Filomat 39:13 (2025), 4395–4409
https://doi.org/10.2298/FIL2513395O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# On SR-fuzzy Sheffer stroke Hilbert algebras

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**Abstract.** It is known that there are some extensions of fuzzy sets that can be used to solve various real-life problems. In this paper, we propose a novel extension of a fuzzy subalgebras, fuzzy ideals, and fuzzy filters. We define SR-fuzzy subalgebras, SR-fuzzy ideals and SR-fuzzy filters on Sheffer stroke Hilbert algebras. Moreover, we provide some examples to illustrate these definitions.

## 1. Introduction

The Sheffer operation, also known as the Sheffer stroke or NAND operator, was first introduced by H. M. Sheffer [31]. In 2002, M. McCune et al. ([23]) applied the Sheffer stroke operation to Boolean algebras and showed that there is no shorter axiom in terms of the Sheffer stroke. Algebraic structures play a prominent role in mathematics with wide-ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. This provides sufficient motivation to researchers to simplify axioms for some different algebraic structures e.g., see [9], [10] and [19]–[29]. In 1950, L. Henkin [14] introduced the notion of an "implicative model", which serves as a model of positive implicative propositional calculus. In 1960, A. Monteiro [24] gave the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, A. Diego ([12]) intensively studied and developed some properties of Hilbert algebras. In 2021, T. Oner et al. ([27]) investigated the relation between the Sheffer stroke and Hilbert algebras. Also, see [18]. In 1965, L. A. Zadeh ([32]) proposed a new theory called fuzzy set theory. Since then several researchers studied various extensions and generalizations of this theory, e.g., intuitionistic fuzzy sets [6], L-fuzzy sets [13], type-2 fuzzy sets [22], interval valued fuzzy sets [15], multi fuzzy sets [30], bipolar-valued fuzzy sets [21] and $m$-polar fuzzy sets [11] etc.. In 2005, Y. B. Jun [17] introduced the concept of $(\alpha, \beta)$-fuzzy ideal of a BCK/BCI-algebra and discussed related results. In 2019, A. Al-Masarwah and A. G. Ahmad [1] focused on combining the concepts of m-polar fuzzy sets and m-polar fuzzy points to introduce a new notion called

2020 *Mathematics Subject Classification*. Primary 06F05, 03G25; Secondary 03G10.
*Keywords*. Sheffer stroke Hilbert algebra, SR-fuzzy subalgebra, SR-fuzzy ideal, SR-fuzzy filter.
Received: 19 September 2024; Revised: 01 March 2025; Accepted: 14 March 2025
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m-polar $(\alpha, \beta)$-fuzzy ideals in BCK/BCI-algebras. They showed that the defined notion is a generalization of fuzzy ideals, bipolar fuzzy ideals, $(\alpha, \beta)$-fuzzy ideals, and bipolar $(\alpha, \beta)$-fuzzy ideals in BCK/BCI-algebras. In 2021, T. Oner et al. [25] explored fuzzy structures in Sheffer stroke Hilbert algebras and defined fuzzy filters. In 2022, T. M. Al-Shami et al. [5] as a novel extension of fuzzy set, defined the concept of SR-Fuzzy sets and compared it with the other types of fuzzy sets. Additionally, they defined a score function for the ranking of SR-Fuzzy sets and applied their results to Decision-Making. In 2023, R. A. Borzooei et al [8] discussed fuzzy weak filters of Sheffer stroke Hilbert algebras and investigated their properties. In 2024, H. S. Kim et al. [20] applied the notion of bipolar-valued fuzzy sets to filters and deductive systems in Sheffer stroke Hilbert algebras. They investigated the conditions under which the bipolar-valued fuzzy set can be a bipolar-valued fuzzy filter. In recent years, fuzzy sets with different powers have been extensively studied, leading to significant advancements in both theory and applications. For instance, $(2, 1)$-fuzzy sets have been explored for their properties, weighted aggregated operators, and applications to multi-criteria decision-making methods [2], $(3, 2)$-Fuzzy sets have been applied to topology and optimal choices, providing a robust framework for handling complex decision-making scenarios [15], $(m, n)$-fuzzy sets, have been developed and applied to multi-criteria decision-making methods, offering greater flexibility in modeling uncertainty [4], $(a, b)$-fuzzy soft sets, have been introduced to address limitations in existing models and enhance their applicability to real-world problems [3], $k_m^n$-rung picture fuzzy sets have been proposed as a powerful extension of traditional fuzzy sets, enabling more nuanced representations of uncertainty and imprecision in decision-making processes [16].

It is now natural to investigate similar argument of extensions and generalizations of the existing fuzzy subsystems of other algebraic structures. It is motivated us to define SR- fuzzy subalgebra, SR-fuzzy ideal and SR-fuzzy filter on Sheffer stroke Hilbert algebras and discuss their main properties.

## 2. Preliminaries

Recall the definitions and results from [5], [27], and [31] for the reader's convenience.

**Definition 2.1.** [31] *Let $H = \langle H, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$(S1)$ $x|y = y|x$
$(S2)$ $(x|x)|(x|y) = x$
$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.$

**Definition 2.2.** [27] *A Sheffer stroke Hilbert algebra is a structure $\langle H, |\rangle$ of type (2), in which H is a non-empty set and $|$ is a Sheffer stroke operation on H such that the following identities are satisfied for all $x, y, z \in H$:*

1. $(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) = x|(x|x),$
2. *If* $x|(y|y) = y|(x|x) = x|(x|x)$, *then* $x = y$.

**Lemma 2.3.** [27] *Let $\langle A, |\rangle$ be a Sheffer stroke Hilbert algebra. Then, there exists a unique $1 \in A$, such that the following identities hold for all $x \in A$:* $1.x|(x|x) = 1$.
$2.x|(1|1) = 1$.
$3.1|(x|x) = 1$.

**Proposition 2.4.** [27] *Let $\langle A, |\rangle$ be a Sheffer stroke Hilbert algebra. Then the binary relation $x \leq y$ if and only if $(x|(y|y)) = 1$ is a partial order on A. Moreover, 1 is greatest element of A.*

**Lemma 2.5.** [27] *Let $\langle A, |\rangle$ be a Sheffer stroke Hilbert algebra with the least element 0 and the greatest element 1. Then* $0|0 = 1$ *and* $1|1 = 0$.

**Definition 2.6.** [27] *A nonempty subset G of a Sheffer stroke Hilbert algebra X is called a subalgebra of X if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.*

**Definition 2.7.** [27] *A nonempty subset $G$ of a Sheffer stroke Hilbert algebra $X$ is called an ideal of $X$ if for all $x, y \in G$*

1. $0 \in G$,
2. $(x|(y|y))|(x|(y|y)) \in G$ *and* $y \in G \Rightarrow x \in G$.

**Definition 2.8.** [5] *Let $X$ be a universal set. Then the pair $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $X$, where $\mu_{\mathcal{S}}$ is a membership grade, and $\gamma_{\mathcal{S}}$ is a non-membership grade, is called an SR-fuzzy set in $X$ if it satisfies $0 \leq (\mu_{\mathcal{S}}(a))^2 + \sqrt{\gamma_{\mathcal{S}}(a)} \leq 1$ for all $a \in X$.*

## 3. SR-fuzzy subalgebras of Sheffer stroke Hilbert algebras

In this section, we introduce the concept of SR-fuzzy subalgebras in the context of Sheffer stroke Hilbert algebras. Unless explicitly stated otherwise, $X$ will denote a Sheffer stroke Hilbert algebra.

**Definition 3.1.** *An SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $X$ is called SR-fuzzy subalgebra of $X$ if it satisfies*

$$(\forall x, y \in X)\left(\begin{array}{c} (\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 \geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\} \\ \sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\} \end{array}\right) \tag{1}$$

**Example 3.2.** *Consider the Sheffer stroke Hilbert algebra in Tabel 1.*

Table 1: Sheffer stroke Hilbert algebra $(X, |)$

| \| | a | b | c | d | e | f | g | h |
|---|---|---|---|---|---|---|---|---|
| a | b | a | d | c | f | e | h | g |
| b | a | a | a | a | a | a | a | a |
| c | d | a | d | a | a | d | d | a |
| d | c | a | a | c | f | g | f | g |
| e | f | a | a | f | f | a | f | a |
| f | e | a | d | g | a | e | d | g |
| g | h | a | d | f | f | d | h | a |
| h | g | a | a | g | a | g | a | g |

*Let $\mu_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}$ be fuzzy sets in $X$ given by the Table 2:*

Table 2: fuzzy sets $\mu_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}$ of $X$

| $X$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $\mu_{\mathcal{S}}(x)$ | 0.1 | 0.6 | 0.5 | 0.4 | 0.6 | 0.2 | 0.3 | 0.4 |
| $\gamma_{\mathcal{S}}(x)$ | 0.2 | 0.1 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.1 |

*Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is given by the Table 3.*

Table 3: SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ of $X$

| $X$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $(\mu_{\mathcal{S}}(x))^2$ | 0.01 | 0.36 | 0.25 | 0.16 | 0.36 | 0.04 | 0.09 | 0.16 |
| $\sqrt{\gamma_{\mathcal{S}}(x)}$ | 0.4472 | 0.3162 | 0.4472 | 0.4472 | 0.4472 | 0.4472 | 0.4472 | 0.3162 |

*where $\sqrt{\gamma_{\mathcal{S}}(x)}$ is calculated to four decimal places. Then the SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy subalgebra of $X$.*

**Definition 3.3.** *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be an SR-fuzzy set in $X$ and $(t, s) \in [0, 1]^2$ be such that $0 \leq t^2 + \sqrt{s} \leq 1$. Consider the following sets:*

$$S(\mu_{\mathcal{S}}, t) = \{x \in X | \mu_{\mathcal{S}}(x) \geq t\}$$

*and*

$$R(\gamma_{\mathcal{S}}, s) = \{x \in X | \gamma_{\mathcal{S}}(x) \leq s\}.$$

**Example 3.4.** *From Example 3.2, we can observe that, if* $t = 0.3$ *and* $s = 0.4$, *then* $0 \le t^2 + \sqrt{t} = 0.09 + 0.6325 = 0.7225 \le 1$ *and*

$$S(\mu_{\mathcal{S}}, 0.3) = \{x \in X | \mu_{\mathcal{S}}(x) \ge 0.3\} = \{c, d, e, f, h\}$$

*and*

$$R(\gamma_{\mathcal{S}}, 0.4) = \{x \in X | \gamma_{\mathcal{S}}(x) \le 0.4\} = \{a, b, c, d, e, f, h\}$$

**Theorem 3.5.** *An SR-fuzzy set* $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ *in* $X$ *is an SR-fuzzy subalgebra of* $X$ *if and only if the nonempty sets* $S(\mu_{\mathcal{S}}, t)$ *and* $R(\gamma_{\mathcal{S}}, s)$ *are subalgebras of* $X$ *for all* $(t, s) \in [0, 1]^2$ *satisfying* $0 \le t^2 + \sqrt{s} \le 1$.

*Proof.* Assume $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $X$. Let $(t, s) \in [0, 1]^2$ such that $0 \le t^2 + \sqrt{s} \le 1$, and let $x, y \in S(\mu_{\mathcal{S}}, t) \cap R(\gamma_{\mathcal{S}}, s)$. By definition:

$$(\mu_{\mathcal{S}}(x))^2 \ge t^2, \quad (\mu_{\mathcal{S}}(y))^2 \ge t^2, \quad \sqrt{\gamma_{\mathcal{S}}(x)} \le \sqrt{s}, \quad \sqrt{\gamma_{\mathcal{S}}(y)} \le \sqrt{s}.$$

Suppose $(x|(y|y))|(x|(y|y)) \notin S(\mu_{\mathcal{S}}, t)$ or $(x|(y|y))|(x|(y|y)) \notin R(\gamma_{\mathcal{S}}, s)$. Then:

$$\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))) < t \quad \text{or} \quad \gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y))) > s.$$

This implies:

$$(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y)))^2 < t^2 \le \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\},$$

or

$$\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} > \sqrt{s} \ge \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\},$$

which contradicts the definition of an SR-fuzzy subalgebra. Hence,

$$(x|(y|y))|(x|(y|y)) \in S(\mu_{\mathcal{S}}, t) \cap R(\gamma_{\mathcal{S}}, s),$$

proving $S(\mu_{\mathcal{S}}, t)$ and $R(\gamma_{\mathcal{S}}, s)$ are subalgebras.

Conversely, assume $S(\mu_{\mathcal{S}}, t)$ and $R(\gamma_{\mathcal{S}}, s)$ are subalgebras for all $(t, s) \in [0, 1]^2$ with $0 \le t^2 + \sqrt{s} \le 1$. Suppose for contradiction that there exist $a, b \in X$ such that:

$$(\mu_{\mathcal{S}}((a|(b|b))|(a|(b|b))))^2 < \min\{(\mu_{\mathcal{S}}(a))^2, (\mu_{\mathcal{S}}(b))^2\},$$

or

$$\sqrt{\gamma_{\mathcal{S}}((a|(b|b))|(a|(b|b)))} > \max\{\sqrt{\gamma_{\mathcal{S}}(a)}, \sqrt{\gamma_{\mathcal{S}}(b)}\}.$$

Define $t = \min\{(\mu_{\mathcal{S}}(a))^2, (\mu_{\mathcal{S}}(b))^2\}$ and $s = \max\{\sqrt{\gamma_{\mathcal{S}}(a)}, \sqrt{\gamma_{\mathcal{S}}(b)}\}$. Then $a, b \in S(\mu_{\mathcal{S}}, t) \cap R(\gamma_{\mathcal{S}}, s)$. However,

$$(a|(b|b))|(a|(b|b)) \notin S(\mu_{\mathcal{S}}, t) \cap R(\gamma_{\mathcal{S}}, s),$$

contradicting the assumption that these sets are subalgebras. Thus, for all $x, y \in X$:

$$(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 \ge \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\},$$

$$\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} \le \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\},$$

proving $\mathcal{S}$ is an SR-fuzzy subalgebra of $X$. □

**Theorem 3.6.** *For a subset* $D$ *of* $X$, *let* $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ *be an SR-fuzzy set in* $X$, *which is defined as follows:*

$$\mu_{\mathcal{S}}(x) = \begin{cases} t & if\ x \in D, \\ 0 & otherwise, \end{cases} \quad \gamma_{\mathcal{S}}(x) = \begin{cases} s & if\ x \in D, \\ 1 & otherwise \end{cases}$$

*where* $(t, s) \in (0, 1] \times [0, 1)$ *with* $0 \le t^2 + \sqrt{s} \le 1$. *Then* $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ *is an SR-fuzzy subalgebra of* $X$ *if and only if* $D$ *is a subalgebra of* $X$.

*Proof.* Assume that $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $X$. Then $S(\mu_{\mathcal{S}}, t)$ and $R(\gamma_{\mathcal{S}}, s)$ are nonempty subalgebra of $X$. Let $x, y \in X$ be such that $x \in D$ and $y \in D$. Then

$$\mu_{\mathcal{S}}(x) = t = \mu_{\mathcal{S}}(y)$$

and

$$\sqrt{\gamma_{\mathcal{S}}(x)} = s = \sqrt{\gamma_{\mathcal{S}}(y)},$$

Thus,

$$(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 \geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\} = t^2$$

and

$$\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\} = \sqrt{s}.$$

Therefore, $\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))) = t$ and $\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} = s$. This shows that $(x|(y|y))|(x|(y|y)) \in D$. Therefore, $D$ is a subalgebra of $X$.

Conversely, let $D$ be a subalgebra of $X$. Let $x, y \in X$. If $x \in D$ and $y \in D$, then $(x|(y|y))|(x|(y|y)) \in D$. Hence

$$(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 = t^2 = \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\}$$

and

$$\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} = \sqrt{s} = \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\}.$$

If $x \notin D$ or $y \notin D$, then $(\mu_{\mathcal{S}}(x))^2 = 0$ and $\sqrt{\gamma_{\mathcal{S}}(x)} = 1$, or $(\mu_{\mathcal{S}}(x))^2 = 0$ and $\sqrt{\gamma_{\mathcal{S}}(y)} = 1$. Thus

$$(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 \geq 0 = \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\}$$

and

$$\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} \leq 1 = \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\}.$$

Therefore, $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $X$. □

**Theorem 3.7.** *Let $X$ be a Sheffer stroke Hilbert algebra with least element 0. If SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $X$ is a SR-fuzzy subalgebra of $X$, then $(\mu_{\mathcal{S}}(0))^2 \geq (\mu_{\mathcal{S}}(x))^2$ and $\sqrt{\gamma_{\mathcal{S}}(0)} \leq \sqrt{\gamma_{\mathcal{S}}(x)}$ for all $x \in X$.*

*Proof.* Given $x \in X$. Then we have

$$(\mu_{\mathcal{S}}(0))^2 = (\mu_{\mathcal{S}}(1|1))^2 = (\mu_{\mathcal{S}}((x|(x|x))|(x|(x|x))))^2 \geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x))^2\} = (\mu_{\mathcal{S}}(x))^2,$$

and

$$\sqrt{\gamma_{\mathcal{S}}(0)} = \sqrt{\gamma_{\mathcal{S}}(1|1)} = \sqrt{\gamma_{\mathcal{S}}((x|(x|x))|(x|(x|x)))} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(x)}\} = \sqrt{\gamma_{\mathcal{S}}(x)},$$

proving the theorem. □

Converse of the above theorem need to be true.

**Example 3.8.** *Consider the Sheffer stroke Hilbert algebra in Tabel 4.*

Table 4: Sheffer stroke Hilbert algebra $(X, |)$

| $\|$ | 0 | a | b | c | d | e | f | g |
|---|---|---|---|---|---|---|---|---|
| 0 | a | 0 | c | b | e | d | g | f |
| a | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| b | c | 0 | c | 0 | 0 | c | c | 0 |
| c | b | 0 | 0 | b | e | f | e | f |
| d | e | 0 | 0 | e | e | 0 | e | 0 |
| e | d | 0 | c | f | 0 | d | c | f |
| f | g | 0 | c | e | e | c | g | 0 |
| g | f | 0 | 0 | f | 0 | f | 0 | f |

*Let $\mu_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}$ be fuzzy sets in $X$ given by the Table 5:*

Table 5: fuzzy sets $\mu_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}$ of $X$

| $X$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|---|
| $\mu_{\mathcal{S}}(x)$ | 0.7 | 0.6 | 0.5 | 0.4 | 0.6 | 0.2 | 0.3 | 0.7 |
| $\gamma_{\mathcal{S}}(x)$ | 0.1 | 0.3 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.1 |

*Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is given by the Table 6.*

Table 6: SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ of $X$

| $X$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|---|
| $(\mu_{\mathcal{S}}(x))^2$ | 0.49 | 0.36 | 0.25 | 0.16 | 0.36 | 0.04 | 0.09 | 0.49 |
| $\sqrt{\gamma_{\mathcal{S}}(x)}$ | 0.3162 | 0.5477 | 0.4472 | 0.4472 | 0.4472 | 0.4472 | 0.4472 | 0.3162 |

*where $\sqrt{\gamma_{\mathcal{S}}(x)}$ is calculated to four decimal places. Then the SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $X$ satisfies*

$$(\mu_{\mathcal{S}}(0))^2 \geq (\mu_{\mathcal{S}}(x))^2 \text{ and } \sqrt{\gamma_{\mathcal{S}}(0)} \leq \sqrt{\gamma_{\mathcal{S}}(x)} \text{ for all } x \in X.$$

*But it is not SR-fuzzy subalgebra of $X$ since*

$$(\mu_{\mathcal{S}}((0|(d|d))|(0|(d|d))))^2 = (\mu_{\mathcal{S}}((0|e)|(0|e)))^2 = (\mu_{\mathcal{S}}(d|d))^2 = (\mu_{\mathcal{S}}(e))^2 = 0.04$$

*and*

$$\min\{(\mu_{\mathcal{S}}(0))^2, (\mu_{\mathcal{S}}(d))^2\} = \min\{0.49, 0.36\} = 0.36$$

*but*

$$(\mu_{\mathcal{S}}((0|(d|d))|(0|(d|d))))^2 \ngeq \min\{(\mu_{\mathcal{S}}(0))^2, (\mu_{\mathcal{S}}(d))^2\}.$$

**Theorem 3.9.** *Let $X$ be a Sheffer stroke Hilbert algebra with least element 0. If a Sheffer stroke Hilbert algebra $(X, |)$ is linearly ordered, then every SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ satisfying $(\mu_{\mathcal{S}}(0))^2 \geq (\mu_{\mathcal{S}}(x))^2$ and $\sqrt{\gamma_{\mathcal{S}}(0)} \leq \sqrt{\gamma_{\mathcal{S}}(x)}$ for all $x \in X$ is an SR-fuzzy subalgebra of $X$.*

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be SR-fuzzy set that satisfies $(\mu_{\mathcal{S}}(0))^2 \geq (\mu_{\mathcal{S}}(x))^2$ and $\sqrt{\gamma_{\mathcal{S}}(0)} \leq \sqrt{\gamma_{\mathcal{S}}(x)}$ for all $x \in X$. For every $x, y \in X$ we have $x \leq y$ or $y \leq x$ since $X$ is linearly ordered. If $x \leq y$ then $x|(y|y) = 1$. Hence

$$(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 = (\mu_{\mathcal{S}}(1|1))^2 = (\mu_{\mathcal{S}}(0))^2 \geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\}$$

and

$$\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} = \sqrt{\gamma_{\mathcal{S}}(1|1)} = \sqrt{\gamma_{\mathcal{S}}(0)} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\}.$$

Hence $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $X$. Similarly, we can prove the result when $y \leq x$. □

**Proposition 3.10.** *For a SR-fuzzy subalgebra of X satisfies*

$$(\forall\, x, y \in X)\left( \left\{ \begin{array}{l} (\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 \geq (\mu_{\mathcal{S}}(y))^2 \\ \sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y))))} \leq \sqrt{\gamma_{\mathcal{S}}(y)} \end{array} \right\} \right) \tag{2}$$

*if and only if* $(\mu_{\mathcal{S}}(0))^2 = (\mu_{\mathcal{S}}(x))^2$ *and* $\sqrt{\gamma_{\mathcal{S}}(0)} = \sqrt{\gamma_{\mathcal{S}}(x)}$.

*Proof.* We have

$$\begin{aligned} (\mu_{\mathcal{S}}(x))^2 &= (\mu_{\mathcal{S}}((x|x)|(x|x)))^2 \\ &= (\mu_{\mathcal{S}}((1|((x|x)|(x|x)))|(1|((x|x)|(x|x)))))^2 \\ &= (\mu_{\mathcal{S}}((x|1)|(x|1)))^2 \\ &= (\mu_{\mathcal{S}}((x|(0|0))|(x|(0|0))))^2 \\ &\geq (\mu_{\mathcal{S}}(0))^2 \end{aligned}$$

and

$$\begin{aligned} \sqrt{\gamma_{\mathcal{S}}(x)} &= \sqrt{\gamma_{\mathcal{S}}((x|x)|(x|x))} \\ &= \sqrt{\gamma_{\mathcal{S}}((1|((x|x)|(x|x)))|(1|((x|x)|(x|x))))} \\ &= \sqrt{\gamma_{\mathcal{S}}((x|1)|(x|1))} \\ &= \sqrt{\gamma_{\mathcal{S}}((x|(0|0))|(x|(0|0)))} \\ &\leq \sqrt{\gamma_{\mathcal{S}}(0)}. \end{aligned}$$

Then by Theorem 3.7, we have $(\mu_{\mathcal{S}}(0))^2 = (\mu_{\mathcal{S}}(x))^2$ and $\sqrt{\gamma_{\mathcal{S}}(0)} = \sqrt{\gamma_{\mathcal{S}}(x)}$. The converse is clear. □

## 4. SR-fuzzy ideals of Sheffer stroke Hilbert algebras

In this section, we introduce the concept of SR-fuzzy ideals in the context of Sheffer stroke Hilbert algebras. Through out this section, $X$ means Sheffer stroke Hilbert algebra with the least element 0.

**Definition 4.1.** *An SR-fuzzy set* $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ *in X is called SR-fuzzy ideal of X if it satisfies: for all* $x, y \in X$

$$\begin{array}{l} (\mu_{\mathcal{S}}(0))^2 \geq (\mu_{\mathcal{S}}(x))^2 \geq \min\{(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2, (\mu_{\mathcal{S}}(y))^2\} \\ \sqrt{\gamma_{\mathcal{S}}(0)} \leq \sqrt{\gamma_{\mathcal{S}}(x)} \leq \max\{\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y))))}, \sqrt{\gamma_{\mathcal{S}}(y)}\} \end{array} \tag{3}$$

**Example 4.2.** *Consider the Sheffer stroke Hilbert algebra in Table 7.*

Table 7: Sheffer stroke Hilbert algebra $(X, |)$

| \| | 0 | 1 | a | b | c | d | e | f |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | b | a | d | c | f | e |
| a | 1 | b | b | 1 | 1 | b | b | 1 |
| b | 1 | a | 1 | a | d | e | d | e |
| c | 1 | d | 1 | d | d | 1 | d | 1 |
| d | 1 | c | b | e | 1 | c | b | e |
| e | 1 | f | b | d | d | b | f | 1 |
| f | 1 | e | 1 | e | 1 | e | 1 | e |

*Let* $\mu_{\mathcal{S}}$ *and* $\gamma_{\mathcal{S}}$ *be fuzzy sets in X given by the Table 8:*

Table 8: fuzzy sets $\mu_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}$ of $X$

| $X$ | 0 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|---|---|
| $\mu_{\mathcal{S}}(x)$ | 0.7 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.7 |
| $\gamma_{\mathcal{S}}(x)$ | 0.05 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.05 |

*Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is given by the Table 9.*

Table 9: SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ of $X$

| $X$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $(\mu_{\mathcal{S}}(x))^2$ | 0.49 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.49 |
| $\sqrt{\gamma_{\mathcal{S}}(x)}$ | 0.2236 | 0.3162 | 0.3162 | 0.3162 | 0.3162 | 0.3162 | 0.3162 | 0.2236 |

*where $\sqrt{\gamma_{\mathcal{S}}(x)}$ is calculated to four decimal places. Then SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy ideal of $X$.*

**Lemma 4.3.** *If $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy ideal of $X$, then*

$$(\forall\, x, y \in X)\left( x \le y \Rightarrow \begin{cases} (\mu_{\mathcal{S}}(x))^2 \ge (\mu_{\mathcal{S}}(y))^2 \\ \sqrt{\gamma_{\mathcal{S}}(x)} \le \sqrt{\gamma_{\mathcal{S}}(y)} \end{cases} \right). \tag{4}$$

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy ideal of $X$ and $x \le y$. Then by Theorem 3.7, we have

$$(\mu_{\mathcal{S}}(x))^2 \ge \min\{(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2, (\mu_{\mathcal{S}}(y))^2\} = \min\{(\mu_{\mathcal{S}}(0))^2, (\mu_{\mathcal{S}}(y))^2\} = (\mu_{\mathcal{S}}(y))^2$$

and

$$\sqrt{\gamma_{\mathcal{S}}(x)} \le \max\{\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))}, \sqrt{\gamma_{\mathcal{S}}(y)}\} = \max\{\sqrt{\gamma_{\mathcal{S}}(0)}, \sqrt{\gamma_{\mathcal{S}}(y)}\} = \sqrt{\gamma_{\mathcal{S}}(y)}$$

for all $x, y \in X$. □

**Theorem 4.4.** *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy set of $X$. Then $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy ideal of $X$ if and only if, for all $x, y, z \in X$*

$$(x|(y|y))|(x|(y|y)) \le z \text{ implies } \begin{cases} (\mu_{\mathcal{S}}(x))^2 \ge \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(z))^2\} \\ \sqrt{\gamma_{\mathcal{S}}(x)} \le \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(z)}\} \end{cases} \tag{5}$$

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy ideal of $X$ and $(x|(y|y))|(x|(y|y)) \le z$. Then

$$(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) = 1|1 = 0.$$

But

$$\begin{aligned}(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 &\ge \min\{(\mu_{\mathcal{S}}(((((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z))))^2, (\mu_{\mathcal{S}}(z))^2\} \\ &= \min\{(\mu_{\mathcal{S}}(0))^2, (\mu_{\mathcal{S}}(z))^2\} \\ &= (\mu_{\mathcal{S}}(z))^2,\end{aligned}$$

and

$$\begin{aligned}\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} &\le \max\{\sqrt{\gamma_{\mathcal{S}}(((((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)))}, \sqrt{\gamma_{\mathcal{S}}(z)}\} \\ &= \max\{\sqrt{\gamma_{\mathcal{S}}(0)}, \sqrt{\gamma_{\mathcal{S}}(z)}\} \\ &= \sqrt{\gamma_{\mathcal{S}}(z)}.\end{aligned}$$

Hence

$$(\mu_{\mathcal{S}}(x))^2 \ge \min\{(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2, (\mu_{\mathcal{S}}(y))^2\} \ge \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(z))^2\}$$

and

$$\sqrt{\gamma_{\mathcal{S}}(x)} \le \sqrt{\gamma_{\mathcal{S}}(x)} \le \max\{\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))}, \sqrt{\gamma_{\mathcal{S}}(y)}\} \le \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(z)}\}.$$

Conversely, let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy set of $X$ satisfies the condition (5).

Since $(0|(x|x))|(0|(x|x)) = ((x|x)|(1|1))|((x|x)|(1|1)) = 1|1 = 0 \le z$, $(\mu_{\mathcal{S}}(0))^2 \ge (\mu_{\mathcal{S}}(x))^2$ and $\sqrt{\gamma_{\mathcal{S}}(0)} \le \sqrt{\gamma_{\mathcal{S}}(x)}$; for all $x \in H$.

Since $((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y) = (x|(y|y))|((x|(y|y))|(x|(y|y))) = 1$, $(x|(x|(y|y)))|(x|(x|(y|y))) \le y$. Since

$$\begin{aligned}(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))) &= (x|(x|(y|y)))|(x|(x|(y|y))) \\ &\le y,\end{aligned}$$

$(\mu_{\mathcal{S}}(x)^2 \ge \min\{(\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2, (\mu_{\mathcal{S}}(y))^2\}$ and $\sqrt{\gamma_{\mathcal{S}}(x)} \ge \min\{\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))}, \sqrt{\gamma_{\mathcal{S}}(y)})\}$ for all $x, y, z \in X$. Thus, $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy ideal of $X$. □

The following theorem can be proved similarly to Theorem 3.5.

**Theorem 4.5.** *An SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $X$ is an SR-fuzzy ideal of $X$ if and only if the nonempty sets $S(\mu_{\mathcal{S}}, t)$ and $R(\gamma_{\mathcal{S}}, s)$ are ideals of $X$ for all $(t, s) \in [0, 1]^2$ be such that $0 \leq t^2 + \sqrt{s} \leq 1$.*

The following theorem can be proved similarly to Theorem 3.6.

**Theorem 4.6.** *For a subset $D$ of $X$, let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be an SR-fuzzy set in $X$, which is defined as follows:*

$$\mu_{\mathcal{S}}(x) = \begin{cases} t & if\ x \in D, \\ 0 & otherwise, \end{cases} \quad \gamma_{\mathcal{S}}(x) = \begin{cases} s & if\ x \in D, \\ 1 & otherwise \end{cases}$$

*where $(t, s) \in (0, 1] \times [0, 1)$ with $0 \leq t^2 + \sqrt{s} \leq 1$. Then $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is an SR-fuzzy ideal of $X$ if and only if $D$ is an ideal of $X$.*

**Theorem 4.7.** *Every SR-fuzzy ideal of a $X$ is a SR-fuzzy subalgebra of $X$.*

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy ideal of $X$. Then

$$\begin{aligned} (\mu_{\mathcal{S}}((x|(y|y))|(x|(y|y))))^2 &\geq \min\{(\mu_{\mathcal{S}}((((x|(y|y))|(x|(y|y)))|(x|x))|(((x|(y|y))|(x|(y|y)))|(x|x))))^2, (\mu_{\mathcal{S}}(x))^2\} \\ &= \min\{(\mu_{\mathcal{S}}(((y|y)|((x|(x|x))|(x|(x|x))))|((y|y)|((x|(x|x))|(x|(x|x))))))^2, (\mu_{\mathcal{S}}(x))^2\} \\ &= \min\{(\mu_{\mathcal{S}}(((y|y)|(1|1))|((y|y)|(1|1))))^2, (\mu_{\mathcal{S}}(x))^2\} \\ &= \min\{(\mu_{\mathcal{S}}(1|1))^2, (\mu_{\mathcal{S}}(x))^2\} \\ &= \min\{(\mu_{\mathcal{S}}(0))^2, (\mu_{\mathcal{S}}(x))^2\} \\ &= (\mu_{\mathcal{S}}(x))^2 \\ &\geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\} \end{aligned}$$

and

$$\begin{aligned} \sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(x|(y|y)))} &\leq \max\{\sqrt{\gamma_{\mathcal{S}}((((x|(y|y))|(x|(y|y)))|(x|x))|(((x|(y|y))|(x|(y|y)))|(x|x)))}, \sqrt{\gamma_{\mathcal{S}}(x)}\} \\ &= \max\{\sqrt{\gamma_{\mathcal{S}}(((y|y)|((x|(x|x))|(x|(x|x))))|((y|y)|((x|(x|x))|(x|(x|x)))))}, \sqrt{\gamma_{\mathcal{S}}(x)}\} \\ &= \max\{\sqrt{\gamma_{\mathcal{S}}(((y|y)|(1|1))|((y|y)|(1|1)))}, \sqrt{\gamma_{\mathcal{S}}(x)}\} \\ &= \max\{\sqrt{\gamma_{\mathcal{S}}(1|1)}, \sqrt{\gamma_{\mathcal{S}}(x)}\} \\ &= \max\{\sqrt{\gamma_{\mathcal{S}}(0)}, \sqrt{\gamma_{\mathcal{S}}(x)}\} \\ &= \sqrt{\gamma_{\mathcal{S}}(x)} \\ &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\}. \end{aligned}$$

Hence $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy ideal of $X$. □

**Definition 4.8.** *We define the following subsets of $X$,*

$$\mu_x = \{x \in X : (\mu_{\mathcal{S}}(x))^2 \geq (\mu_{\mathcal{S}}(x_t))^2\},$$

*and*

$$\gamma_x = \{x \in X : \sqrt{\gamma_{\mathcal{S}}(x)} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}\},$$

*for all $x, x_t \in X$.*

**Theorem 4.9.** *If $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy ideal of $X$, then $\mu_x$ and $\gamma_x$ are ideals of $X$, where $x \in X$.*

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy ideal of $X$. Let $y_2 \in \mu_x$ and $y_3 \in \gamma_x$. Then

$$(\mu_{\mathcal{S}}(y_2))^2 \geq (\mu_{\mathcal{S}}(x_t))^2 \text{ and } \sqrt{\gamma_{\mathcal{S}}(y_3)} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}.$$

This shows that

$$(\mu_{\mathcal{S}}((y_2|(x_2|x_2))|(y_2|(x_2|x_2))))^2 \geq (\mu_{\mathcal{S}}(y_2))^2 \geq (\mu_{\mathcal{S}}(x_t))^2$$

and

$$\sqrt{\gamma_{\mathcal{S}}((y_3|(x_3|x_3))|(y_3|(x_3|x_3)))} \leq \sqrt{\gamma_{\mathcal{S}}(y_3)} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}.$$

Therefore, $(y_2|(x_2|x_2))|(y_2|(x_2|x_2)) \in \mu_x$ and $(y_3|(x_3|x_3))|(y_3|(x_3|x_3)) \in \gamma_x$.
Let $x_2, (y_2|(x_2|x_2))|(y_2|(x_2|x_2)) \in \mu_x$ and $x_3, (y_3|(x_3|x_3))|(y_3|(x_3|x_3)) \in \gamma_x$. Then

$$(\mu_{\mathcal{S}}(x_2))^2 \geq (\mu_{\mathcal{S}}(x_t))^2,\ (\mu_{\mathcal{S}}((y_2|(x_2|x_2))|(y_2|(x_2|x_2))))^2 \geq (\mu_{\mathcal{S}}(x_t))^2$$

and

$$\sqrt{\gamma_{\mathcal{S}}(x_3)} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)},\ \sqrt{\gamma_{\mathcal{S}}((y_3|(x_3|x_3))|(y_3|(x_3|x_3)))} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}.$$

It follows that

$$(\mu_{\mathcal{S}}(y_2))^2 \geq \min\{(\mu_{\mathcal{S}}(x_2))^2, (\mu_{\mathcal{S}}((y_2|(x_2|x_2))|(y_2|(x_2|x_2))))^2\} \geq (\mu_{\mathcal{S}}(x_t))^2$$

and

$$\sqrt{\gamma_{\mathcal{S}}(y_3)} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x_3)}, \sqrt{\gamma_{\mathcal{S}}((y_3|(x_3|x_3))|(y_3|(x_3|x_3)))}\} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}.$$

Hence $y_2 \in \mu_{\mathcal{S}}(x_t)$ and $y_3 \in \gamma_{\mathcal{S}}(x_t)$. Therefore, $\mu_x$ and $\gamma_x$ are ideals of $X$. □

**Theorem 4.10.** *If $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy subalgebra of $X$, then $\mu_x$ and $\gamma_x$ are subalgebras of $X$, where $a, x_t \in X$.*

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy subalgebra of $X$. Let $x_2, y_2 \in \mu_x$ and $x_3, y_3 \in \gamma_x$. Then $\mu_{\mathcal{S}}(x_2) \geq \mu_{\mathcal{S}}(x_t)$, $(\mu_{\mathcal{S}}(y_2))^2 \geq (\mu_{\mathcal{S}}(x_t))^2$ and
$\sqrt{\gamma_{\mathcal{S}}(x_3)} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}$, $\sqrt{\gamma_{\mathcal{S}}(y_3)} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}$. Then

$$(\mu_{\mathcal{S}}(x_2|(y_2|y_2))|(x_2|(y_2|y_2))))^2 \geq \min\{(\mu_{\mathcal{S}}(x_2))^2, (\mu_{\mathcal{S}}(y_2))^2\} \geq (\mu_{\mathcal{S}}(x_t))^2$$

and

$$\sqrt{\gamma_{\mathcal{S}}(x_3|(y_3|y_3))|(x_3|(y_3|y_3)))} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x_3)}, \sqrt{\gamma_{\mathcal{S}}(y_3)}\} \leq \sqrt{\gamma_{\mathcal{S}}(x_t)}.$$

Hence $(x_2|(y_2|y_2))|(x_2|(y_2|y_2)) \in \mu_x$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3)) \in \gamma_x$. Thus, $\mu_x$ and $\gamma_x$ are subalgebras of $X$. □

## 5. SR-fuzzy filters of Sheffer stroke Hilbert algebras

In this section, we introduce the concept of SR-fuzzy filters in the context of Sheffer stroke Hilbert algebras.

**Definition 5.1.** *An SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $X$ is called SR-fuzzy deductive system of $X$ if it satisfies*

$$(\forall x \in X)\left( (\mu_{\mathcal{S}}(1))^2 \geq (\mu_{\mathcal{S}}(x))^2,\ \sqrt{\gamma_{\mathcal{S}}(1)} \leq \sqrt{\gamma_{\mathcal{S}}(x)} \right), \tag{6}$$

$$(\forall x, y \in X)\left( \begin{array}{c} (\mu_{\mathcal{S}}(y))^2 \geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x|(y|y))))^2\} \\ \sqrt{\gamma_{\mathcal{S}}(y)} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(x|(y|y)))}\} \end{array} \right). \tag{7}$$

**Definition 5.2.** *An SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $X$ is called SR-fuzzy filter of $X$ if it satisfies* (6) *and*

$$(\forall x \in X)\left( (\mu_{\mathcal{S}}(x|(y|y))))^2 \geq (\mu_{\mathcal{S}}(y))^2,\ \sqrt{\gamma_{\mathcal{S}}(x|(y|y)))} \leq \sqrt{\gamma_{\mathcal{S}}(y)} \right), \tag{8}$$

$$(\forall x, y \in X)\left( \begin{array}{c} (\mu_{\mathcal{S}}(((x|(y|z))|(y|z)))^2 \geq \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(z))^2\} \\ \sqrt{\gamma_{\mathcal{S}}(((x|(y|z))|(y|z)))} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(z)}\} \end{array} \right). \tag{9}$$

**Example 5.3.** *Consider the Sheffer stroke Hilbert algebra in Table 10.*

Table 10: Sheffer stroke Hilbert algebra $(X, |)$

| \| | 0 | 1 | a | b | c | d | e | f |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | b | a | d | c | f | e |
| a | 1 | b | b | 1 | d | e | d | e |
| b | 1 | a | 1 | a | 1 | a | a | 1 |
| c | 1 | d | d | 1 | d | 1 | d | 1 |
| d | 1 | c | e | a | 1 | c | a | e |
| e | 1 | f | d | a | d | a | f | 1 |
| f | 1 | e | e | 1 | 1 | e | 1 | e |

*Let $\mu_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}$ be fuzzy sets in X given by the Table 11:*

Table 11: fuzzy sets $\mu_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}$ of $X$

| $X$ | 0 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|---|---|
| $\mu_{\mathcal{S}}(x)$ | 0.6 | 0.7 | 0.7 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| $\gamma_{\mathcal{S}}(x)$ | 0.06 | 0.05 | 0.05 | 0.06 | 0.06 | 0.06 | 0.06 | 0.06 |

*Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is given by the Table 12.*

Table 12: SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ of $X$

| $X$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $(\mu_{\mathcal{S}}(x))^2$ | 0.36 | 0.49 | 0.49 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 |
| $\sqrt{\gamma_{\mathcal{S}}(x)}$ | 0.2449 | 0.2236 | 0.2236 | 0.2449 | 0.2449 | 0.2449 | 0.2449 | 0.2449 |

*where $\sqrt{\gamma_{\mathcal{S}}(x)}$ is calculated to four decimal places. Then the SR-fuzzy set $\mathcal{S}$ satisfies the conditions of Definition 5.2, making it an SR-fuzzy filter of X.*

**Theorem 5.4.** *Given a SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in X, the following are equivalent to each other.*

1. *$\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy deductive system of X.*
2. *$\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy filter of X.*

*Proof.* Assume that $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy deductive system of $X$ and let $x, y, z \in X$. Note that $y|((x|(y|y))|(x|(y|y))) = 1$. Then

$$\begin{aligned}
\sqrt{\gamma_{\mathcal{S}}(x|(y|y))} &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(y|((x|(y|y))|(x|(y|y))))}\} \\
&= \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(1)}\} \\
&= \sqrt{\gamma_{\mathcal{S}}(y)}
\end{aligned}$$

and

$$\begin{aligned}
(\mu_{\mathcal{S}}(x|(y|y)))^2 &\geq \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(y|((x|(y|y))|(x|(y|y)))))^2\} \\
&= \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(1))^2\} \\
&= (\mu_{\mathcal{S}}(y))^2.
\end{aligned}$$

Note that

$$\begin{aligned}
y|(((y|z)|z)|((y|z)|z)) &= y|(((y|z)|((z|z)|(z|z)))|((y|z)|((z|z)|(z|z)))) \\
&= (y|z)|((y|((z|z)|(z|z)))|(y|((z|z)|(z|z)))) \\
&= (y|z)|((y|z)|(y|z)) \\
&= 1.
\end{aligned}$$

It follows that

$$\begin{aligned}\sqrt{\gamma_{\mathcal{S}}((y|z)|z)} &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(y|(((y|z)|z)|((y|z)|z)))}\}\\ &= \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(1)}\}\\ &= \sqrt{\gamma_{\mathcal{S}}(y)}\end{aligned}$$

and

$$\begin{aligned}(\mu_{\mathcal{S}}((y|z)|z))^2 &\geq \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(y|(((y|z)|z)|((y|z)|z))))^2\}\\ &= \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(1))^2\}\\ &= (\mu_{\mathcal{S}}(y))^2.\end{aligned}$$

Since $z|(((y|z)|(y|z))|((y|z)|(y|z))) = z|(y|z) = (y|z)|z$, we obtain

$$\begin{aligned}\sqrt{\gamma_{\mathcal{S}}((y|z)|(y|z))} &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(z)}, \sqrt{\gamma_{\mathcal{S}}(z|(((y|z)|(y|z))|((y|z)|(y|z))))}\}\\ &= \max\{\sqrt{\gamma_{\mathcal{S}}(z)}, \sqrt{\gamma_{\mathcal{S}}((y|z)|z)}\}\\ &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(z)}, \sqrt{\gamma_{\mathcal{S}}(y)}\}\end{aligned}$$

and

$$\begin{aligned}(\mu_{\mathcal{S}}((y|z)|(y|z)))^2 &\geq \min\{(\mu_{\mathcal{S}}(z))^2, (\mu_{\mathcal{S}}(z|(((y|z)|(y|z))|((y|z)|(y|z)))))^2\}\\ &= \min\{(\mu_{\mathcal{S}}(z))^2, (\mu_{\mathcal{S}}((y|z)|z))^2\}\\ &\geq \min\{(\mu_{\mathcal{S}}(z))^2, (\mu_{\mathcal{S}}(y))^2\}.\end{aligned}$$

Hence

$$\begin{aligned}&\sqrt{\gamma_{\mathcal{S}}((x|(y|z))|(y|z))}\\ &= \sqrt{\gamma_{\mathcal{S}}((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z))))}\\ &\leq \sqrt{\gamma_{\mathcal{S}}((y|z)|(y|z))}\\ &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(z)}, \sqrt{\gamma_{\mathcal{S}}(y)}\}\end{aligned}$$

and

$$\begin{aligned}&(\mu_{\mathcal{S}}((x|(y|z))|(y|z)))^2\\ &= (\mu_{\mathcal{S}}((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z)))))^2\\ &\geq (\mu_{\mathcal{S}}((y|z)|(y|z)))^2\\ &\geq \min\{(\mu_{\mathcal{S}}(z))^2, (\mu_{\mathcal{S}}(y))^2\}.\end{aligned}$$

Therefore, $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy filter of $X$.

Conversely, assume that $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy filter of $X$ and let $x, y, z \in X$. If we replace $y$, $z$ and $x$ with $x$, $x|(y|y)$ and $y$, respectively, in (9), then

$$\begin{aligned}\sqrt{\gamma_{\mathcal{S}}(y)} &= \sqrt{\gamma_{\mathcal{S}}(((x|x)|(1|1))|(y|y))}\\ &= \sqrt{\gamma_{\mathcal{S}}(((x|x)|((y|(y|y))|(y|(y|y))))|(y|y))}\\ &= \sqrt{\gamma_{\mathcal{S}}(((((x|x)|y)|((x|x)|y))|(y|y))|(y|y))}\\ &= \sqrt{\gamma_{\mathcal{S}}((y|((x|x)|y))|((x|x)|y))}\\ &= \sqrt{\gamma_{\mathcal{S}}(((((x|x)|y)|y)|y)|(((x|x)|y)|y))}\\ &= \sqrt{\gamma_{\mathcal{S}}((y|(x|(x|(y|y))))|(x|(x|(y|y))))}\\ &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(x|(y|y))}\}\end{aligned}$$

and

$$\begin{array}{rcl}(\mu_{\mathcal{S}}(y))^2 & = & (\mu_{\mathcal{S}}(((x|x)|(1|1))|(y|y)))^2 \\ & = & (\mu_{\mathcal{S}}(((x|x)|((y|(y|y))|(y|(y|y))))|(y|y)))^2 \\ & = & (\mu_{\mathcal{S}}(((((x|x)|y)|((x|x)|y))|(y|y))|(y|y)))^2 \\ & = & (\mu_{\mathcal{S}}((y|((x|x)|y))|((x|x)|y)))^2 \\ & = & (\mu_{\mathcal{S}}((((((x|x)|y)|y)|y)|(((x|x)|y)|y)))^2 \\ & = & (\mu_{\mathcal{S}}((y|(x|(x|(y|y))))|(x|(x|(y|y)))))^2 \\ & \geq & \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x|(y|y)))^2\}'\end{array}$$

Consequently, $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy deductive system of $X$. □

**Proposition 5.5.** *Every SR-fuzzy filter $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ of $X$ satisfies*

$$(\forall x, y \in X)\left(\begin{array}{c}\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(y|y))} \leq \sqrt{\gamma_{\mathcal{S}}(x)} \\ (\mu_{\mathcal{S}}((x|(y|y))|(y|y)))^2 \geq (\mu_{\mathcal{S}}(x))^2\end{array}\right), \tag{10}$$

*and*

$$(\forall\, x, y \in X)\left(\ x \leq y \ \Rightarrow \left\{\begin{array}{c}(\mu_{\mathcal{S}}(y))^2 \geq (\mu_{\mathcal{S}}(x))^2 \\ \sqrt{\gamma_{\mathcal{S}}(y)} \leq \sqrt{\gamma_{\mathcal{S}}(x)}\end{array}\right.\right). \tag{11}$$

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy filter of $X$. Then

$$\begin{array}{rcl}\sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(y|y))} & = & \sqrt{\gamma_{\mathcal{S}}((y|(x|x))|(x|x))} \\ & \geq & \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(x)}\} \\ & = & \sqrt{\gamma_{\mathcal{S}}(x)},\end{array}$$

and

$$\begin{array}{rcl}(\mu_{\mathcal{S}}((x|(y|y))|(y|y)))^2 & = & (\mu_{\mathcal{S}}((y|(x|x))|(x|x)))^2 \\ & \leq & \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x))^2\} \\ & = & (\mu_{\mathcal{S}}(x))^2\end{array}$$

for all $x, y \in X$. Therefore, (10) is valid. Let $x, y \in X$ be such that $x \leq y$. Then $x|(y|y) = 1$, and so

$$\begin{array}{rcl}\sqrt{\gamma_{\mathcal{S}}(y)} & = & \sqrt{\gamma_{\mathcal{S}}(1|(y|y))} \\ & = & \sqrt{\gamma_{\mathcal{S}}((x|(y|y))|(y|y))} \\ & \leq & \sqrt{\gamma_{\mathcal{S}}(x)}\end{array}$$

and

$$\begin{array}{rcl}(\mu_{\mathcal{S}}(y))^2 & = & (\mu_{\mathcal{S}}(1|(y|y)))^2 \\ & = & (\mu_{\mathcal{S}}((x|(y|y))|(y|y)))^2 \\ & \geq & (\mu_{\mathcal{S}}(x))^2,\end{array}$$

proving the proposition. □

**Theorem 5.6.** *A SR-fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ in $L$ is a SR-fuzzy filter of $X$ if and only if it satisfies the condition* (11) *and*

$$(\forall x, y \in X)\left(\begin{array}{c}\sqrt{\gamma_{\mathcal{S}}((x|y)|(x|y))} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(x)}\} \\ (\mu_{\mathcal{S}}((x|y)|(x|y))^2 \geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\}\end{array}\right). \tag{12}$$

*Proof.* Let $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ be a SR-fuzzy filter of $X$. Then the condition (11) is valid. Then

$$\begin{aligned}\sqrt{\gamma_{\mathcal{S}}((x|y)|(x|y))} &= \sqrt{\gamma_{\mathcal{S}}(((1|1)|(x|y))|(x|y))}\\ &\leq \max\{\sqrt{\gamma_{\mathcal{S}}(x)}, \sqrt{\gamma_{\mathcal{S}}(y)}\}\end{aligned}$$

and $(\mu_{\mathcal{S}}((x|y)|(x|y)))^2 = (\mu_{\mathcal{S}}(((1|1)|(x|y))|(x|y)))^2 \geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\}$ for all $x, y \in X$.

Conversely, assume that $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ satisfies (11) and (12). Since $x \leq 1$ and $y \leq x|(y|y)$ for all $x, y \in X$, $\sqrt{\gamma_{\mathcal{S}}(1)} \leq \sqrt{\gamma_{\mathcal{S}}(x)}$, $(\mu_{\mathcal{S}}(1))^2 \leq (\mu_{\mathcal{S}}(x))^2$, $\sqrt{\gamma_{\mathcal{S}}(x|(y|y))} \leq \sqrt{\gamma_{\mathcal{S}}(y)}$, and $(\mu_{\mathcal{S}}(x|(y|y)))^2 \leq (\mu_{\mathcal{S}}(y))^2$. So we have $\sqrt{\gamma_{\mathcal{S}}((x|(y|z))|(y|z))} \leq \sqrt{\gamma_{\mathcal{S}}((y|z)|(y|z))} \leq \max\{\sqrt{\gamma_{\mathcal{S}}(y)}, \sqrt{\gamma_{\mathcal{S}}(z)}\}$. and $(\mu_{\mathcal{S}}((x|(y|z))|(y|z)))^2 \leq (\mu_{\mathcal{S}}((y|z)|(y|z)))^2 \leq \min\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(z))^2\}$ for all $x, y \in X$. Therefore, $\mathcal{S} = (\mu_{\mathcal{S}}, \gamma_{\mathcal{S}})$ is a SR-fuzzy filter of $X$. □

## 6. Conclusion and future work

We introduced a novel extension of fuzzy structures, namely SR-fuzzy subalgebra, SR-fuzzy ideal, and SR-fuzzy filter, within the framework of Sheffer stroke Hilbert algebras. By leveraging the concept of SR-fuzzy sets, we generalized existing fuzzy structures and provided illustrative examples to demonstrate the applicability and relevance of these definitions. Our work builds on the foundational studies of Sheffer stroke operations, Hilbert algebras, and various extensions of fuzzy set theory, contributing to the ongoing exploration of algebraic structures and their applications in diverse fields such as computer science, information sciences, and decision-making. The proposed SR-fuzzy structures not only extend the existing theory but also provide a new perspective for analyzing fuzzy subsystems in algebraic systems. By pursuing the following directions, the proposed SR-fuzzy structures can be further developed and applied to a wide range of mathematical and practical problems, contributing to the advancement of both theoretical and applied research in fuzzy algebra and related fields.

- The concepts of SR-fuzzy subalgebra, SR-fuzzy ideal, and SR-fuzzy filter can be extended to other algebraic structures, such as BCK/BCI-algebras, BE/CI-algebras, MV-algebras, and residuated lattices, etc., to explore their properties and applications.
- The SR-fuzzy sets and their associated structures can be applied to real-world problems in decision-making, control engineering, and information sciences, leveraging their ability to handle uncertainty and imprecision.
- A detailed comparison of SR-fuzzy structures with other fuzzy extensions, such as intuitionistic fuzzy sets, bipolar-valued fuzzy sets, and m-polar fuzzy sets, can provide deeper insights into their advantages and limitations.
- Future research can focus on developing computational algorithms and score functions for ranking SR-fuzzy sets, enabling their practical implementation in decision-making processes.
- Further investigation into the properties of SR-fuzzy subalgebras, ideals, and filters, such as homomorphisms, congruences, and quotient structures, can enrich the theoretical framework and enhance its applicability.
- Combining SR-fuzzy sets with other fuzzy extensions or hybrid models (e.g., neutrosophic sets or soft sets) may lead to the development of more robust and versatile tools for handling complex real-life problems.

## Acknowledgments

The authors would like to express their sincere gratitude to the editors and anonymous reviewers for their invaluable comments and constructive feedback, which significantly contributed to the enhancement of this paper.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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