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On the characterisations of ruled surface pairs according to the Sabban frame

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Abstract. In this study, we introduce ruled surface pairs according to the Sabban frame of the curve defined on a unit sphere. The first, the conditions for each couple of two ruled surface pairs to be simultaneosly developable and minimal are investigated. Then, the asymptotic, geodesic and line of curvature of the parameter curves of the ruled surface pairs are simultaneously characterised. In addition, we examined whether the direction curves and the striction curves of the ruled surface pairs coincide. Finally, examples of the ruled surface pairs are given, and their graphs are drawn.

1. Introduction

Surfaces are one of the most important basic subjects of differential geometry, and we encounter this subject in almost every differential geometry book [5, 16, 19, 20, 24, 28, 29]. Among the surfaces, the most interesting ones are the ruled surfaces, which were introduced for the the first time by G. Monge. In contrast, a ruled surface is a surface that can be traced by moving a line in space and can be used in various fields such as computer-aided design (CAD), electric discharge machining, kinematics, etc [2, 11, 23]. In some areas of engineering, the importance of the theory of ruled surfaces is obvious. Examples of ruled surfaces are cylinders, cones, conical surfaces, right conics and helicoids. The ruled surfaces are also the subject of work of architecture. For example, ruled surfaces can be seen in many famous buildings such as the Ciechanow Water Tower, the Kobe Port Tower and the Shuckhov Tower. In addition to the visibility of ruled surfaces in the real world, the theoretical development of these surfaces is still going on in depth. The best known examples are cylinders and cones. The widespread use of ruled surfaces has led researchers to question some of their question some of their characterisations. Topics such as the developability and minimality of surfaces and the characterisation of curves on a ruled surface are the most researched topics [1, 4, 6, 9, 14, 17, 21, 25, 26, 30, 32]. Hu et al. redesigned ruled surfaces by reference to Bezier curves [13]. Normal and binormal ruled surfaces were introduced by Özsoy. The surfaces in which the base curve was assigned to be a curve in the W direction, and by providing the characteristics of these surfaces, they worked out the cases of the base curve as an asymptotic, geodesic and curvature line [22]. The ruled surface and its characterisations, obtained by the evolution of a polynomial space curves, were studied by

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Eren et al. in [7]. In the near future; Has, Yilmaz and Ayvaci ruled surfaces studied with conformable fractional calculus and the ruled surface is rearranged concerning the conformable surface definition and geometric properties investigated [12]. Li et al. studied the simultaneous characterisations of partner ruled surfaces using Flc-frame [18]. Ersoy et al. proposed a series of spatial quaternionic expressions for surfaces, utilising these expressions to derive the striction curves and draills [10]. Sabban frame was the first defined by Koenderink and there are various studies on this frame [15]. Using the Sabban frame, Tasköprü and Tosun studied special Smarandache curves on [27]. Şenyurt and Çalışkan studied special Smarandache curves according to the Sabban frame with the help of a spherical indicatrix of a curve and they gave some characterisations of Smarandache curves [3]. Yakut and Kizilay investigated the evolution of the curve with respect to the modified orthogonal Sabban frame [31]. Also, very recently, Eren et al. have used the Flc frame of polynomial curves in Euclidean 3-space to express necessary and sufficient conditions for any ruled surface of polynomial curves to have constant angles and to study the characterisations of these surfaces [8]. In this study, we have introduced ruled surface pairs according to the Sabban frame of the curve defined on a unit sphere. Then, we have simultaneously provided the conditions for each ruled surface pairs to be developable or minimal by considering the main curvatures with the Sabban frame invariants. Such conditions have also been linked to the characterisations of parametric curves such as asymptotic, geodesic or line of curvature. An example has been given at the end of the paper with the corresponding figures of the generated ruled surfaces pairs.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a C^3 space curve in Euclidean 3-space. Then, the Frenet frame $\{T(s), N(s), B(s)\}$ of the curve $\alpha(s)$ is given by

$$T' = \kappa N, \;\; N' = -\kappa T + \tau B, \;\; B' = -\tau N.$$

where T, N, B are tangent vector, principal normal, binormal vectors and κ , τ denote the curvature and torsion of the curve $\alpha(s)$, respectively.
The sphere of radius $r = 1$ and with center in the origin in the space E^3 is defined by

$$S^2 = \{X = (x_1, x_2, x_3) | \langle X, X\rangle = 1\} \; .$$

Now, we give a new frame different from the Frenet frame. Let γ be a unit speed spherical curve. We denote s as the arc-length parameter of γ . Let us denote $t(s)$=$\gamma'(s)$, and we call $t(s)$ a unit tangent vector of γ . We now set a vector $d(s)$=$\gamma(s) \times t(s)$ along γ . This frame $\{\gamma(s), t(s), d(s)\}$ is called the Sabban frame of γ on S^2 . This gives us the following spherical Frenet formulae

$$\begin{cases} \gamma' = t, \\ t' = -\gamma + \kappa_g d, \\ d' = -\kappa_g t. \end{cases} \tag{1}$$

where is called the geodesic curvature of κ_g on S^2 and $\kappa_g = \langle t', d\rangle$ [4].
The parametrization of

$$\Psi(s, v) = \alpha(s) + vX(s),$$

is called a ruled surface. The curve $\alpha(s)$ is the base curve and X(s) is its ruling.
The striction curve and the distribution parameter of a ruled surface are given by the following equations

$$\mu(s) = \alpha(s) - \frac{\langle \alpha'(s), X'(s)\rangle}{||X'(s)||^2} X(s) \tag{2}$$

and

$$\delta = \frac{\det(\alpha'(s), X(s), X'(s))}{||X'(s)||^2} \tag{3}$$

respectively [20]. Furthermore, the distribution parameter gives a characteristic for the ruled surface and it is known that the ruled surface whose distribution parameter disappears can be developable. The unit normal vector field of this surface is defined as follows:

$$U = \frac{\Psi_s \times \Psi_v}{\|\Psi_s \times \Psi_v\|}. \tag{4}$$

The coefficents of the the first end the second fundamental form of this surface, respectively, is found by

$$E = \left\langle \frac{\partial \Psi}{\partial s}, \frac{\partial \Psi}{\partial s} \right\rangle, \ F = \left\langle \frac{\partial \Psi}{\partial s}, \frac{\partial \Psi}{\partial v} \right\rangle, \ G = \left\langle \frac{\partial \Psi}{\partial v}, \frac{\partial \Psi}{\partial v} \right\rangle, e = \left\langle \frac{\partial^2 \Psi}{\partial s^2}, U \right\rangle, \ f = \left\langle \frac{\partial^2 \Psi}{\partial s \partial v}, U \right\rangle, \ g = \left\langle \frac{\partial^2 \Psi}{\partial v^2}, U \right\rangle. \tag{5}$$

The Gaussian curvature and the mean curvature of the ruled surface are given as follows:

$$K = \frac{eg - f^2}{EG - F^2}, H = \frac{1}{2} \frac{Eg - 2fF + Ge}{EG - F^2}. \tag{6}$$

Also, the surface $\Psi(s, v)$ is

- developable surface if and only if the Gaussian curvature vanishes at all points,
- minimal surface if and only if the mean curvature vanishes at all points.

3. On the characterisations of the ruled surface pairs according to the Sabban frame

In this section, we study the characterisations of the pairs of ruled surface generated by the vectors of the Sabban frame of the given curve on the unit sphere.

3.1. γt-ruled surface pairs

Definition 3.1. *Let γ be a unit speed spherical curve and be Sabban frame $\{\gamma(s), t(s), d(s)\}$ of the spherical space curve. The two ruled surfaces defined by*

$$\begin{cases} \Phi_\gamma^t(s, v) = t(s) + v\gamma(s), \\ \Phi_t^\gamma(s, v) = \gamma(s) + vt(s). \end{cases} \tag{7}$$

are called γt-ruled surface pairs with the Sabban frame of the spherical curve.

The normal vector fields of the γt-ruled surface pairs, respectively, are found by

$$U_\gamma^t = \frac{\kappa_g t - vd}{\sqrt{\kappa_g^2 + v^2}}, \quad U_t^\gamma = \frac{-\kappa_g \gamma - d}{\sqrt{1 + \kappa_g^2}}. \tag{8}$$

If the unit normal vector vanishes at any point of a surface $\Phi(s, v)$, i.e. at any points, then these points are called the singular points of the surface. So the following result is obvious.

Corollary 3.2. *Let Φ_γ^t and Φ_t^γ be γt-ruled surface pairs,*

- *The ruled surface $\Phi_\gamma^t(s, v)$ is regular at points $\kappa_g \neq 0$ and $v \neq 0$,*
- *The ruled surface $\Phi_t^\gamma(s, v)$ is regular at every point.*

Theorem 3.3. *Let Φ_γ^t and Φ_t^γ be γt- ruled surface pairs,*

- *the base curve of γt-ruled surface pairs coincides with the striction curve,*

- *the ruled surface $\Phi^t_\gamma(s,v)$ is not developable,*
- *the ruled surface $\Phi^\gamma_t(s,v)$ is always developable.*

Proof.
- If Eq. (1) Sabban frame derivative formulae and Eq. (2) definition of striction curve are taken into account and necessary calculations are made for γt- ruled surface pairs, it is easily seen that the striction curve of both surfaces is equal to the base curve.
- From Eq. (3), the distribution parameter of $\Phi^\gamma_t(s,v)$ is $\delta_{\Phi^t_\gamma} = \kappa_g$. The necessary and sufficient condition for the distribution parameter to be 0 is that $\kappa_g = 0$. But this contradicts the regularity of the surface. Then the ruled surface $\Phi^\gamma_t(s,v)$ is not developable.
- From Eq. (3), the distribution parameter of the ruled surface $\Phi^\gamma_t(s,v)$ is $\delta_{\Phi^\gamma_t} = 0$. Therefore the ruled surface $\Phi^\gamma_t(s,v)$ is always developable.

□

Theorem 3.4. *Let Φ^t_γ and Φ^γ_t be γt- ruled surface pairs,*

- *the ruled surface $\Phi^t_\gamma(s,v)$ is minimal if and only if*

$$2\kappa_g \; + \sqrt{\kappa_g^2+v^2}((1-\kappa_g^2)\kappa_g - v(\kappa_g+\kappa_g')) = 0.$$

- *the ruled surface $\Phi^\gamma_t(s,v)$ is minimal if and only if $\kappa_g = cons\tan t$.*

Proof.
- By differentiating the first Eq. (7), with respect to s and v, respectively; using Sabban frame derivative formulas and Eq. (5), we find the components of the the first fundamental form of the Φ^t_γ ruled surface as follows:

$$E_{t\gamma} = 1+\kappa_g^2+v^2,\;\; F_{t\gamma} = -1,\;\; G_{t\gamma} = 1. \tag{9}$$

If the necessary the second order derivatives of the first Eq.(7), with respect to s and v expressed in Eq. (5), are found and the scalar product of these derivatives with the normal vector field Eq. (8), is calculated, we have the component of the the second fundamental form of the Φ^t_γ ruled surface Φ^t_γ as follows:

$$e_{t\gamma} = \frac{(1-\kappa_g^2)\kappa_g - v(\kappa_g+\kappa_g')}{\sqrt{\kappa_g^2+v^2}},\;\; f_{t\gamma} = \frac{\kappa_g}{\sqrt{\kappa_g^2+v^2}},\;\; g_{t\gamma} = 0. \tag{10}$$

Thus, by substituting Eqs.(9) and (10), into Eq. (6), the mean curvature of the ruled surface Φ^t_γ is calculated by

$$H_{t\gamma} = \frac{2\kappa_g \;\; + (1-\kappa_g^2)\kappa_g\sqrt{\kappa_g^2+v^2} - v(\kappa_g+\kappa_g')\sqrt{\kappa_g^2+v^2}}{2(\kappa_g^2+v^2)^{3/2}} \tag{11}$$

Since the necessary and sufficient condition for the surface to be minimal is $H_{t\gamma} = 0$, the condition stated in the theorem is fulfilled.

- Similarly, if the derivatives required for the second part of Eq. (7), are calculated and the related equations are used, the coefficients of the first and the second basis form of the Φ^γ_t ruled surface are found as follows:

$$E_{\gamma t} = 1+v^2\kappa_g^2+v^2,\;\; F_{\gamma t} = 1,\;\; G_{\gamma t} = 1, \tag{12}$$

and

$$e_{\gamma t} = \frac{-v\kappa_g'}{\sqrt{1+\kappa_g^2}}, \;\; f_{\gamma t} = 0, \;\; g_{\gamma t} = 0. \tag{13}$$

Thus, by substituting Eqs. (12) and (13), into Eq. (6), the mean curvature of the Φ_t^γ ruled surface is calculated by

$$H_{\gamma t} = \frac{-v\kappa_g'}{2(1+\kappa_g^2)^2}. \tag{14}$$

Since the necessary and sufficient condition for the surface to be minimal is $H_{\gamma t} = 0$, the condition stated in the theorem is fulfilled. □

Theorem 3.5. *Let Φ_γ^t and Φ_t^γ be γt-ruled surface pairs, the v-parameter curve of the γt-ruled surface pairs are at the same time*

- *geodesic,*
- *asymptotic.*

Proof. Let Φ_γ^t and Φ_t^γ be γt -ruled surface pairs. Since $\left(\Phi_\gamma^t\right)_{vv} \times U_\gamma^t$=0 and $\left(\Phi_t^\gamma\right)_{vv} \times U_t^\gamma = 0$, the v-parameter curves of the γt -ruled surface pairs are at the same time geodesic. On the other hand, $g_{t\gamma} = \left\langle \left(\Phi_\gamma^t\right)_{vv}, U_\gamma^t \right\rangle = 0$ and $g_{\gamma t} = \left\langle \left(\Phi_t^\gamma\right)_{vv}, U_t^\gamma \right\rangle = 0$, the v-parameter curves of the γt-ruled surface pairs are at the same time asymptotic. □

Theorem 3.6. *Let Φ_γ^t and Φ_t^γ be γt-ruled surface pairs, the s and v-parameter curves of the γt-ruled surface pairs are at the same time not lines of curvature.*

Proof. Let Φ_γ^t and Φ_t^γ be γt -ruled surface pairs. The necessary and sufficient conditions for the s and v-parameter curves to be line of curvature are $F_{t\gamma} = f_{t\gamma} = 0$ and $F_{\gamma t} = f_{\gamma t} = 0$, and when Eqn. (10) and (13), are considered, it is seen that these conditions are not fulfilled. Thus, the proof of the theorem is completed. □

Theorem 3.7. *Let Φ_γ^t and Φ_t^γ be γt-ruled surface pairs,*

- *the s-parameter curve of the Φ_γ^t ruled surface is not geodesic,*
- *the s-parameter curve of the Φ_t^γ ruled surface is geodesic if and only if $v = -\frac{1}{\kappa_g^2}$ and $\kappa_g^3 + \kappa_g + \kappa_g' = 0$,*
- *the s-parameter curve of the Φ_γ^t ruled surface is an asymptotic if and only if $v = \frac{(1-\kappa_g^2)\kappa_g}{\kappa_g+\kappa_g'}$,*
- *the s-parameter curve of the Φ_t^γ ruled surface is an asymptotic if and only if $\kappa_g = cons\tan t$.*

Proof.

- Let Φ_γ^t and Φ_t^γ be γt -ruled surface pairs. In order for the s-parameter curves to be geodesic on Φ_γ^t ruled surface, the following relations should hold:

  $$\left(\Phi_\gamma^t\right)_{ss} \times U_\gamma^t = \frac{1}{\sqrt{\kappa_g^2+v^2}}\left(-v(1-\kappa_g^2) - \kappa_g(\kappa_g+\kappa_g'), -v, -\kappa_g\right) = 0.$$

  With some manipulations on this, we get $v(1-\kappa_g^2) = 0, v = 0, \; \kappa_g = 0$. However, the points satisfying the above relations are the singular ones. Therefore, the s-parameter curves can not be geodesic on the Φ_γ^t ruled surface.

- On the other hand, in order for the s-parameter curves to be geodesic on Φ_t^γ ruled surface, the following relations should hold:
$\left(\Phi_t^\gamma\right)_{ss} \times U_t^\gamma = \frac{1}{\sqrt{1+\kappa_g^2}}\left((v + v^2\kappa_g^2), -\kappa_g(\kappa_g + v\kappa_g') - v, -\kappa_g\left(1 + v\kappa_g^2\right)\right) = 0.$
With some manipulations on this, we get $1 + v\kappa_g^2 = 0, -\kappa_g(\kappa_g + v\kappa_g') - v = 0, \kappa_g\left(1 + v\kappa_g^2\right) = 0$. If necessary calculations are made from these equations, $v = -\frac{1}{\kappa_g^2}$ and $\kappa_g^3 + \kappa_g + \kappa_g' = 0$ are obtained.

- the s-parameter curves are an asymptotic on the Φ_γ^t ruled surface if and only if $e_{t\gamma} = 0$. Therefore,
$$e_{t\gamma} = \frac{(1-\kappa_g^2)\kappa_g - v(\kappa_g+\kappa_g')}{\sqrt{\kappa_g^2+v^2}} = 0 \Leftrightarrow v = \frac{(1-\kappa_g^2)\kappa_g}{\kappa_g+\kappa_g'}.$$

- the s-parameter curves are an asymptotic on the Φ_t^γ ruled surface if and only if $e_{\gamma t} = 0$. Therefore,
$$e_{\gamma t} = \frac{-v\kappa_g'}{\sqrt{1+\kappa_g^2}} = 0 \Leftrightarrow \kappa_g' = 0 \Leftrightarrow \kappa_g = cons\tan t.$$

□

Corollary 3.8. *Let Φ_γ^t and Φ_t^γ be γt-ruled surface pairs,*

- *If the s-parameter curve of the Φ_γ^t ruled surface is an asymptotic, the Φ_γ^t ruled surface can not be minimal. Conversely, if the Φ_γ^t ruled surface is minimal, the s-parameter curve of the Φ_γ^t ruled surface can not be asymptotic.*
- *The s-parameter curve of the Φ_t^γ ruled surface is an asymptotic if and only if the Φ_t^γ ruled surface is minimal.*

3.2. γd-ruled surface pairs

Definition 3.9. *Let γ be a unit speed spherical curve and be Sabban frame $\{\gamma(s), t(s), d(s)\}$ of the spherical space curve. The two ruled surfaces defined by*

$$\begin{cases} \Phi_d^\gamma(s,v) = \gamma(s) + v d(s), \\ \Phi_\gamma^d(s,v) = d(s) + v\gamma(s). \end{cases} \tag{15}$$

are called γd-ruled surface pairs with the Sabban frame of the spherical curve.

The normal vector fields of the γd-ruled surface pairs, respectively, are found by

$$U_\gamma^d = \frac{(v-\kappa_g)d}{v-\kappa_g}, \quad U_d^\gamma = \frac{(1-v\kappa_g)\gamma}{1-v\kappa_g}. \tag{16}$$

Corollary 3.10. *Let Φ_d^γ and Φ_γ^d be γd-ruled surface pairs,*

- *the ruled surface $\Phi_d^\gamma(s,v)$ is regular at the points $1 - v\kappa_g \neq 0$,*
- *the ruled surface$\Phi_\gamma^d(s,v)$ is regular at the points where $v \neq \kappa_g$.*

Theorem 3.11. *Let Φ_γ^d and Φ_d^γ be γd- ruled surface pairs,*

- *the base curve of γd-ruled surface pairs does not coincides with the striction curve,*
- *γd-ruled surface pairs are always developable.*

Proof.
- If Eq. (1), Sabban frame derivative formulae and Eq. (2), definition of striction curve are taken into account and necessary calculations are made for the γd- ruled surface pairs, it is easily seen that the striction curve of both surfaces is not equal to the base curve.

- From Eq. (3), the distribution parameters of the γd-ruled surface pairs are $\delta_{\Phi_d^\gamma} = \delta_{\Phi_\gamma^d} = 0$. Therefore, the γd-ruled surface pairs are always developable.

□

Theorem 3.12. *Let Φ_γ^d and Φ_d^γ be γd- ruled surface pairs,*

- *the ruled surface $\Phi_d^\gamma(s,v)$ is not minimal,*
- *the ruled surface $\Phi_\gamma^d(s,v)$ is minimal if and only if $\kappa_g = 0$.*

Proof. • By differentiating the the first Eq. (15), with respect to s and v, respectively; using Sabban frame derivative formulas and Eq. (5), we find the components of the the first fundamental form of the Φ_d^γ ruled surface as follows:

$$E_{\gamma d} = (1 - v\kappa_g)^2, \;\; F_{\gamma d} = 0, \;\; G_{\gamma d} = 1. \tag{17}$$

If the necessary the second order derivatives of the first Eq. (15), with respect to s and v expressed in Eq. (5), are found and the scalar product of these derivatives with the normal vector field Eq. (16), is calculated, we have the component of the the second fundamental form of the Φ_d^γ ruled surface as follows:

$$e_{\gamma d} = \kappa_g v - 1, \;\; f_{\gamma d} = 0, \;\; g_{\gamma d} = 0. \tag{18}$$

Thus, by substituting Eqs.(17) and (18), into Eq. (6), the mean curvature of the Φ_d^γ ruled surface is calculated by

$$H_{\gamma d} = \frac{1}{2(v\kappa_g - 1)}. \tag{19}$$

Since the necessary and sufficient condition for the Φ_d^γ ruled surface to be minimal is $H_{\gamma d} = 0$, this is not possible. So, the Φ_d^γ ruled surface can not be minimal.

- Similarly, if the necessary calculations are made for the Φ_γ^d ruled surface,

$$E_{d\gamma} = (v - \kappa_g)^2, \;\; F_{d\gamma} = 0, \;\; G_{d\gamma} = 1, \tag{20}$$

and

$$e_{d\gamma} = \kappa_g(v - \kappa_g), \;\; f_{d\gamma} = 0, \;\; g_{d\gamma} = 0 \tag{21}$$

is obtained. Thus,

$$H_{d\gamma} = \frac{\kappa_g}{2(v - \kappa_g)}. \tag{22}$$

, and since it is possible for this surface to be minimal with $\kappa_g = 0$. Thus the proof is completed.

□

Theorem 3.13. *Let Φ_d^γ and Φ_γ^d be γd-ruled surface pairs, the s-parameter curve of the γd-ruled surface pairs are,*

- *none geodesic,*
- *none asymptotic,*
- *lines of curvature.*

Proof. • Let Φ_{γ}^{d} and Φ_{d}^{γ} be γd -ruled surface pairs. In order for the s-parameter curves to be geodesic on the Φ_{γ}^{d} ruled surface, the following relations should hold: $\left(\Phi_{d}^{\gamma}\right)_{ss} \times U_{d}^{\gamma}$=$(0,\kappa_g(1 - v\kappa_g), v\kappa_g') = 0$. With some manipulations on this, we get $\kappa_g(1 - v\kappa_g) = 0$, and $v\kappa_g'$=0.This contradicts the regularity of Φ_{γ}^{d} ruled surface. Therefore the s-parameter curves can not be geodesic. Similarly $\left(\Phi_{\gamma}^{d}\right)_{ss} \times U_{\gamma}^{d} = (-\kappa_g', v - \kappa_g, 0) = 0$, and hence $\kappa_g' = 0$, and $v - \kappa_g = 0$. This contradicts the regularity of the Φ_{γ}^{d} ruled surface. Therefore the s-parameter curves can not be geodesic.

- For the s-parameter curves of the γd-ruled surface pairs to be asymptotic, $e_{d\gamma} = \left\langle \left(\Phi_{\gamma}^{d}\right)_{ss}, U_{\gamma}^{d} \right\rangle = 0$ and $e_{\gamma d} = \left\langle \left(\Phi_{d}^{\gamma}\right)_{ss}, U_{d}^{\gamma} \right\rangle = 0$. From Eqs. (18) and (21), the conditions for $e_{d\gamma} = 0$ and $e_{\gamma d} = 0$ can not be fulfilled since this would contradict the regularity of the surface. Thus the s-parameter curves of the γd-ruled surface pairs are not asymptotic.
- Let Φ_{d}^{γ} and Φ_{γ}^{d} be γd-ruled surface pairs. The necessary and sufficient conditions for the s and v-parameter curves to be line of curvature are $F_{d\gamma} = f_{d\gamma} = 0$ and $F_{\gamma d} = f_{\gamma d} = 0$, and when Eqn. (18) and (20), are considered, it is seen that these conditions are fulfilled. Thus, the proof of the theorem is completed.

□

3.3. *td-ruled surface pairs*

Definition 3.14. *Let γ be a unit speed spherical curve and be Sabban frame $\{\gamma(s), t(s), d(s)\}$ of the spherical space curve. The two ruled surfaces defined by*

$$\begin{cases} \Phi_{d}^{t}(s, v) = t(s) + vd(s), \\ \Phi_{t}^{d}(s, v) = d(s) + vt(s). \end{cases} \tag{23}$$

are called td-ruled surface pairs with the Sabban frame of the spherical curve.

The normal vector fields of the *td*-ruled surface pairs, respectively, are found by

$$U_{d}^{t} = \frac{-v\kappa_g\gamma + t}{\sqrt{1 + \kappa_g^2 v^2}}, \quad U_{t}^{d} = \frac{-\kappa_g\gamma - d}{\sqrt{1 + \kappa_g^2}}. \tag{24}$$

Corollary 3.15. *Let Φ_{d}^{t} and Φ_{t}^{d} be td-ruled surface pairs,*

- *the ruled surface $\Phi_{d}^{t}(s, v)$ is regular at every points,*
- *the ruled surface $\Phi_{t}^{d}(s, v)$ is regular at every point.*

Theorem 3.16. *Let Φ_{d}^{t} and Φ_{t}^{d} be td- ruled surface pairs,*

- *the base curve of td-ruled surface pairs coincides with the striction curve,*
- *the ruled surface $\Phi_{d}^{t}(s, v)$ is always developable,*
- *the ruled surface $\Phi_{t}^{d}(s, v)$ is always developable.*

Proof. • If Eq. (1), Sabban frame derivative formulae and Eq. (2), definition of striction curve are taken into account and necessary calculations are made for *td*-ruled surface pairs, it is easily seen that the striction curve of both surfaces is equal to the base curve.

- From Eq. (3), the distribution parameter of $\Phi_{d}^{t}(s, v)$ is $\delta_{\Phi_{d}^{t}} = 0$. Since $\delta_{\Phi_{d}^{t}} = 0$, the surface is developable.

- From Eq. (3), the distribution parameter of $\Phi_t^d(s,v)$ is $\delta_{\Phi_t^d} = 0$. Therefore the surface is always developable.

□

Theorem 3.17. *Let Φ_d^t and Φ_t^d be td- ruled surface pairs,*

- *the ruled surface $\Phi_d^t(s,v)$ is minimal if and only if*

  $\kappa_g^2 - 1 - v\kappa_g' - v^2\kappa_g^2 = 0.$

- *the ruled surface $\Phi_t^d(s,v)$ is minimal if and only if $\kappa_g = cons\tan t$.*

Proof.

- By differentiating the first Eq. (23) with respect to s and v, respectively; using Sabban frame derivative formulas and Eq. (5), we find the components of the first fundamental form of the Φ_d^t ruled surface as follows:

$$E_{td} = 1 + \kappa_g^2 + v^2\kappa_g^2, \;\; F_{td} = \kappa_g, \;\; G_{td} = 1. \tag{25}$$

If the necessary the second order derivatives of the first Eq. (23), with respect to s and v expressed in Eq. (5), are found and the scalar product of these derivatives with the normal vector field Eq. (24), is calculated, we have the component of the the second fundamental form of the Φ_d^t ruled surface as follows:

$$e_{td} = \frac{-1 - \kappa_g^2 - v\kappa_g' - v^2\kappa_g^2}{\sqrt{1 + \kappa_g^2 v^2}}, \;\; f_{td} = \frac{-\kappa_g}{\sqrt{1 + \kappa_g^2 v^2}}, \;\; g_{td} = 0. \tag{26}$$

Thus, by substituting Eqs. (25) and (26) into Eq (6), the mean curvature of the ruled surface Φ_d^t is calculated by

$$H_{td} = \frac{-1 + \kappa_g^2 - v\kappa_g' - v^2\kappa_g^2}{2(1 + \kappa_g^2 v^2)^{3/2}}. \tag{27}$$

Since the necessary and sufficient condition for the surface to be minimal is $H_{td} = 0$, the condition stated in the theorem is fulfilled.

- Similarly, if the derivatives required for the the second part of Eq. (23), are calculated and the related equations are used, the coefficients of the the first and the second basis form of the Φ_t^d ruled surface are found as follows:

$$E_{dt} = \kappa_g^2 + v^2\kappa_g^2 + v^2, \;\; F_{dt} = -\kappa_g, \;\; G_{dt} = 1. \tag{28}$$

and

$$e_{dt} = \frac{-v\kappa_g'}{\sqrt{1 + \kappa_g^2}}, \;\; f_{dt} = 0, \;\; g_{dt} = 0. \tag{29}$$

Thus, by substituting Eqs. (28) and (29), into Eq. (6), the mean curvature of the Φ_t^d ruled surface is calculated by

$$H_{dt} = \frac{\kappa_g'}{2v(1 + \kappa_g^2)^{3/2}}. \tag{30}$$

Since the necessary and sufficient condition for the surface to be minimal is $H_{dt} = 0$, the condition stated in the theorem is fulfilled. □

Theorem 3.18. *Let Φ_d^t and Φ_t^d be td-ruled surface pairs, the v-parameter curve of the td-ruled surface pairs are at the same time*

- *geodesic,*
- *asymptotic.*

Proof. Let Φ_d^t and Φ_t^d be *td*-ruled surface pairs. Since $\left(\Phi_d^t\right)_{vv} \times U_d^t=0$ and $\left(\Phi_t^d\right)_{vv} \times U_t^d = 0$, the v-parameter curves of the *td*-ruled surface pairs are at the same time geodesic. On the other hand,
$g_{td} = \left\langle \left(\Phi_d^t\right)_{vv}, U_d^t \right\rangle = 0$ and $g_{dt} = \left\langle \left(\Phi_t^d\right)_{vv}, U_t^d \right\rangle = 0$, the v-parameter curves of the *td*-ruled surface pairs are at the same time asymptotic. □

Theorem 3.19. *Let Φ_d^t and Φ_t^d be td-ruled surface pairs, the s and v-parameter curves of the td-ruled surface pairs are lines of curvature if and only if $\kappa_g = 0$.*

Proof. Let Φ_d^t and Φ_t^d be *td* -ruled surface pairs. The necessary and sufficient conditions for the s and v-parameter curves to be line of curvature are $F_{td} = f_{td} = 0$ and $F_{dt} = f_{dt} = 0$, and when Eqs.(25), (26), (28) and (29), must be $\kappa_g = 0$. Thus, the proof of the theorem is completed. □

Theorem 3.20. *Let Φ_d^t and Φ_t^d be td ruled surface pairs,*

- *the s-parameter curve of the Φ_d^t ruled surface is geodesic if and only if $\kappa_g = 0$.*
- *the s-parameter curve of the Φ_t^d ruled surface is not geodesic,*
- *the s-parameter curve of the Φ_d^t ruled surface is an asymptotic if and only if $1 + \kappa_g^2 + v\kappa_g' + v^2\kappa_g^2 = 0$.*
- *the s-parameter curve of the Φ_t^d ruled surface asymptotic is an if and only if $\kappa_g = 0$.*

Proof. This theorem's proof is similar to the proof of Theorem 3.7. □

Corollary 3.21. *If the s and v-parameter curves of the td-ruled surface pairs are line of curvature, the surface cannot be minimal. Conversely, if the surface is minimal, the s and v-parameter curves cannot be line of curvature.*

Corollary 3.22. *The s-parameter curve of the Φ_d^t ruled surface is geodesic if and only if it is a line of curvature.*

Corollary 3.23. *If the s-parameter curve of the Φ_d^t ruled surface is an asymptotic, it cannot be geodesic and cannot be a line of curvature. Conversely, if the s-parameter curve of the Φ_d^t ruled surface is geodesic or a line of curvature, it cannot be asymptotic.*

Corollary 3.24. *The s-parameter curve of the Φ_t^d ruled surface is an asymptotic if and only if it is a line of curvature.*

Corollary 3.25. *The Φ_t^d ruled surface is minimal if and only if the s and v-parameter curves are an asymptotic or line of curvature.*

4. Example

Example 4.1. Let us consider spherical curve $\gamma(s) = \left(\cos(s), \sin(s)\cos(s), \sin^2(s)\right)$. It is obvious that the Sabban frame of γ:

$$\begin{cases} \gamma(s) = \left(\cos(s), \sin(s)\cos(s), \sin^2(s)\right), \\ t(s) = \left(-\sin(s), \cos(2s), \sin(2s)\right), \\ d(s) = \left(\sin^2(s), -\sin(s)(\sin^2(s) + 2\cos^2(s)), \cos^3(s)\right). \end{cases}$$

In view of the Sabban frame and Sabban formulae, we have a geodesic curvature of γ:

$$\kappa_g = -\cos(s)\sin^2(s) + 2\sin^3(s)\sin(2s) + 4\sin(s)\cos^2(s)\sin(2s) + 2\cos^3(s)\cos(2s).$$

The ruled surfaces $\Phi_{\gamma}^{t}(s,v)$, $\Phi_{t}^{\gamma}(s,v)$, $\Phi_{d}^{\gamma}(s,v)$, $\Phi_{\gamma}^{d}(s,v)$, $\Phi_{d}^{t}(s,v)$ and $\Phi_{t}^{d}(s,v)$ we obtain that

$$\Phi_{\gamma}^{t}(s,v) = \left(\tfrac{3}{5}\sin(s) + \tfrac{3v}{5\sqrt{2}}\left(\cos(s)-\sin(s)\right), \tfrac{3}{5}\cos(s) - \tfrac{3v}{5\sqrt{2}}\left(\sin(s)+\cos(s)\right), \tfrac{4}{5}s + \tfrac{4v}{5\sqrt{2}}\right),$$

$$\Phi_{t}^{\gamma}(s,v) = \left(\tfrac{3}{5}\sin(s) + v\tfrac{675}{\sqrt{2}}\cos(s), \tfrac{3}{5}\cos(s) - v\tfrac{675}{\sqrt{2}}\sin(s), \tfrac{4}{5}s + v\tfrac{200}{\sqrt{2}}\right),$$

$$\Phi_{d}^{\gamma}(s,v) = \left(\tfrac{3}{5}\sin(s) + \tfrac{3v}{5\sqrt{2}}\left(\tfrac{4}{5}\cos(s)-\sin(s)\right), \tfrac{3}{5}\cos(s) - \tfrac{3v}{5\sqrt{2}}\left(\tfrac{4}{5}\sin(s)+\cos(s)\right), \tfrac{4}{5}s - v\tfrac{12}{25\sqrt{2}}\right),$$

$$\Phi_{\gamma}^{d}(s,v) = \left(\tfrac{3}{5}\sin(s) + \tfrac{3v}{25\sqrt{3}}\left(9\cos(s)-\sin(s)\right), \tfrac{3}{5}\cos(s) - \tfrac{3v}{25\sqrt{3}}\left(9\sin(s)+\cos(s)\right), \tfrac{4}{5}s + v\tfrac{8}{25\sqrt{3}}\right),$$

$$\Phi_{d}^{t}(s,v) = \left(\tfrac{3}{5}\sin(s) + \tfrac{v}{5\sqrt{2}}\left(3\cos(s)+4\sin(s)\right), \tfrac{3}{5}\cos(s) + \tfrac{v}{5\sqrt{2}}\left(-3\sin(s)+4\cos(s)\right), \tfrac{4}{5}s + v\tfrac{4}{5\sqrt{2}}\right),$$

$$\Phi_{t}^{d}(s,v) = \left(\tfrac{3}{5}\sin(s) - v\tfrac{1}{25\sqrt{2}}\cos(s), \tfrac{3}{5}\cos(s) + v\tfrac{1}{25\sqrt{2}}\sin(s), \tfrac{4}{5}s + v\tfrac{32}{25\sqrt{2}}\right).$$

The following figure 1-3 shows the graphic of this ruled surfaces for $0 \le s \le 2\pi$, $0 \le v \le \frac{\pi}{2}$.

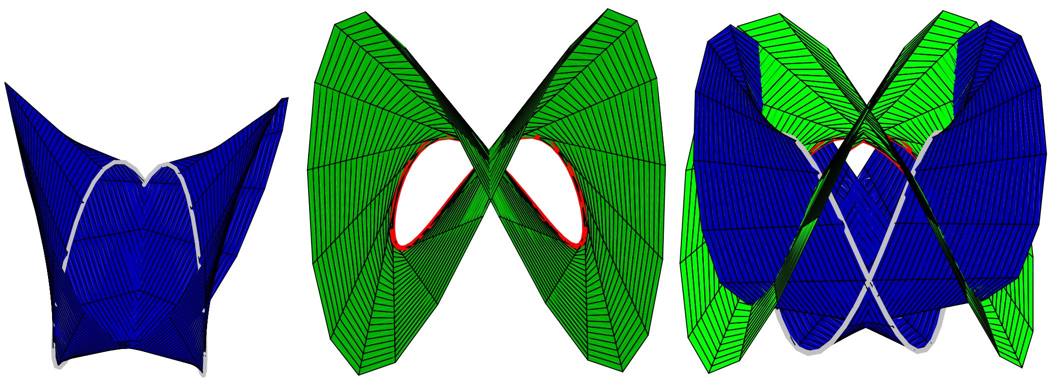

Figure 1: $t\gamma$-ruled surface pairs (Φ_{γ}^{t} ruled surface, Φ_{t}^{γ} ruled surface, Φ_{γ}^{t} and Φ_{t}^{γ} ruled surfaces).

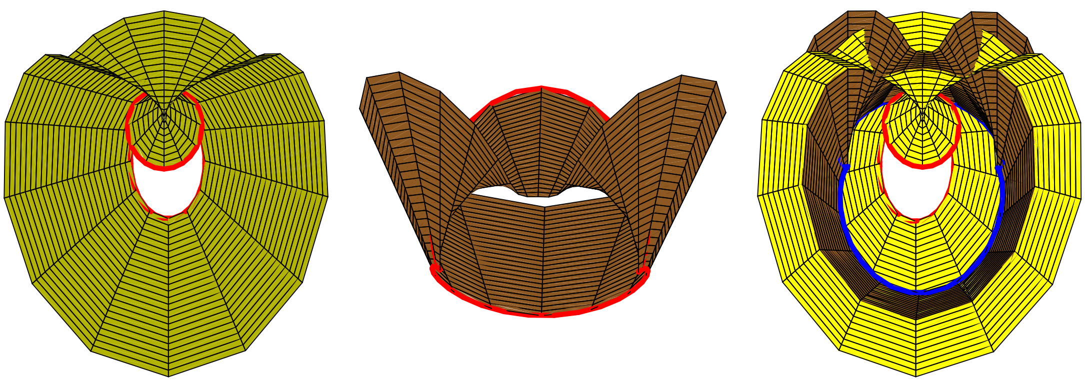

Figure 2: γd-ruled surface pairs(Φ_{d}^{γ} ruled surface, Φ_{γ}^{d} ruled surface, Φ_{d}^{γ} and Φ_{γ}^{d} ruled surfaces).

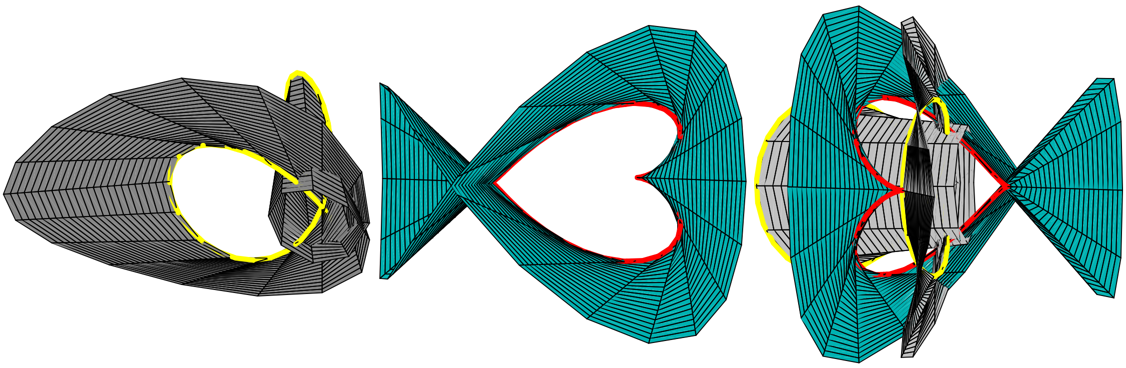

Figure 3: *td*-ruled surface pairs (Φ_d^t ruled surface, Φ_t^d ruled surface, Φ_d^t and Φ_t^d ruled surfaces).

5. Conclusion

In this paper, we define ruled surface pairs of a given curve on the unit sphere according to the Sabban frame and obtain important results on the characterisation of the surface. In particular, this study provides new information as ruled surface pairs have not been analysed in this context before. Also, we study the relation between the surface being developable and minimal when the parameter curves of the ruled surface pairs are special curves (geodesic, asymptotic and line of curvature). Finally, by providing examples and accompanying graphical representations of the pairs of ruled surfaces, this study is an original contribution that will provide a valuable basis for future research in this area. This work can be considered in Minkowski space, Galilean space or various higher dimensional spaces.

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