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$*\,(g,e)$ quasi-normal ring and dual $*\,(g,e)$ quasi-normal ring

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Abstract. Let R be a $*$-ring and $g, e \in E(R)$, the set of idempotents of R. The ring R is said to be a (resp. dual) $*\,(g,e)$ quasi-normal ring if $gae = 0$ implies (resp. $g^*Rae = 0$) $gaRe^* = 0$. We prove that R is (resp. dual) $*\,(g,e)$ quasi-normal if and only if (resp. $g^*R(1-g)Re = 0$) $gR(1-e)Re^* = 0$. As by-products, we give a $*$-ring, which is clean, almost clean, $*$-clean, almost $*$-clean, $*$-regular and unit regular. Moreover, we use some matrix rings to describe (dual) $*\,(g,e)$ quasi-normal rings. Finally, we consider the relations between (dual) $*\,(g,e)$ quasi-normal rings and other generalized inverses.

1. Introduction

Throughout the paper, all rings are associative with identity. The symbols $\mathbb{Z}_2$, $E(R)$, $Z(R)$, $U(R)$, $M_2(R)$ and $T_2(R)$ stand for the ring of integers modulo the positive integer 2, the set of all idempotents, the center, the set of all invertible elements of R, 2×2 matrix ring over R, and 2×2 upper triangular matrix ring over R, respectively. In a ring R, an idempotent $e \in E(R)$ is called left (resp. right) semicentral if $ae = eae$ (resp. $ea = eae$) for each $a \in R$. The set of all left (resp. right) semicentral idempotents in R is denoted by $S_l(R)$ (resp. $S_r(R)$). In [7], Lam said an idempotent e in a ring R q-central if $eR(1-e)Re = 0$. In particular, R is q-abelian if each idempotent in R is q-central. In [1] and [22], q-central idempotents and q-abelian rings are called inner Peirce trivial idempotents and quasi-normal rings, respectively. Furthermore, in [22], Wei defined quasi-normal rings, which are q-abelian rings. We need to point out that in [5, 18, 20], q-central idempotents and q-abelian rings are seen as "2-central rings" and "2-Abelian rings", respectively. Then Meng et al. defined and studied e-symmetric rings, weakly e-symmetric rings, (g,e)-symmetric rings in [8, 9, 11, 13].

A ring R is called an involution ring (or a $*$-ring) if there exists a bijection $* : R \to R, a \mapsto a^*$ such that for any $a, b \in R$,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*.$$

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In a $*$-ring, idempotents also play an important role. For examples, the concepts of $*$-regular rings [3], $*$-clean rings and almost $*$-clean rings [21] are related to idempotents or projections. Inspired the previous works, in this paper, we define $*$ (g,e) quasi-normal rings and dual $*$ (g,e) quasi-normal rings. Many properties of these two rings are obtained.

2. $*$ (g,e) quasi-normal ring

In this section, we will give the definitions of $*$ (g,e) quasi-normal rings and dual $*$ (g,e) quasi-normal rings. Moreover, many properties of these two kinds of rings are studied.

Definition 2.1. *Let R be a $*$-ring and $g,e \in E(R)$. The ring R is called a $*$ (g,e) quasi-normal ring if $gae = 0$ implies $gaRe^* = 0$ for any $a \in R$.*

Theorem 2.2. *Let R be a $*$-ring and $g,e \in E(R)$. Then the following statements are equivalent:*
(1) R is $*$ (g,e) quasi-normal;
(2) $gR(1-e)Re^* = 0$;
(3) $gabe^* = gaebe^*$ for any $a,b \in R$.

Proof. (1)⇒(2) Since $ga(1-e)e = 0$ for any $a \in R$, $ga(1-e)Re^* = 0$, which implies $gR(1-e)Re^* = 0$.
(2)⇒(3) It is obvious.
(3)⇒(1) By hypothesis, assume $gae = 0$, then $gaRe^* = (gae)Re^* = 0$. Hence, R is $*$ (g,e) quasi-normal. □

Definition 2.3. *Let R be a $*$-ring and $e \in E(R)$. The idempotent e is said to be $*$-q-central if $e^*R(1-e)Re^* = 0$. The set of all $*$-q-central idempotents of R is denoted by q^*-idem(R). In particular, R is called $*$-quasi-normal if $e \in q^*$-idem(R) for each $e \in E(R)$.*

Remark 2.4. *Let R be a $*$-ring and $e \in E(R)$. Then $e \in q^*$-idem(R) if and only if R is a $*$ (e^*,e) quasi-normal ring.*

Recall that in a $*$-ring, an element $a \in R$ is called a partial isometry if $a = aa^*a$ [16]. The symbol R^{PI} stands for the set of all partial isometries of R.

Proposition 2.5. *Let R be a $*$-ring and $e \in R^{PI} \cap S_l(R)$. Then R is $*$ (g,e) quasi-normal for any $g \in E(R)$.*

Proof. On one hand, $e = ee^*e$ implies $gR(1-e)Re^* = gR(1-e)Re^*ee^* \subseteq gR((1-e)Re)e^*$. On the other hand, $e \in S_l(R)$ shows $(1-e)Re = 0$. Hence, $gR(1-e)Re^* = 0$. □

By Proposition 2.5, we have the following.

Theorem 2.6. *Let R be a $*$-ring and $e \in S_l(R)$. Then R is $*$ $(1-e,e)$ quasi-normal if and only if $e \in R^{PI}$.*

Proof. ⇒ By hypothesis, $(1-e)R(1-e)Re^* = 0$. Hence, $(1-e)Re^* = 0$, i.e., $eae^* = ae^*$. Taking $a = e^*$, then $ee^* = e^*$, so $e^*(ee^*) = e^*e^* = e^*$.
⇐ It follows from Proposition 2.5. □

Similar to Proposition 2.5 and Theorem 2.6, we have the following.

Proposition 2.7. *Let R be a $*$-ring. Then following statements hold.*
(1) If $e \in R^{PI} \cap S_r(R)$, then R is $*$ $(g, 1-e^*)$ quasi-normal for any $g \in E(R)$.
(2) If $e \in S_r(R)$, then R is $*$ $(e^*, 1-e^*)$ quasi-normal if and only if $e \in R^{PI}$.

Recall that a $(*)$ ring is called (g,e) quasi-normal if $gR(1-e)Re = 0$, where $g,e \in E(R)$.

Theorem 2.8. *Let R be a $*$-ring, $g \in E(R)$ and $e \in E(R) \cap R^{PI}$. Then R is (g,e) quasi-normal if and only if R is $*$ (g,e) quasi-normal.*

Proof. ⇒ By hypothesis, $e = ee^*e$ and $gR(1-e)Re = 0$. Hence, $gR(1-e)Re^*e \subseteq gR(1-e)Re = 0$. It follows that $gR(1-e)Re^* = (gR(1-e)Re^*e)e^* = 0$.
⇐ Similarly, one can prove this conclusion. □

Remark 2.9. *In Propositions 2.5, 2.7 and Theorems 2.6, 2.8, if we replace R^{PI} by $PE(R)$, the new conclusions all hold.*

Note that an idempotent e in a $*$-ring R is $*$-q-central implies $e = ee^*e$. Hence, by Theorem 2.8, we have the following.

Proposition 2.10. *Let R be a $*$-ring, $g \in E(R)$ and $e \in q^*$-idem(R). Then R is (g, e) quasi-normal if and only if R is $*$ (g, e) quasi-normal.*

Corollary 2.11. *Let R be a $*$ quasi-normal ring. Then R is (g, e) quasi-normal if and only if R is $*$ (g, e) quasi-normal for any $g, e \in E(R)$.*

An involution $*$ in a ring is called proper if any nonzero element $a \in R$, $aa^* = 0$ implies $a = 0$.

Proposition 2.12. *Let R be a $*$ ring and $g, e, e^*e \in E(R)$ with $*$ being proper. Then R is (g, e) quasi-normal if and only if R is $*$ (g, e) quasi-normal.*

Proof. It is sufficient to show $e = ee^*e$, and the rest proof follows from the proof of Theorem 2.8. In fact,

$$
\begin{aligned}
(e^* - e^*ee^*)(e^* - e^*ee^*)^* &= (e^* - e^*ee^*)(e - ee^*e) \\
&= e^*e - e^*ee^*e - e^*ee^*e + e^*ee^*ee^*e \\
&= 0.
\end{aligned}
$$

Hence, by $*$ is proper, $e = ee^*e$. □

Remark 2.13. *In Proposition 2.12, if e^*e is replaced by ee^*, then the conclusion also holds.*

Recall that an element a in a $*$-ring R is said to be normal if $aa^* = a^*a$ [4]. The set of all normal elements of R is denoted by R^{Nor}. It is easy to see that if $e \in E(R) \cap R^{Nor}$, then $ee^*, e^*e \in E(R)$. Hence, by Proposition 2.12, we infer the following.

Proposition 2.14. *Let R be a $*$-ring and $e \in E(R) \cap R^{Nor}$. Then R is (g, e) quasi-normal if and only if R is $*$ (g, e) quasi-normal.*

An element $a \in R$ is said to be regular if $a \in aRa$ [2]. The set of all regular elements in R is denoted by R^{reg}. An element $b \in R$ is called a inner inverse of a if $a = aba$, which is denoted by a^-. In general, if $a \in R^{reg}$, the inner inverse of a is not unique. The set of all inner inverses of a is written by $a\{1\}$. Obviously, $E(R) \subseteq R^{reg}$. A ring R is called regular if each element in R is regular. A $*$-ring R is said to be $*$-regular if R is regular and the involution of R is proper, or equivalently if R is regular and for any $x \in R$, there is a projection $p \in R$ such that $xR = pR$ [3].

Proposition 2.15. *Let R be a $*$-regular ring, $e \in PE(R)$, $g \in E(R)$ and $gR = e^*R$. Then $e \in q^*$-idem(R) if and only if R is $*$ (g, e) quasi-normal.*

In the following, we consider a $*$-regular ring. Let $R_0 = \mathbb{Z}_2[x]/(x^3 - 1)$ and define $* : \alpha_0 + \alpha_1 x + \alpha_2 x^2 \mapsto \alpha_0 + \alpha_2 x + \alpha_1 x^2$, where $\alpha_2, \alpha_1, \alpha_2 \in \mathbb{Z}_2$. Then it is easy to check that R_0 is a $*$-ring. Moreover, note that for any $a = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \in R_0$, we have

$$
\begin{aligned}
a^2 &= (\alpha_0 + \alpha_1 x + \alpha_2 x^2)^2 \\
&= \alpha_0^2 + \alpha_1^2 x^2 + \alpha_2^2 x \\
&= \alpha_0 + \alpha_2 x + \alpha_1 x^2 \\
&= (\alpha_0 + \alpha_1 x + \alpha_2 x^2)^* \\
&= a^*.
\end{aligned}
$$

Lemma 2.16. $E(R_0) = PE(R_0) = \{0, 1, x + x^2, 1 + x + x^2\}$.

Proof. Let $a = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \in R_0$, $\alpha_i \in \mathbb{Z}_2$, $i = 0, 1, 2$. Then

$$a^* = (\alpha_0 + \alpha_1 x + \alpha_2 x^2)^* = \alpha_0 + \alpha_2 x + \alpha_1 x^2.$$

$a^* = a$ implies that $\alpha_1 = \alpha_2$. Thus, $E(R_0) = PE(R_0) = \{0, 1, x + x^2, 1 + x + x^2\}$. □

Remark 2.17. *In R_0, taking $e = x$, then $(1 - e)e^* = (1 - x)x^2 = x^2 - 1 \neq 0$. However, $(1 - e)e = 0$. This shows that a ring is (g, e) quasi-normal can not yield that it is $*$ (g, e) quasi-normal.*

Next, we prove that R_0 is regular.

Theorem 2.18. $R_0^{reg} = R_0$.

Proof. Notice that the number of the elements in R_0 is finite, so we can consider each element of R_0. Let $a = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \in R_0$.

Case I: 0, 1. It is obvious that $0, 1 \in R_0^{reg}$, $0\{1\} = R_0$ and $1\{1\} = \{1\}$.

Case II: x. By a straightforward computation, we have

$$\begin{aligned} xax &= x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2) \\ &= \alpha_1 + \alpha_2 x + \alpha_0 x^2 \\ &= x. \end{aligned}$$

Then $a = x^2$, which shows that $x \in R_0^{reg}$, and $x\{1\} = \{x^2\}$. Similar to the case of x, one can check that $x^2 \in R_0^{reg}$, and $x^2\{1\} = \{x\}$.

Case III: $1 + x$. By a direct calculation, we have

$$\begin{aligned} (1 + x)a(1 + x) &= (1 + x^2)(\alpha_0 + \alpha_1 x + \alpha_2 x^2) \\ &= (\alpha_0 + \alpha_1) + (\alpha_1 + \alpha_2)x + (\alpha_0 + \alpha_2)x^2 \\ &= 1 + x. \end{aligned}$$

Thus, we get the following equalities

$$\begin{cases} \alpha_0 + \alpha_1 = 1, \\ \alpha_1 + \alpha_2 = 1, \\ \alpha_0 + \alpha_2 = 0. \end{cases} \tag{1}$$

It is easy to compute that (1) has the following two solutions:

$$\begin{cases} \alpha_0 = \alpha_2 = 0 \\ \alpha_1 = 1, \end{cases} \qquad \begin{cases} \alpha_0 = \alpha_2 = 1 \\ \alpha_1 = 0. \end{cases}$$

It follows that $1 + x \in R_0^{reg}$, and $1 + x\{1\} = \{x, 1 + x^2\}$. Similarly, it is not difficult to check that $1 + x^2, x + x^2 \in R_0^{reg}$, and $1 + x^2\{1\} = \{x^2, 1 + x\}$, $x + x^2\{1\} = \{1, x + x^2\}$.

Case IV: $1 + x + x^2$. It is easy to compute

$$\begin{aligned} (1 + x + x^2)a(1 + x + x^2) &= (1 + x + x^2)(\alpha_0 + \alpha_1 x + \alpha_2 x^2) \\ &= (\alpha_0 + \alpha_1 + \alpha_2)(1 + x + x^2). \end{aligned}$$

Hence, we get the following equality

$$\alpha_0 + \alpha_1 + \alpha_2 = 1,$$

which shows that $1 + x + x^2 \in R_0^{reg}$, and $1 + x + x^2\{1\} = \{1, x, x^2, 1 + x + x^2\}$. Thus, the proof is completed. □

Proposition 2.19. *R_0 is a $*$-regular ring.*

Proof. By Theorem 2.18, R_0 is regular, it suffices to show that the involution $*$ in R_0 is proper. In fact, for any $a = \alpha_0 + \alpha_1 x + \alpha_2 x^2$, we have $a^* = \alpha_0 + \alpha_1 x^2 + \alpha_2 x$. A direct computation shows

$$\begin{aligned} aa^* &= (\alpha_0 + \alpha_1 x + \alpha_2 x^2)(\alpha_0 + \alpha_1 x^2 + \alpha_2 x) \\ &= \alpha_0^2 + \alpha_0\alpha_1 x + \alpha_0\alpha_2 x \\ &+ \alpha_0\alpha_1 x^2 + \alpha_1^2 + \alpha_1\alpha_2 x \\ &+ \alpha_0\alpha_2 x + \alpha_1\alpha_2 x^2 + \alpha_2^2 \\ &= (\alpha_0^2 + \alpha_1^2 + \alpha_2^2) + (\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2)x \\ &= (\alpha_0 + \alpha_1 + \alpha_2) + (\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2)x. \end{aligned}$$

If $aa^* = 0$, then either $a = 0$ or $\alpha_i = 0$ and $\alpha_j = \alpha_k = 1$, where $\{i, j, k\} = \{0, 1, 2\}$. Assume that $\alpha_i = 0$ and $\alpha_j = \alpha_k = 1$, then $\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2 = 1 \neq 0$, a contradiction. Hence, the involution $*$ in R_0 is proper. □

Remark 2.20. *In R_0, taking any $g = e \in E(R_0)$, then $gR_0 = e^*R_0$, which illustrates Proposition 2.15. Moreover, if we take $g \neq e \in E(R_0)$, then $gR_0 \neq e^*R_0$.*

A ring is clean (resp. almost clean) if its every element can be written as the sum of a unit (resp. regular element) and an idempotent [17] (resp. [12]). As a generalization, a $*$-ring is $*$-clean (resp. almost $*$-clean) if its every element can be written as the sum of a unit (resp. regular element) and a projection [21]. Here, we first give a result about regular elements in a $*$-ring.

Lemma 2.21. *Let R be a $*$-ring. Then $U(R) \subseteq R^{reg}$. Moreover, if $a \in U(R)$, then $a\{1\} = \{a^{-1}\}$.*

Proposition 2.22. *$U(R_0) = \{1, x, x^2\}$.*

Proof. It is evident to see that $1, x, x^2 \in U(R_0)$, and $1^{-1} = 1, x^{-1} = x^2, (x^2)^{-1} = x$. In order to prove that $U(R_0) = \{1, x, x^2\}$, it is enough to show that $1 + x, 1 + x^2, x + x^2, 1 + x + x^2$ are not invertible. We first consider $1 + x$, assume that $1 + x \in U(R_0)$, then by Lemma 2.21, we know that $(1 + x)^{-1} = x$ or $(1 + x)^{-1} = 1 + x^2$. However, by a direct computation,

$$(1 + x)x = (1 + x)(1 + x^2) = x + x^2 \neq 1,$$

a contradiction. Hence, $1 + x \notin U(R_0)$. Similarly, we can get the following results

$$(1 + x^2)x^2 = (1 + x^2)(1 + x) = (x + x^2)^2 = x + x^2 \neq 1,$$

and

$$(1 + x + x^2)x = (1 + x + x^2)x^2 = (1 + x + x^2)^2 = 1 + x + x^2 \neq 1.$$

This completes the proof. □

By Proposition 2.22, we have the following.

Proposition 2.23. *R_0 is clean, almost clean, $*$-clean and almost $*$-clean.*

An element a in a ring is called unit regular if $a = aua$ for some invertible element u. In particular, a ring is said to be unit regular if each element is unit regular [6].

Proposition 2.24. *R_0 is unit regular.*

Remark 2.25. *In [14], the author proposed a question that can we find a $*$-regular ring being not unit regular. It is a pity that the example R_0 is false.*

Let R be a ring and $g, e \in E(R)$. Then ring R is called dual (g, e) quasi-normal if $gR(1-g)Re = 0$.

Definition 2.26. *Let R be a $*$-ring and $g, e \in E(R)$. The ring R is said to be dual $*$ (g, e) quasi-normal if $gae = 0$ implies $g^*R(1-g)Re = 0$.*

Similar to the condition of $*$ (g, e) quasi-normal rings, we have the following.

Proposition 2.27. *Let R be a $*$-ring and $g, e \in E(R)$. Then*
(1) The following statements are equivalent:
(a) R is dual $*$ (g, e) quasi-normal;
(b) $gR^*(1-g)Re = 0$;
(c) $g^*abe = g^*agbe$ for any $a, b \in R$.
(2) $e \in q^*$-idem(R) if and only if R is dual $*$ (e, e^*) quasi-normal.
(3) If $g \in R^{PI} \cap S_r(R)$, then R is dual $*$ (g, e) quasi-normal.
(4) If $e \in S_r(R)$, then R is dual $*$ $(e, 1-e)$ quasi-normal if and only if $e \in R^{PI}$.
(5) If $g \in R^{PI} \cap S_l(R)$, then R is dual $*$ $(1-g^*, e)$ quasi-normal.
(6) If $e \in S_l(R)$, then R is dual $*$ $(1-e^*, e^*)$ if and only if $e \in R^{PI}$.
(7) If $g \in E(R) \cap R^{PI}$, then R is dual (g, e) quasi-normal if and only if R is dual $*$ (g, e) quasi-normal.
(8) In (3), (4), (5), (6), if R^{PI} is replaced by $PE(R)$, then the results also hold.
(9) If $g \in q^*$-idem(R), then R is dual (g, e) quasi-normal if and only if R is dual $*$ (g, e) quasi-normal.
(10) If R is $*$-quasi-normal, then R is dual (g, e) quasi-normal if and only if R is dual $*$ (g, e) quasi-normal.
(11) If the involution $*$ in R is proper and $gg^* \in E(R)$ (or $g^*g \in E(R)$), then R is dual (g, e) quasi-normal if and only if R is dual $*$ (g, e) quasi-normal.
(12) If $g \in E(R) \cap R^{Nor}$. then R is dual (g, e) quasi-normal if and only if R is dual $*$ (g, e) quasi-normal.

3. Relations with matrix rings

In this section, we will use matrix rings to characterize $*$ (g, e) quasi-normal and dual $*$ (g, e) quasi-normal rings. Let R be a $*$-ring, throughout the following in this section, for $g, e \in E(R)$, we always consider $G = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$ and $E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$.

Let R be a $*$-ring and $M_2(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in R\}$. Define $* : M_2(R) \to M_2(R), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$. By [23, Proposition 2], we know that $M_2(R)$ is a $*$-ring.

Proposition 3.1. *Let R be a $*$-ring and $g, e \in E(R)$. Then R is $*$ (g, e) (resp. dual $*$ (g, e)) quasi-normal if and only if $M_2(R)$ is $*$ (G, E) (resp. dual $*$ (G, E)) quasi-normal.*

Proof. $\Rightarrow$ Since R is $*$ (g, e) quasi-normal, $gR(1-e)Re^* = 0$. For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} f & h \\ i & j \end{pmatrix} \in M_2(R)$. A straightforward computation shows

$$\begin{aligned} GA(1-E)BE^* &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix} \begin{pmatrix} f & h \\ i & j \end{pmatrix} \begin{pmatrix} e^* & 0 \\ 0 & e^* \end{pmatrix} \\ &= \begin{pmatrix} (ga(1-e)f + gb(1-e)i)e^* & (ga(1-e)h + gb(1-e)j)e^* \\ (gc(1-e)f + gd(1-e)i)e^* & (gc(1-e)h + gd(1-e)j)e^* \end{pmatrix}. \end{aligned}$$

By $gR(1-e)Re^* = 0$, we have $GA(1-E)BE^* = 0$. Hence, $M_2(R)$ is $*$ (G, E) quasi-normal.
$\Leftarrow$ By "$\Rightarrow$",

$$0 = GA(1-E)BE^* = \begin{pmatrix} (ga(1-e)f + gb(1-e)i)e^* & (ga(1-e)h + gb(1-e)j)e^* \\ (gc(1-e)f + gd(1-e)i)e^* & (gc(1-e)h + gd(1-e)j)e^* \end{pmatrix}.$$

This shows $gR(1-e)Re^* = 0$. Hence, R is $*$ (g,e) quasi-normal.

Similarly, one can prove that the case "dual" holds. □

Let R be a $*$-ring and $T_2(R) = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a,b,c \in R\}$. Define $* : T_2(R) \to T_2(R), \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a^* & 0 \\ 0 & c^* \end{pmatrix}$. Then it is easy to check that $T_2(R)$ is a $*$-ring.

Proposition 3.2. *Let R be a $*$-ring and $g,e \in E(R)$. Then R is $*$ (g,e) (resp. dual $*$ (g,e)) quasi-normal if and only if $T_2(R)$ is $*$ (G,E) (resp. dual $*$ (g,e)) quasi-normal.*

Proof. ⇒ For any $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, B = \begin{pmatrix} d & f \\ 0 & h \end{pmatrix} \in T_2(R)$. By a direct computation, we have

$$\begin{aligned} GA(1-E)BE^* &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix}\begin{pmatrix} d & f \\ 0 & h \end{pmatrix}\begin{pmatrix} e^* & 0 \\ 0 & e^* \end{pmatrix} \\ &= \begin{pmatrix} ga(1-e)de^* & (ga(1-e)f + gb(1-e)h)e^* \\ 0 & gc(1-e)he^* \end{pmatrix}. \end{aligned}$$

Since $gR(1-e)Re^* = 0$, $GA(1-E)BE^* = 0$. It follows that $T_2(R)$ is $*$ (G,E) quasi-normal.

⇐ By "⇒", we have

$$0 = GA(1-E)BE^* = \begin{pmatrix} ga(1-e)de^* & (ga(1-e)f + gb(1-e)h)e^* \\ 0 & gc(1-e)he^* \end{pmatrix},$$

which implies $gR(1-e)Re^* = 0$. Hence, R is $*$ (g,e) quasi-normal.

Similarly, one can check that the case "dual" holds. □

Remark 3.3. *In $M_2(R)$, if we define $* : M_2(R) \to M_2(R), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix}$, then $M_2(R)$ is a $*$-ring. Moreover, in this case, the revised Proposition 3.1 is true.*

Let R be a $*$-ring and $L_2(R) = \{\begin{pmatrix} a & 0 \\ b & a-b \end{pmatrix} | a,b \in R\}$. Define $* : L_2(R) \to L_2(R)$, $\begin{pmatrix} a & 0 \\ b & a-b \end{pmatrix} \mapsto \begin{pmatrix} a^* & 0 \\ 0 & a^* \end{pmatrix}$. One can easily check that $L_2(R)$ is a $*$-ring.

Proposition 3.4. *Let R be a $*$-ring and $g,e \in E(R)$. Then R is $*$ (g,e) (resp. dual $*$ (g,e)) quasi-normal if and only if $L_2(R)$ is $*$ (G,E) (resp. dual $*$ (g,e)) quasi-normal.*

Proof. ⇒ For any $A = \begin{pmatrix} a & 0 \\ b & a-b \end{pmatrix}, B = \begin{pmatrix} c & 0 \\ d & c-d \end{pmatrix} \in L_2(R)$. It is easy to compute

$$\begin{aligned} GA(1-E)BE^* &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} a & 0 \\ b & a-b \end{pmatrix}\begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix}\begin{pmatrix} c & 0 \\ d & c-d \end{pmatrix}\begin{pmatrix} e^* & 0 \\ 0 & e^* \end{pmatrix} \\ &= \begin{pmatrix} ga(1-e)ce^* & 0 \\ (gb(1-e)c + g(a-b)(1-e)d)e^* & g(a-b)(1-e)(c-d)e^* \end{pmatrix}. \end{aligned}$$

By $gR(1-e)Re^* = 0$, we have $GA(1-E)BE^* = 0$. Hence, $L_2(R)$ is $*$ (G,E) quasi-normal.

⇐ By "⇒", we have

$$0 = GA(1-E)BE^* = \begin{pmatrix} ga(1-e)ce^* & 0 \\ (gb(1-e)c + g(a-b)(1-e)d)e^* & g(a-b)(1-e)(c-d)e^* \end{pmatrix}.$$

It follows that $gR(1-e)Re^*=0$, and so R is $*$ (g,e) quasi-normal.

Similarly, one can show that the case "dual" is true. □

Let R be a $*$-ring and $V_2(R)=\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a,b\in R\}$. Define $*: V_2(R)\to V_2(R)$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\mapsto \begin{pmatrix} a^* & 0 \\ 0 & a^* \end{pmatrix}$. It is not difficult to check that $V_2(R)$ is a $*$-ring.

Proposition 3.5. *Let R be a $*$-ring and $g,e\in E(R)$. Then R is $*$ (g,e) (resp. dual $*$ (g,e)) quasi-normal if and only if $V_2(R)$ is $*$ (G,E) (resp. dual $*$ (g,e)) quasi-normal.*

Proof. ⇒ For any $A=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B=\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\in V_2(R)$. One can easily compute

$$\begin{aligned} GA(1-E)BE^* &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\begin{pmatrix} e^* & 0 \\ 0 & e^* \end{pmatrix} \\ &= \begin{pmatrix} ga(1-e)ce^* & (ga(1-e)d+gb(1-e)c)e^* \\ 0 & ga(1-e)ce^* \end{pmatrix}. \end{aligned}$$

Since $gR(1-e)Re^*=0$, $GA(1-E)BE^*=0$. Hence, $V_2(R)$ is $*$ (G,E) quasi-normal.

⇐ By "⇒",

$$0=GA(1-E)BE^*=\begin{pmatrix} ga(1-e)ce^* & (ga(1-e)d+gb(1-e)c)e^* \\ 0 & ga(1-e)ce^* \end{pmatrix},$$

which implies $gR(1-e)Re^*=0$, and hence R is $*$ (g,e) quasi-normal.

Similarly, one can prove that the condition "dual" is true. □

Let R be a ring and $V_2^{(f)}(R)=\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a,b\in R\}$, where $f\in E(R)$. For any $A=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $B=\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}$, where $a,b,c,d\in R$. Define the addition and multiplication of $V_2^{(f)}(R)$ as follows:

$$A+B=\begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix},\quad AB=\begin{pmatrix} ac & ad+bc-fbd \\ 0 & ac \end{pmatrix}.$$

It is not difficult to check that $V_2^{(f)}(R)$ is a ring if and only if $f\in Z(R)$. Furthermore, if R is a $*$-ring, $f\in E(R)\cap Z(R)$, define $*: V_2^{(f)}(R)\to V_2^{(f)}(R)$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\mapsto\begin{pmatrix} a^* & 0 \\ 0 & a^* \end{pmatrix}$, then one can prove that $V_2^{(f)}(R)$ is a $*$-ring.

Theorem 3.6. *Let R be a $*$-ring, $g,e\in E(R)$ and $f\in E(R)\cap Z(R)$. Then R is $*$ (g,e) (resp. dual $*$ (g,e)) quasi-normal if and only if $V_2^{(f)}(R)$ is $*$ (G,E) (resp. dual $*$ (g,e)) quasi-normal.*

Proof. ⇒ For any $A=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B=\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\in V_2^{(f)}(R)$. A straightforward computation shows

$$\begin{aligned} GA(1-E)BE^* &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\begin{pmatrix} e^* & 0 \\ 0 & e^* \end{pmatrix} \\ &= \begin{pmatrix} ga(1-e)ce^* & (ga(1-e)d+gb(1-e)c)e^*-fgb(1-e)de^* \\ 0 & ga(1-e)ce^* \end{pmatrix}. \end{aligned}$$

By $gR(1-e)Re^*=0$, we have $GA(1-E)BE^*=0$. Hence, $V_2^{(f)}(R)$ is $*$ (G,E) quasi-normal.

⇐ By "⇒",

$$0 = GA(1-E)BE^* = \begin{pmatrix} ga(1-e)ce^* & (ga(1-e)d + gb(1-e)c)e^* - fgb(1-e)de^* \\ 0 & ga(1-e)ce^* \end{pmatrix}.$$

It follows that $gR(1-e)Re^* = 0$. Hence, R is $*$ (g,e) quasi-normal.

Similarly, we can show that the case "dual" holds. □

Let R be a ring and $T_2^{(f)}(R) = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a,b,c \in R\}$, where $f \in E(R)$. For any $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ 0 & g \end{pmatrix}$, where $a,b,c,d,e,g \in R$. In $T_2^{(f)}(R)$, addition and multiplication are determined by

$$A + B = \begin{pmatrix} a+d & b+e \\ 0 & c+g \end{pmatrix}, \quad AB = \begin{pmatrix} ad & ae+bgf \\ 0 & cg \end{pmatrix}.$$

It has been shown that $T_2^{(f)}(R)$ is a ring if and only if $f \in S_l(R)$ [13, Theorem 3.4]. In particular, if R is a $*$-ring, $f \in S_l(R)$, and define $* : T_2^{(f)}(R) \to T_2^{(f)}(R), \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a^* & 0 \\ 0 & c^* \end{pmatrix}$, then $T_2^{(f)}(R)$ is a $*$-ring.

Theorem 3.7. *Let R be a $*$-ring, $g,e \in E(R)$ and $f \in S_l(R)$. Then R is $*$ (g,e) (resp. dual $*$ (g,e)) quasi-normal if and only if $T_2^{(f)}(R)$ is $*$ (G,E) (resp. dual $*$ (g,e)) quasi-normal.*

Proof. ⇒ For any $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, B = \begin{pmatrix} d & h \\ 0 & i \end{pmatrix} \in T_2^{(f)}(R)$. It is easy to compute

$$\begin{aligned} GA(1-E)BE^* &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix}\begin{pmatrix} d & h \\ 0 & i \end{pmatrix}\begin{pmatrix} e^* & 0 \\ 0 & e^* \end{pmatrix} \\ &= \begin{pmatrix} ga(1-e)de^* & (ga(1-e)h + gb(1-e)fif)e^*f \\ 0 & gc(1-e)ie^* \end{pmatrix}. \end{aligned}$$

Since $f \in S_l(R)$, $GA(1-E)BE^*$ can be reduced as

$$\begin{pmatrix} ga(1-e)de^* & (ga(1-e)h + gb(1-e)i)e^*f \\ 0 & gc(1-e)ie^* \end{pmatrix}.$$

Hence, $gR(1-e)Re^* = 0$ shows $GA(1-E)BE^* = 0$. It follows that $T_2^{(f)}(R)$ is $*$ (G,E) quasi-normal.

⇐ By "⇒",

$$0 = GA(1-E)BE^* = \begin{pmatrix} ga(1-e)de^* & (ga(1-e)h + gb(1-e)i)e^*f \\ 0 & gc(1-e)ie^* \end{pmatrix}.$$

It follows that $gR(1-e)Re^* = 0$, and hence R is $*$ (g,e) quasi-normal.

Similarly, we can prove that the condition "dual" holds. □

Let R be a ring and $f \in E(R)$. Write ${}^{(f)}T_2(R) = T_2(R)$ as a set. In ${}^{(f)}T_2(R)$, addition and multiplication are given by

$$A + B = \begin{pmatrix} a+d & b+e \\ 0 & c+g \end{pmatrix}, \quad AB = \begin{pmatrix} ad & fae+bg \\ 0 & cg \end{pmatrix}.$$

It has been proven that ${}^{(f)}T_2(R)$ is a ring if and only if $f \in S_r(R)$ [13, Theorem 3.5]. Moreover, if R is a $*$-ring, $f \in S_r(R)$ and define $* : {}^{(f)}T_2(R) \to {}^{(f)}T_2(R), \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a^* & 0 \\ 0 & c^* \end{pmatrix}$, then ${}^{(f)}T_2(R)$ is a $*$-ring.

Theorem 3.8. *Let R be a $*$-ring, $g, e \in E(R)$ and $f \in S_r(R)$. Then R is $*\,(g, e)$ (resp. dual $*\,(g, e)$) quasi-normal if and only if ${}^{(f)}T_2(R)$ is $*\,(G, E)$ (resp. dual $*\,(g, e)$) quasi-normal.*

Proof. $\Rightarrow$ For any $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, B = \begin{pmatrix} d & h \\ 0 & i \end{pmatrix} \in {}^{(f)}T_2(R)$. A direct computation implies

$$\begin{aligned} GA(1-E)BE^* &= \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix}\begin{pmatrix} d & h \\ 0 & i \end{pmatrix}\begin{pmatrix} e^* & 0 \\ 0 & e^* \end{pmatrix} \\ &= \begin{pmatrix} ga(1-e)de^* & f(ga(1-e)h + gb(1-e)i)e^* \\ 0 & gc(1-e)ie^* \end{pmatrix}. \end{aligned}$$

By $gR(1-e)Re^* = 0$, we have $GA(1-E)BE^* = 0$. Hence, ${}^{(f)}T_2(R)$ is $*\,(G, E)$ quasi-normal.

$\Leftarrow$ By "$\Rightarrow$",

$$0 = GA(1-E)BE^* = \begin{pmatrix} ga(1-e)de^* & f(ga(1-e)h + gb(1-e)i)e^* \\ 0 & gc(1-e)ie^* \end{pmatrix}.$$

It follows that $gR(1-e)Re = 0$. Hence, R is $*\,(g, e)$ quasi-normal.

Similarly, we can show that the case "dual" is true. □

4. Relations with other generalized inverses

In this section, we will discuss the relations among $*\,(g, e)$, dual $*\,(g, e)$ quasi-normal rings and other generalized inverses.

Theorem 4.1. *Let R be a $*$-ring, $g, e \in E(R)$ and $f \in g\{1\} \cap E(R)$. Then R is $*\,(g, e)$ quasi-normal if and only if R is $*\,(fg, e)$ quasi-normal.*

Proof. $\Rightarrow$ By assumption, $g = gfg$ and $gR(1-e)Re^* = 0$. Hence, $(fg)^2 = f(gfg) = fg$, which shows $fg \in E(R)$. Moreover, it is clear $fgR(1-e)Re^* = 0$. It follows that R is $*\,(fg, e)$ quasi-normal.

$\Leftarrow$ The proof follows from $gR(1-e)Re^* = g(fgR(1-e)Re^*) = 0$. □

By Theorem 4.1, we have the following.

Proposition 4.2. *Let R be a $*$-ring, $g, e \in E(R)$ and $f \in e\{1\} \cap E(R)$. Then R is dual $*\,(g, e)$ quasi-normal if and only if R is dual $*\,(g, ef)$ quasi-normal.*

Definition 4.3. *An element a in a $*$-ring R is called Moore-Penrose invertible (MP-invertible) (see [19]) if there exists $x \in R$ such that $axa = a$, $xax = x$, $(ax)^* = ax$, $(xa)^* = xa$.*

The element x in Definition 4.3 is is said to be the Moore-Penrose inverse of a, which is unique if it exists and written by $a^\dagger$. The set of all MP-invertible elements in R is denoted by $R^\dagger$.

Theorem 4.1 and Proposition 4.2 implies the following.

Corollary 4.4. *Let R be a $*$-ring and $g, e \in E(R) \cap R^\dagger$. Then R is $*\,(g, e)$ (resp. dual $*\,(g, e)$) quasi-normal if and only if R is $*\,(g^\dagger g, e)$ (resp. dual $(g, ee^\dagger)$) quasi-normal.*

In particular, in Definition 4.3, if we only consider (1) $axa = a$, (2) $xax = x$, then all elements satisfying (1) and (2) in R are denoted by $a\{1, 2\}$, and x is called a $\{1, 2\}$-inverse of a, and denoted by $a^{(1,2)}$, the set of all $\{1, 2\}$-inverses of R is denoted by $R^{\{1,2\}}$ [19].

Proposition 4.5. *Let R be a $*$-ring, $g \in E(R) \cap R^{\{1,2\}}$, and $e \in E(R)$. Then for any $g^{(1,2)} \in E(R)$, R is $*\,(g, e)$ quasi-normal if and only if R is $*\,(g^{(1,2)}, e)$ quasi-normal.*

Proof. $\Rightarrow$ Since $g^{(1,2)}R(1-e)Re^* = g^{(1,2)}gg^{(1,2)}R(1-e)Re^* \subseteq g^{(1,2)}gR(1-e)Re^*$, by hypothesis, $g^{(1,2)}R(1-e)Re^* = 0$.
$\Leftarrow$ The proof is similar to the proof of "$\Rightarrow$". □

Similarly, we have the following.

Proposition 4.6. *Let R be a $*$-ring, $e \in E(R) \cap R^{\{1,2\}}$, and $g \in E(R)$. Then for any $e^{(1,2)} \in E(R)$, R is dual $*$ (g,e) quasi-normal if and only if R is dual $*$ $(g, e^{(1,2)})$ quasi-normal.*

Corollary 4.7. *Let R be a $*$-ring, $g, e \in E(R)$. Then*
(1) If $g^\dagger \in E(R)$, then R is $*$ (g,e) quasi-normal if and only if R is $*$ $(g^\dagger, e)$ quasi-normal.
(2) If $e^\dagger \in E(R)$, then R is dual $*$ (g,e) quasi-normal if and only if R is $*$ $(g, e^\dagger)$ quasi-normal.

Remark 4.8. *The conclusions in Corollaries 4.4 and 4.7 can be generalized to the case of "g (or e) is an SEP element". For the concept of SEP elements, one can refer to [15].*

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Conflict of Interest

The authors declared that they have no conflict of interest.

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