$ and $\varepsilon_2 = \frac{\rho_z}{4a_2^2 N_2}$, we end up with

$$\begin{aligned}
F'(t) \leq &- \left[N\beta - N_1 C_1 - \frac{4\left(a_2 N_2\right)^2}{\rho_z} h_0^2 - K(0) \right] \int_0^1 u_t^2 dx + \left[\frac{N_1}{2} h_0 + \frac{4\left(a_2 N_2\right)^2}{\rho_z} h_0^2 \right] \int_0^1 h \circ u_t dx \\
&- \left[N_1 \left(a_3 - \frac{a_2^2}{a_1} \right) - N_2 C_2 \left(\frac{\rho_z}{4a_2^2 N_2} \right) - \frac{a_3}{4} \right] \int_0^1 u_x^2 dx - \left[\frac{N_2 a_2^2}{2\rho_u} - 2a_1 \right] \int_0^1 z_x^2 dx \\
&- \frac{\rho_z}{2} \int_0^1 z_t^2 dx + \frac{N\rho_u}{2} \int_0^1 h' \Box u_t dx - \delta I_4(t) - \int_0^1 \int_0^t h(t-r) u_t^2(r) dr dx,
\end{aligned} \tag{59}$$

where, for all $t \geq t_0 > 0$,

$$\int_0^t h(r) dr \geq \int_0^{t_0} h(r) dr = h_0.$$

Now, we choose N_2 large enough such that

$$N_2 > \frac{4\rho_u a_1}{a_2^2},$$

then, we select N_1 large enough such that

$$N_1\left(a_3 - \frac{a_2^2}{a_1}\right) - N_2 C_2\left(\frac{\rho_z}{4a_2^2 N_2}\right) - \frac{a_3}{4} > 0.$$

Finally

$$N\beta - N_1 C_1 - \frac{4\,(a_2 N_2)^2}{\rho_z} h_0^2 - K(0) > 0.$$

As a result, the relation (59) becomes

$$F'(t) \leq -C_3 \int_0^1 \left(u_t^2 + u_x^2 + z_t^2 + z_x^2 + h * u_t^2\right) dx - C_4 I_4(t),\ \forall t > 0, \tag{60}$$

where C_3 and C_4 are positive constants. We also have from (32), after using Young's inequality, that

$$\begin{aligned} E(t) &\leq \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + (a_1 + a_2)\, z_x^2 + \rho_u u_t^2 + (a_3 + a_2)\, u_x^2\right] dx + \frac{\rho_u}{2}\int_0^1 \left(\int_0^t h(t-r) u_t^2(x,r) dr\right) dx \\ &\leq \lambda_1 \int_0^1 \left[z_t^2 + z_x^2 + u_t^2 + u_x^2 + \left(\int_0^t h(t-r) u_t^2(x,r) dr\right)\right] dx,\ \lambda_1 > 0. \end{aligned} \tag{61}$$

The combination of (60) and (61) results in

$$F'(t) \leq -C_5\left[E(t) + I_4(t)\right],\ C_5 > 0. \tag{62}$$

From (62) and the right side of (56), we obtain

$$F(t) \leq C_6 e^{-\frac{C_5}{\alpha_2} t},\ t > 0, \tag{63}$$

where C_6 is a positive constant. Which yields the desired result (58) by using the other side of the equivalence relation again. □

Acknowlegement

I appreciate the anonymous referees for the very careful reading and correction of several flaws.

References

[1] J. Atkin and R. E. Craine, *Continuum theories of mixtures: Basic theory and historical development*, Q. J. Mech. Appl. Math. **29** (1976), 209–243.

[2] A. Bedford and D. S. Drumheller, *Theories of immiscible and structured mixtures*, International Journal of Engineering Science, **vol. 21, no. 8** (1983), pp. 863–960.

[3] R. M. Bowen and J. C. Wiese, *Diffusion in mixtures of elastic materials*, Int. J. Eng. Sci.**7** (1969), 689–722.

[4] C. Truesdell and R. Toupin, *The Classical Field Theories*, Springer Berlin Heidelberg, pp. 226–858, (1960).

[5] A. E. Green and P. M. Naghdi, *A dynamical theory of interacting continua*, Int. J. Eng. Sci. **3(2)** (1965), 231–241.

[6] Eringen, A.C.: *A continuum theory of swelling porous elastic soil*s. Int. J. Eng. Sci. **32(8)** (1994), 1337–1349.

[7] Quintanilla, R.: *Existence and exponential decay in the linear theory of viscoelastic mixtures*. Euro. J. Mech. A/Solids **24(2)** (2005), 311–324.

[8] T. A. Apalara, *General stability result of swelling porous elastic soils with a viscoelastic damping*, Zeitschrift für Angewandte Mathematik und Physik, **vol. 71, no. 6** (2020), p. 200.
[9] Alves, M.S., Muñoz Rivera, J.E., Quintanilla, R.: *Exponential decay in a thermoelastic mixture of solids. Int. J. Solids Struct*. **46(7-8)** (2009), 1659–1666.
[10] Ieşan, D.: *On the theory of mixtures of thermoelastic solids*. J. Thermal stress. **14(4)** (1991), 389–408.
[11] Quintanilla, R.: *Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation*. J. Comput. Appl. Math **145(2)** (2002), 525–533.
[12] A. J. A. Ramos, M. M. Freitas, D. S. Almeida Jr, A. S. Noe, M. D. J. Santos, *Stability results for elastic porous media swelling with nonlinear damping*, J. Math. Phys., **61** (2020), 101-505.
[13] Wang, J.M., Guo, B.Z.: *On the stability of swelling porous elastic soils with fluid saturation by one internal damping*. IMA J. Appl. Math. **71(4)** (2006), 565–582.
[14] Quintanilla, R. *Exponential Stability of Solutions of Swelling Porous Elastic Soils*. Meccanica **39** (2004), 139–145.
[15] A. Choucha, S. M. Boulaaras, D. Ouchenane, B. B. Cherif, and M. Abdalla, *Exponential stability of swelling porous elastic with a viscoelastic damping and distributed delay term*, Journal of Function Spaces, **vol. 2021**, Article ID 5581634, **8 pages**, (2021).
[16] A. J. A. Ramos, D. S. Almeida Júnior, M. M. Freitas, A. S. Noé, and M. J. Dos Santos , *Stabilization of swelling porous elastic soils with fluid saturation and delay time terms*, J. Math. Phys. **62** (2021), 021507.
[17] Al-Mahdi, A.M.; Al-Gharabli, M.M.; Alahyane, M. *Theoretical and Computational Results of a Memory-Type Swelling Porous-Elastic System*. Math. Comput. Appl. **27, 27** (2022).
[18] Tatar NE., *Stability for the damped wave equation with neutral delay*. Math Nach. **290** (2017), 2401-2412.
[19] Kerbal, S., Tatar, N.E., *Exponential stabilization of a neutrally delayed viscoelastic Timoshenko beam*, Turk J Math. **43** (2019), 595–611.
[20] Dridi, H., Saci, M. & Djebabla, *A. General decay of Bresse system by modified thermoelasticity of type III*. Ann Univ Ferrara **68** (2022), 203–222.
[21] B. Feng, *Global well-posedness and stability for a viscoelastic plate equation with a time delay*, Mathematical Problems in Engineering Article ID585021 2015 , 1-11, (2015).
[22] Lions. J.-L., *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, 1969.