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Nonlinear ∗-Lie n-type derivations on ∗-algebras

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Abstract. Let $\mathcal{M}$ be an ∗-algebra containing a non-trivial projection with unit I. In this paper, we study the characterization of nonlinear ∗-Lie type derivations on ∗-algebras. For any $S, T \in \mathcal{M}$, a product $[S,T]_* = ST - T^*S$ is called ∗-Lie product. In this article it is shown that, if a map $\Theta : \mathcal{M} \longrightarrow \mathcal{M}$ (not necessarily linear) satisfies $\Theta(q_n(S_1, S_2, \ldots, S_n)) = \sum_{i=1}^{2} q_n(S_1, \ldots, S_{i-1}, \Theta(S_i), S_{i+1}, \ldots, S_n)(n \geq 3)$ for all $S_1, S_2, \ldots, S_n \in \mathcal{M}$, then Θ is additive. Moreover, if $\Theta(iI)$ is self- adjoint, then Θ is an additive ∗-derivation. As an application, we can also apply our result on von Neumann algebras, standard operator algebras and prime ∗-algebras.

1. Introduction

Let $\mathcal{M}$ ba an ∗-algebra over the field $\mathbb{C}$. The expressions $[S,T] = ST - TS$ and $[S,T]_* = ST - T^*S$ for any $S, T \in \mathcal{M}$, represent the Lie product and the Lie ∗-product. The study of these products has gained significant attention in various research areas, as highlighted in the works of numerous authors, including [1, 4, 5, 14, 18, 21].

Let $\mathcal{M}$ be an additive mapping. Then $\Theta : \mathcal{M} \to \mathcal{M}$ is said to be additive derivation if $\Theta(ST) = \Theta(S)T + S\Theta(T)$ for every pair of elements $S, T \in \mathcal{M}$. Furthermore, if Θ also fulfills the condition $\Theta(S^*) = \Theta(S)^*$ for every $S \in \mathcal{M}$, we call Θ is an additive ∗-derivation. In other way, let $\Theta : \mathcal{M} \to \mathcal{M}$ be a non additive mapping, then we call Θ, nonlinear Lie derivation or ∗-Lie derivation if it satisfies the condition

$$\Theta([S,T]) = [\Theta(S), T] + [S, \Theta(T)]$$

or

$$\Theta([S,T]_*) = [\Theta(S), T]_* + [S, \Theta(T)]_*$$

for all $S, T \in \mathcal{M}$. This concept of a nonlinear Lie derivation or Lie ∗-derivation can be extended naturally. Specifically, Θ is called a nonlinear Lie triple derivation or nonlinear ∗-Lie triple derivation if it meets the condition

$$\Theta([[S,T],U]) = [[\Theta(S),T],U] + [[S,\Theta(T)],U] + [[S,T],\Theta(U)]$$

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or

$$\Theta([[S,T]_*,U]_*) = [[\Theta(S),T]_*,U]_* + [[S,\Theta(T)]_*,U]_* + [[S,T]_*,\Theta(U)]_*$$

for all $S,T,U \in \mathcal{M}$. Based on the definition of $*$-Lie derivation, Jing [16] gave the complete characterization of nonlinear $*$-Lie derivation on standard operator algebra and proved that every nonlinear $*$-Lie derivation is linear and inner $*$-derivation. To continue the study of characterization of the Lie derivation to Lie-triple derivation. Li et al. [20] studied nonlinear skew Lie triple derivation on factor von Neumann algebra and proved that every nonlinear skew Lie triple derivation on factors is an addtive $*$-derivation. Similarly, Kong et al. [17] concentrated on characterizing a kind of non-global nonlinear skew Lie triple derivations Θ on factor von Neumann algebras satisfying

$$\Theta([[S,T]_*,U]_*) = [[\Theta(S),T]_*,U]_* + [[S,\Theta(T)]_*,U]_* + [[S,T]_*,\Theta(U)]_*$$

for all $S,T,U \in \mathcal{M}$ with $S^*T^*U = 0$.

In recent years, several researchers have explored Lie n-derivations across various types of algebras (see [22], [23] and related references). In [19], the authors proved that a map Θ between two-factor von Neumann algebras is a $*$-ring isomorphism if and only if $\Theta([a,b]_*) = [\Theta(a),\Theta(b)_*]$, where $[a,b]_* = ab - ba^*$. In [7], Ferreira and Costa extended these new products and defined two other types of applications, named multiplicative $*$-Jordan n-map and multiplicative $*$-Lie n-map and used it to impose conditions such that a map between C^*-algebras is a $*$-ring isomorphism. Further, Andrade et al. [3] study the characterization of multiplicative $*$-Lie-type maps and as application, they obtained the result on alternative W^*-algebras. In [2], the authors provide the characterization of multiplicative $*$-Jordan-type maps on alternative algebras.

Many authors have studied Lie- type derivations in structure like $*$-algebras, matrix rings, and even more general structures like alternative algebras see[15]-[9]. Building on the concepts of Lie derivation and Lie triple derivation, we were inspired to explore similar questions in the context of nonlinear $*$-Lie-type derivations on $*$-algebras. For a fixed positive integer n, where $n \geq 2$, we define polynomials sequence as

$$\begin{aligned} q_1(S_1) &= S_1, \\ q_2(S_1,S_2) &= [q_1(S_1),S_2]_* = [S_1,S_2]_*, \\ q_3(S_1,S_2,S_3) &= [q_2(S_1,S_2),S_3]_* = [[S_1,S_2]_*,S_3]_*, \\ &\cdots\cdots.. \\ q_n(S_1,S_2,\ldots,S_n) &= [q_{n-1}(S_1,S_2,\ldots,S_{n-1}),S_n]_*. \end{aligned}$$

The polynomial $q_n(S_1,S_2,\ldots,S_n)$ is known as $(n-1)^{th}$ commutator.

The definition of nonlinear $*$-Lie type derivations is first presented. A map $\Theta : \mathcal{S} \to \mathcal{S}$ that is additive is known as Lie n-derivation or n-type derivation, if the following is satisfied:

$$\Theta(q_n(S_1,S_2,\ldots,S_n)) = \sum_{i=1}^{n} q_n(S_1,\ldots,S_{i-1},\Theta(S_i),S_{i+1},\ldots,S_n)$$

for all $S_1,S_2,\ldots,S_n \in \mathcal{S}$. More generally, removing the additivity of Θ, we get Θ is a nonlinear $*$-Lie n-derivation. It is evident that all derivations are Lie derivations, and every Lie derivation is, in turn, a Lie triple derivation.

2. Main Result

Now take a projection $\mathcal{P}_1 \in \mathcal{M}$ and let $\mathcal{P}_2 = I - \mathcal{P}_1$. We write $\mathcal{M}_{jk} = \mathcal{P}_j\mathcal{M}\mathcal{P}_k$ for $j,k = 1,2$. Then by the Peirce decomposition of $\mathcal{M}$, we have $\mathcal{M} = \mathcal{M}_{11} \oplus \mathcal{M}_{12} \oplus \mathcal{M}_{21} \oplus \mathcal{M}_{22}$. Note that any operator $S \in \mathcal{M}$ can be expressed as $S = S_{11} + S_{12} + S_{21} + S_{22}$ and $S^*_{jk} \in \mathcal{M}_{kj}$ for any $S_{jk} \in \mathcal{M}_{jk}$.

Theorem 2.1. *Let $\mathcal{M}$ be a $*$-algebra having unit I that contains a nontrivial projection $\mathcal{P}$ such that:*

$$X\mathcal{M}\mathcal{P} = 0 \implies X = 0 \tag{∇}$$

$$X\mathcal{M}(I-\mathcal{P}) = 0 \implies X = 0. \tag{$\triangle$}$$

If $\Theta : \mathcal{M} \to \mathcal{M}$ satisfies

$$\Theta\left(q_n\left(S_1, S_2, \ldots, S_n\right)\right) = \sum_{i=1}^{n} q_n\left(S_1, \ldots, S_{i-1}, \Theta\left(S_i\right), S_{i+1}, \ldots, S_n\right)$$

for all $S_1, S_2, \ldots, S_n \in \mathcal{S}$, then Θ is additive. Moreover, if $\Theta(iI)$ is self- adjoint, then Θ is an additive $*$-derivation.

The proof is organized in a series of lemmas. Since the sequence q_n is defined as:

$$q_n(S_1, S_2, S_3, ..., S_n) \quad := \quad [[...[[S_1, S_2]_*, S_3]_*, ..., S_{n-1}]_*, S_n]_*.$$

Lemma 2.2. *For any, $S \in \mathcal{M}$ and for any integer $n \geq 2$, we have*

$$q_n(S, \mathcal{P}_1, \ldots, \mathcal{P}_1) = \mathcal{P}_2 S \mathcal{P}_1 + (-1)^{n-1}\mathcal{P}_1 S \mathcal{P}_2. \tag{1}$$

$$q_n(S, \mathcal{P}_2, \ldots, \mathcal{P}_2) = \mathcal{P}_1 S \mathcal{P}_2 + (-1)^{n-1}\mathcal{P}_2 S \mathcal{P}_1. \tag{2}$$

Lemma 2.3. $\Theta(0) = 0$.

Proof. It is trivial to prove that

$$\begin{aligned}\Theta(0) &= \Theta(q_n(0, 0, \ldots, 0)) \\ &= q_n(\Theta(0), 0, \ldots, 0) + q_n(0, \Theta(0), \ldots, 0) + ... + q_n(0, 0, ..., \Theta(0)) \\ &= 0.\end{aligned}$$

□

Lemma 2.4. *For any $S_{11} \in \mathcal{M}_{11}, S_{12} \in \mathcal{M}_{12}, S_{21} \in \mathcal{M}_{21}, S_{22} \in \mathcal{M}_{22}$, we have*

$$\begin{aligned}\Theta(S_{11} + S_{12}) &= \Theta(S_{11}) + \Theta(S_{12}) \\ \Theta(S_{21} + S_{22}) &= \Theta(S_{21}) + \Theta(S_{22}).\end{aligned}$$

Proof. For any $S_{11} \in \mathcal{M}_{11}, S_{12} \in \mathcal{M}_{12}$, Let $M = \Theta(S_{11} + S_{12}) - (\Theta(S_{11}) + \Theta(S_{12}))$. We have

$$\begin{aligned}&\Theta(q_n(S_{11} + S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\ &\quad = q_n(\Theta(S_{11} + S_{12}), \mathcal{P}_2, \ldots, \mathcal{P}_2) + q_n(S_{11} + S_{12}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2) \\ &\qquad + \cdots + q_n(S_{11} + S_{12}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).\end{aligned}$$

Now, it is easy to see that $q_n(S_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$ and using Lemma 2.3, we have

$$\begin{aligned}&\Theta(q_n(S_{11} + S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\ &\quad = \Theta(q_n(S_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) + \Theta(q_n(S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\ &\quad = q_n(\Theta(S_{11}) + \Theta(S_{12}), \mathcal{P}_2, \ldots, \mathcal{P}_2) + q_n(S_{11} + S_{12}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2) \\ &\qquad + \cdots + q_n(S_{11} + S_{12}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).\end{aligned}$$

Above two relations implies that $q_n(M, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Now, using Lemma 2.2, we have $\mathcal{P}_1 M \mathcal{P}_2 + (-1)^{n-1}\mathcal{P}_2 M \mathcal{P}_1 = 0$. By multiplying $\mathcal{P}_1$ on both sides, we get $\mathcal{P}_1 M \mathcal{P}_2 = 0$. Hence, it follows from (∇) and ($\triangle$) , we obtain $M_{12} = 0$.

Similarly, by multiplying $\mathcal{P}_2$ on both sides and using (∇) and (△), we get $M_{21} = 0$.
Now, it is observe that $q_n(S_{11}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Using Lemma 2.3, we have

$$\begin{aligned}\Theta(q_n(S_{11}+S_{12}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) &= q_n(\Theta(S_{11}+S_{12}), X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(S_{11}+S_{12}, \Theta(X_{21}), \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(S_{11}+S_{12}, X_{21}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2)\\ &\quad +\cdots+q_n(S_{11}+S_{12}, X_{21}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).\end{aligned}$$

Whereas,

$$\begin{aligned}&\Theta(q_n(S_{11}+S_{12}, X_{21}, \mathcal{P}_2, \ldots \mathcal{P}_2))\\ &= \Theta(q_n(S_{11}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) + \Theta(q_n(S_{12}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2))\\ &= q_n(\Theta(S_{11})+\Theta(S_{12}), X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(S_{11}+S_{12}, \Theta(X_{21}), \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(S_{11}+S_{12}, X_{21}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2)\\ &\quad +\cdots+q_n(S_{11}+S_{12}, X_{21}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).\end{aligned}$$

It follows from above two expressions that $q_n(M, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Now using Lemma 2.2, (∇) and (△) implies that $M_{22} = 0$.
Again, we have $q_n(X_{12}, S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Using Lemma 2.3, we have

$$\begin{aligned}&\Theta(q_n(X_{12}, S_{11}+S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2))\\ &= q_n(\Theta(X_{12}), S_{11}+S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2) + q_n(X_{12}, \Theta(S_{11}+S_{12}), \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(X_{12}, S_{11}+S_{12}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2)+\cdots+q_n(X_{12}, S_{11}+S_{12}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).\end{aligned}$$

On the other hand, we get

$$\begin{aligned}&\Theta(q_n(X_{12}, S_{11}+S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2))\\ &= \Theta(q_n(X_{12}, S_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) + \Theta(q_n(X_{12}, S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2))\\ &= q_n(\Theta(X_{12}), S_{11}+S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2) + q_n(X_{12}, \Theta(S_{11})+\Theta(S_{12}), \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(X_{12}, S_{11}+S_{12}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2)+\cdots+q_n(X_{12}, S_{11}+S_{12}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).\end{aligned}$$

Which will give us $q_n(X_{12}, M, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Now, using Lemma 2.3, (∇) and (△), we obtain $X_{12}M\mathcal{P}_2 - \mathcal{P}_1M^\star X_{12} = 0$. Since $M_{22} = 0$. Therefore, we get $M_{11} = 0$. Hence, we have $M = 0$, i.e.,

$$\Theta(S_{11}+S_{12}) = \Theta(S_{11}) + \Theta(S_{12}).$$

The other case can be prove analogously. This concludes the proof. □

Lemma 2.5. *For any $S_{11} \in \mathcal{M}_{11}, S_{12} \in \mathcal{M}_{12}, S_{21} \in \mathcal{M}_{21}, S_{22} \in \mathcal{M}_{22}$, We have*

$$\Theta(S_{11}+S_{12}+S_{21}+S_{22}) = \Theta(S_{11}) + \Theta(S_{12}) + \Theta(S_{21}) + \Theta(S_{22})$$

Proof. Let $M = \Theta(S_{11}+S_{12}+S_{21}+S_{22}) - \Theta(S_{11}) - \Theta(S_{12}) - \Theta(S_{21}) - \Theta(S_{22})$. Now, it is easily seen that $q_n(S_{11}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = q_n(S_{12}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Using Lemma 2.2 and Lemma 2.4, we obtain

$$\begin{aligned}&\Theta(q_n(S_{11}+S_{12}+S_{21}+S_{22}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2))\\ &= \Theta(q_n(S_{11}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) + \Theta(q_n(S_{12}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2))\\ &\quad +\Theta(q_n(S_{21}+S_{22}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2))\\ &= q_n(\Theta(S_{11})+\Theta(S_{12})+\Theta(S_{21})+\Theta(S_{22}), X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(S_{11}+S_{12}+S_{21}+S_{22}, \Theta(X_{21}), \mathcal{P}_2, \ldots, \mathcal{P}_2)\\ &\quad +q_n(S_{11}+S_{12}+S_{21}+S_{22}, X_{21}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2)\\ &\quad +\cdots+q_n(S_{11}+S_{12}+S_{21}+S_{22}, X_{21}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Theta(q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{21},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&= q_n(\Theta(S_{11}+S_{12}+S_{21}+S_{22}),X_{21},\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(S_{11}+S_{12}+S_{21}+S_{22},\Theta(X_{21}),\mathcal{P}_2\ldots,\mathcal{P}_2)\\
&\quad+q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{21},\Theta(\mathcal{P}_2),\ldots,\mathcal{P}_2)\\
&\quad+\cdots+q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{21},\mathcal{P}_2,\ldots,\Theta(\mathcal{P}_2)).
\end{aligned}$$

Which will gives us $q_n(M,X_{21},\mathcal{P}_2,\ldots,\mathcal{P}_2)=0$. Now, using Lemma 2.2, we obtain $-X_{21}^*M\mathcal{P}_2+(-1)^{n-1}\mathcal{P}_2MX_{21}=0$. By left multiplying with $\mathcal{P}_2$, on both sides and using (∇) and ($\triangle$) , we get $M_{22}=0$.
Again, $q_n(S_{22},X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2)=q_n(S_{21},X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2)=0$. and using Lemma 2.2, Lemma 2.4, we have

$$\begin{aligned}
&\Theta(q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&= \Theta(q_n(S_{11}+S_{12},X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2))+\Theta(q_n(S_{21},X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&\quad+\Theta(q_n(S_{22},X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&= q_n(\Theta(S_{11})+\Theta(S_{12})+\Theta(S_{21})+\Theta(S_{21})+\Theta(S_{22}),X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(S_{11}+S_{12}+S_{21}+S_{22},\Theta(X_{12}),\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{12},\Theta(\mathcal{P}_2),\ldots,\mathcal{P}_2)\\
&\quad+\cdots+q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{12},\mathcal{P}_2\ldots,\Theta(\mathcal{P}_2))
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Theta(q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&= q_n(\Theta(S_{11}+S_{12}+S_{21}+S_{22}),X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(S_{11}+S_{12}+S_{21}+S_{22},\Theta(X_{12}),\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{12},\Theta(\mathcal{P}_2),\ldots,\mathcal{P}_2)\\
&\quad+\cdots+q_n(S_{11}+S_{12}+S_{21}+S_{22},X_{12},\mathcal{P}_2,\ldots,\Theta(\mathcal{P}_2)).
\end{aligned}$$

On comparing the above two equations we get, $q_n(M,X_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2)=0$. Now, using Lemma 2.2, we obtain $\mathcal{P}_1MX_{12}+(-1)^{n-1}(-X_{12}^*M\mathcal{P}_1)=0$. By left multiplying $\mathcal{P}_1$ on both sides, we get $\mathcal{P}_1MX_{12}=0$. Hence, it follows from (∇)and ($\triangle$), we get $M_{11}=0$.
Now, since $q_n(\mathcal{P}_2,S_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2,)=q_n(\mathcal{P}_2,S_{11},\mathcal{P}_2,\ldots,\mathcal{P}_2)=0$ and using Lemma 2.2, Lemma 2.4, we have

$$\begin{aligned}
&\Theta(q_n(\mathcal{P}_2,S_{11}+S_{12}+S_{21}+S_{22},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&= \Theta(q_n(\mathcal{P}_2,S_{11},\mathcal{P}_2,\ldots,\mathcal{P}_2))+\Theta(q_n(\mathcal{P}_2,S_{12},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&\quad+\Theta(q_n(\mathcal{P}_2,S_{21}+S_{22},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&= q_n(\Theta(\mathcal{P}_2),S_{11}+S_{12}+S_{21}+S_{22},\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(\mathcal{P}_2,\Theta(S_{11})+\Theta(S_{12})+\Theta(S_{21})+\Theta(S_{22}),\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(\mathcal{P}_2,S_{11}+S_{12}+S_{21}+S_{22},\Theta(\mathcal{P}_2),\ldots,\mathcal{P}_2)\\
&\quad+\cdots+q_n(\mathcal{P}_2,S_{11}+S_{12}+S_{21}+S_{22},\mathcal{P}_2,\ldots,\Theta(\mathcal{P}_2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Theta(q_n(\mathcal{P}_2,S_{11}+S_{12}+S_{21}+S_{22},\mathcal{P}_2,\ldots,\mathcal{P}_2))\\
&= q_n(\Theta(\mathcal{P}_2),S_{11}+S_{12}+S_{21}+S_{22},\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(\mathcal{P}_2,\Theta(S_{11}+S_{12}+S_{21}+S_{22}),\mathcal{P}_2,\ldots,\mathcal{P}_2)\\
&\quad+q_n(\mathcal{P}_2,S_{11}+S_{12}+S_{21}+S_{22},\Theta(\mathcal{P}_2),\ldots,\mathcal{P}_2)\\
&\quad+\cdots+q_n(\mathcal{P}_2,S_{11}+S_{12}+S_{21}+S_{22},\mathcal{P}_2,\ldots,\Theta(\mathcal{P}_2)).
\end{aligned}$$

Which will give us, $q_n(\mathcal{P}_2,M,\mathcal{P}_2,\ldots,\mathcal{P}_2)=0$. Now, using Lemma 2.2, we obtain $-\mathcal{P}_1M^*\mathcal{P}_2+(-1)^{n-1}(\mathcal{P}_2M\mathcal{P}_1)=0$. By left multiplying with $\mathcal{P}_2$ on both sides and using (∇),($\triangle$), we get $M_{21}=0$.

Now for M_{12}, using the fact that $q_n(\mathcal{P}_1, S_{22}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = q_n(\mathcal{P}_1, S_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. and Lemma 2.2, Lemma2.4, we have

$$\begin{aligned}
&\Theta(q_n(\mathcal{P}_1, S_{11} + S_{12} + S_{21} + S_{22}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\
&\quad= \Theta(q_n(\mathcal{P}_1, S_{11} + S_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\
&\qquad+\Theta(q_n(\mathcal{P}_1, S_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\
&\qquad+\Theta(q_n(\mathcal{P}_1, S_{22}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\
&\quad= q_n(\Theta(\mathcal{P}_1), S_{11} + S_{12} + S_{21} + S_{22}, \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\qquad+q_n(\mathcal{P}_1, \Theta(S_{11}) + \Theta(S_{12}) + \Theta(S_{21}) + \Theta(S_{22}), \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\qquad+q_n(\mathcal{P}_1, S_{11} + S_{12} + S_{21} + S_{22}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2) \\
&\qquad+ \cdots + q_n(\mathcal{P}_1, S_{11} + S_{12} + S_{21} + S_{22}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2))
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Theta(q_n(\mathcal{P}_1, S_{11} + S_{12} + S_{21} + S_{22}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\
&\quad= q_n(\Theta(\mathcal{P}_1), S_{11} + S_{12} + S_{21} + S_{22}, \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\qquad+q_n(\mathcal{P}_1, \Theta(S_{11} + S_{12} + S_{21} + S_{22}), \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\qquad+q_n(\mathcal{P}_1, S_{11} + S_{12} + S_{21} + S_{22}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2) \\
&\qquad+ \cdots + q_n(\mathcal{P}_1, S_{11} + S_{12} + S_{21} + S_{22}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).
\end{aligned}$$

On comparing the above two equations, we get $q_n(\mathcal{P}_1, M, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Now using Lemma 2.2, we obtain $\mathcal{P}_1 M \mathcal{P}_2 + (-1)^{n-1}\mathcal{P}_2 M^* \mathcal{P}_1 = 0$. By left multiplying with $\mathcal{P}_1$ on both side and using (∇),($\triangle$) we get $M_{12} = 0$. Hence, $M = 0$. □

Lemma 2.6. *For any $S_{11}, T_{11} \in \mathcal{M}_{11}$ and $S_{22}, T_{22} \in \mathcal{M}_{22}$, we have*

$$\begin{aligned}
\Theta(S_{11} + T_{11}) &= \Theta(S_{11}) + \Theta(T_{11}). \\
\Theta(S_{22} + T_{22}) &= \Theta(S_{22}) + \Theta(T_{22}).
\end{aligned}$$

Proof. *Let $M = \Theta(S_{11} + T_{11}) - \Theta(S_{11}) - \Theta(T_{11})$. Now using the fact $q_n(\mathcal{P}_2, S_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$, we have*

$$\begin{aligned}
\Theta(q_n(\mathcal{P}_2, S_{11} + T_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) &= \Theta(q_n(\mathcal{P}_2, S_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) + \Theta(q_n(\mathcal{P}_2, T_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) \\
&= q_n(\Theta(\mathcal{P}_2), S_{11} + T_{11}, \ldots, \mathcal{P}_2) \\
&\quad+q_n(\mathcal{P}_2, \Theta(S_{11}) + \Theta(T_{11}), \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\quad+q_n(\mathcal{P}_2, S_{11} + T_{11}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2) \\
&\quad+ \cdots + q_n(\mathcal{P}_2, S_{11} + T_{11}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).
\end{aligned}$$

Calculating in another way, we get

$$\begin{aligned}
\Theta(q_n(\mathcal{P}_2, S_{11} + T_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) &= q_n(\Theta(\mathcal{P}_2), S_{11} + T_{11}, \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\quad+q_n(\mathcal{P}_2, \Theta(S_{11} + T_{11}), \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\quad+q_n(\mathcal{P}_2, S_{11} + T_{11}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2) \\
&\quad+ \cdots + q_n(\mathcal{P}_2, S_{11} + T_{11}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).
\end{aligned}$$

On comparing the above two expressions, we get $q_n(\mathcal{P}_2, M, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Using Lemma 2.2, ($\nabla$) and ($\triangle$), we get $M_{12} = 0$. Similarily we can obtain $M_{21} = 0$. Now, for M_{22}, using $q_n(S_{11}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$ and Lemma 2.4,

$$\begin{aligned}
\Theta(q_n(S_{11} + T_{11}, X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2)) &= q_n(\Theta(S_{11}) + \Theta(T_{11}), X_{21}, \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\quad+q_n(S_{11} + T_{11}, \Theta(X_{21}), \mathcal{P}_2, \ldots, \mathcal{P}_2) \\
&\quad+q_n(S_{11} + T_{11}, X_{21}, \Theta(\mathcal{P}_2), \ldots, \mathcal{P}_2) \\
&\quad+ \cdots + q_n(S_{11} + T_{11}, X_{21}, \mathcal{P}_2, \ldots, \Theta(\mathcal{P}_2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\Theta(q_n(S_{11}+T_{11},X_{21},\mathcal{P}_2,\dots,\mathcal{P}_2)) &= q_n(\Theta(S_{11}+T_{11}),X_{21},\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &+q_n(S_{11}+T_{11},\Theta(X_{21}),\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &+q_n(S_{11}+T_{11},X_{21},\Theta(\mathcal{P}_2),\dots,\mathcal{P}_2)\\ &+\cdots+q_n(S_{11}+T_{11},X_{21},\mathcal{P}_2,\dots,\Theta(\mathcal{P}_2))\end{aligned}$$

Comparing the above equations, we obtain $q_n(M,X_{21},\mathcal{P}_2,\dots,\mathcal{P}_2)=0$. Which after further solving and using Lemma 2.2, (∇) and (△) gives us $M_{22}=0$. Similarly, we can show that $M_{11}=0$. Hence,

$$\Theta(S_{11}+T_{11})=\Theta(S_{11})+\Theta(T_{11}).$$

□

Lemma 2.7. *For any $S_{12},T_{12}\in\mathcal{M}_{12}$ and $S_{21},T_{21}\in\mathcal{M}_{21}$, we have*

$$\begin{aligned}\Theta(S_{12}+T_{12}) &= \Theta(S_{12})+\Theta(T_{12}).\\ \Theta(S_{21}+T_{21}) &= \Theta(S_{21})+\Theta(T_{21}).\end{aligned}$$

Proof. Let $M=\Theta(S_{12}+T_{12})-\Theta(S_{12})-\Theta(T_{12})$. Now, using $q_n(\mathcal{P}_2,S_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)=0$ and Lemma 2.3, we have

$$\begin{aligned}\Theta(q_n(\mathcal{P}_2,S_{12}+T_{12},\mathcal{P}_2,\dots,\mathcal{P}_2) &= \Theta(q_n(\mathcal{P}_2,S_{12},\mathcal{P}_2,\dots,\mathcal{P}_2))+\Theta(q_n(\mathcal{P}_2,T_{12},\mathcal{P}_2,\dots,\mathcal{P}_2))\\ &= q_n(\Theta(\mathcal{P}_2),S_{12}+T_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &\quad+q_n(\mathcal{P}_2,\Theta(S_{12})+\Theta(T_{12}),\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &\quad+q_n(\mathcal{P}_2,S_{12}+T_{12},\Theta(\mathcal{P}_2),\dots,\mathcal{P}_2)\\ &\quad+\cdots+q_n(\mathcal{P}_2,S_{12}+T_{12},\mathcal{P}_2,\dots,\Theta(\mathcal{P}_2)).\end{aligned}$$

Whereas,

$$\begin{aligned}\Theta(q_n(\mathcal{P}_2,S_{12}+T_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)) &= q_n(\Theta(\mathcal{P}_2),S_{12}+T_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &+q_n(\mathcal{P}_2,\Theta(S_{12}+T_{12}),\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &+q_n(\mathcal{P}_2,S_{12}+T_{12},\Theta(\mathcal{P}_2),\dots,\mathcal{P}_2)\\ &+\cdots+q_n(\mathcal{P}_2,S_{12}+T_{12},\mathcal{P}_2,\dots,\Theta(\mathcal{P}_2)).\end{aligned}$$

From above two expressions, we get $q_n(\mathcal{P}_2,M,\mathcal{P}_2,\dots,\mathcal{P}_2)=0$. Now, by using Lemma 2.2, (∇) and (△), we get $M_{12}=0$. Similarily by using same approach, we can obtain $M_{21}=0$ Now, for M_{11}. It is easily seen that $q_n(S_{12},X_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)=0$, and using Lemma 2.3, we have

$$\begin{aligned}\Theta(q_n(S_{12}+T_{12},X_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)) &= \Theta(q_n(S_{12},X_{12},\mathcal{P}_2,\dots,\mathcal{P}_2))+\Theta(q_n(T_{12},X_{12},\mathcal{P}_2,\dots,\mathcal{P}_2))\\ &= q_n(\Theta(S_{12})+\Theta(T_{12}),X_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &\quad+q_n(S_{12}+T_{12},\Theta(X_{12}),\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &\quad+q_n(S_{12}+T_{12},X_{12},\Theta(\mathcal{P}_2),\dots,\mathcal{P}_2)\\ &\quad+\cdots+q_n(S_{12}+T_{12},X_{12},\mathcal{P}_2,\dots,\Theta(\mathcal{P}_2)).\end{aligned}$$

Calculating in another way,

$$\begin{aligned}\Theta(q_n(S_{12}+T_{12},X_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)) &= q_n(\Theta(S_{12}+T_{12}),X_{12},\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &+q_n(S_{12}+T_{12},\Theta(X_{12}),\mathcal{P}_2,\dots,\mathcal{P}_2)\\ &+q_n(S_{12}+T_{12},X_{12},\Theta(\mathcal{P}_2),\dots,\mathcal{P}_2)\\ &+\cdots+q_n(S_{12}+T_{12},X_{12},\mathcal{P}_2,\dots,\Theta(\mathcal{P}_2)).\end{aligned}$$

From above two equations, we get $q_n(M, X_{12}, \mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$. Now, using Lemma 2.2, we get $\mathcal{P}_1 M X_{12} + (-1)^{n-1} X_{12}^* M \mathcal{P}_1 = 0$. By multiplying $\mathcal{P}_2$ on both sides, we get $\mathcal{P}_1 M X_{12} = 0$. Therefore, by using ($\nabla$) and ($\triangle$), we get, $M_{11} = 0$. Similarly, we can show that $M_{22} = 0$. Hence,

$$\Theta(S_{12} + T_{12}) = \Theta(S_{12}) + \Theta(T_{12}).$$

In the similar way, one can easily show that

$$\Theta(S_{21} + T_{21}) = \Theta(S_{21}) + \Theta(T_{21}).$$

□

Lemma 2.8. *Θ is additive.*

Proof. Let $S, T \in \mathcal{M}$ and write $S = \sum_{i,j=1}^{2} S_{ij},\ T = \sum_{i,j=1}^{2} T_{ij}$. Then by using Lemma 2.4 - 2.7, we have

$$\begin{aligned}
\Theta(S+T) &= \Theta(\sum_{i,j=1}^{2} S_{ij} + \sum_{i,j=1}^{2} T_{ij}) \\
&= \Theta\left(\sum_{i,j=1}^{2} (S_{ij} + T_{ij})\right) \\
&= \sum_{i,j=1}^{2} \Theta(S_{ij} + T_{ij}) \\
&= \sum_{i,j=1}^{2} \Theta(S_{ij}) + \Theta(T_{ij}) \\
&= \Theta(\sum_{i,j=1}^{2} S_{ij}) + \Theta(\sum_{i,j=1}^{2} T_{ij}) \\
&= \Theta(S) + \Theta(T).
\end{aligned}$$

□

Now in further lemmas, we will prove that Θ is an $*$- derivation.

Lemma 2.9. $\Theta(I)^* = \Theta(I)$.

Proof. Since $q_n(I, I, iI, \ldots, iI) = 0$. Therefore,

$$\begin{aligned}
0 &= \Theta(q_n(I, I, iI, \ldots, iI)) \\
&= q_n(\Theta(I), I, iI, \ldots, iI) + q_n(I, \Theta(I), iI, \ldots, iI) \\
&= 2^{n-1} i^{n-1} (\Theta(I) - \Theta(I)^*).
\end{aligned}$$

Which gives $\Theta(I)^* = \Theta(I)$. □

Lemma 2.10. *If $\Theta(iI)^* = \Theta(iI)$, then $\Theta(iI) = \Theta(I) = 0$.*

Proof. Using the fact that, $q_n(I, iI, \ldots, iI) = 2^n i^n I$ and Lemma 2.9, we obtain

$$\begin{aligned}
\Theta(2^n i^n I) &= q_n(\Theta(I), iI, \ldots, iI) + q_n(I, \Theta(iI), \ldots, iI) \\
&\quad + \cdots + q_n(I, iI, \ldots, \Theta(iI)) \\
2^n \Theta(i^n I) &= 2^n i^n \Theta(I) + 2^{n-1} i^{n-1} (\Theta(iI) - \Theta(iI)^*)(n-1).
\end{aligned}$$

Since, $\Theta(iI) = \Theta(iI)^*$, so

$$\Theta(iI) = i\Theta(I).$$

Taking adjoint on both sides of the above relation, we obtain

$$\Theta(iI)^* = -i\Theta(I).$$

Since Θ is self - adjoint. On combining the last two relations, we obtain $\Theta(iI) = 0$ and $\Theta(I) = 0$. □

Lemma 2.11. *$\Theta(iS) = i\Theta(S)$ for any $S \in \mathcal{M}$.*

Proof. As, $q_n(S, iI, \dots, iI) = 2^n i^n S$

$$\begin{aligned}\Theta(2^n i^n S) &= \Theta(q_n(S, iI, \dots, iI)) \\ &= q_n(\Theta(S), iI \dots, iI) + q_n(S, \Theta(iI) \dots, iI) \\ &\quad + \dots + q_n(S, iI, \dots, \Theta(iI)) \\ &= 2^n i^n \Theta(S).\end{aligned}$$

Hence, $\Theta(iS) = i\Theta(S)$. □

Lemma 2.12. *Θ preserves star.*

Proof. Observes that $q_n(iI, S, iI, \dots, iI) = 2^{n-1} i^n (S - S^*)$, using $\Theta(iI) = \Theta(I) = 0$, we have

$$\begin{aligned}\Theta(2^{n-1} i^n (S - S^*)) &= \Theta(q_n(iI, S, iI, \dots, iI)) \\ &= q_n(iI, \Theta(S), iI, \dots, iI) \\ &= 2^{n-1} i^n (\Theta(S) - \Theta(S)^*).\end{aligned}$$

Which implies, $\Theta(S^*) = \Theta(S)^*$. □

Lemma 2.13. *Θ is a derivation i.e., $\Theta(ST) = \Theta(S)T + S\Theta(T)$ for all $S, T \in \mathcal{M}$.*

Proof. Observe that $q_n(S, T, iI, \dots, iI) = 2^{n-1} i^{n-1} (ST - T^*S)$ for any $S, T \in \mathcal{M}$ and using Lemmas 2.9 - 2.11, we obtain

$$\begin{aligned}2^{n-1} i^{n-1} \Theta(ST - T^*S) &= \Theta(q_n(S, T, iI, \dots, iI)) \\ &= q_n(\Theta(S), T, iI, \dots, iI) + q_n(S, \Theta(T), iI \dots, iI) \\ &= 2^{n-1} i^{n-1} (\Theta(S)T + S\Theta(T) - \Theta(T)^*S - T^*\Theta(S)).\end{aligned}$$

Therefore,

$$\Theta(ST - T^*S) = \Theta(S)T + S\Theta(T) - \Theta(T)^*S - T^*\Theta(S). \tag{3}$$

Equation (3) implies that,

$$\begin{aligned}\Theta(ST + T^*S) &= \Theta((-iS)(iT) - (iT)^*(-iS)) \\ &= \Theta(-iS)(iT) + (-iS)\Theta(iT) - \Theta(iT)^*(-iS) - (iT)^*\Theta(-iS).\end{aligned}$$

Hence,

$$\Theta(ST + T^*S) = \Theta(S)T + S\Theta(T) + \Theta(T)^*S + T^*\Theta(S) \tag{4}$$

On combining (3) and (4), we get

$$\Theta(ST) = \Theta(S)T + S\Theta(T).$$

□

3. Applications

As an applications, corollaries given below arise directly from Theorem 2.1:

Corollary 3.1. *Consider $\mathcal{N}$ to be a standard operator algebra (SOA) on an infinite dimensional (ID) complex Hilbert space (CHS) $\mathcal{H}$ that contain identity operator $\mathfrak{Z}$. Further, assume $\mathcal{N}$ is closed with respect to adjoint operation. Map Θ from $\mathcal{N}$ to $\mathcal{N}$ is defined in such a way that*

$$\Theta\left(q_n\left(S_1, S_2, \ldots, S_n\right)\right) = \sum_{i=1}^{n} q_n\left(S_1, \ldots, S_{i-1}, \Theta\left(S_i\right), S_{i+1}, \ldots, S_n\right)$$

for all $S_1, S_2, \ldots, S_n \in \mathcal{S}$, then Θ is additive. Moreover, if $\Theta(iI)$ is self- adjoint, then Θ is an additive $*$-derivation.

Proof. A prime algebra, denoted as $\mathcal{N}$, which is a conventional operator algebra, directly results from the Hahn-Banach theorem. As this algebra, $\mathcal{N}$ inherently meets criteria as seen in equations (∇) and ($\triangle$). Therefore, we infer that the previously discussed map Ω is an additive $*$-derivation, based on Theorem 2.1. □

Corollary 3.2. *Let $\mathcal{N}$ is a factor von Neumann algebra (VNA) having $dim\mathcal{N} \geq 2$. Defining a mapping Θ from $\mathcal{N}$ to $\mathcal{N}$ so that*

$$\Theta\left(q_n\left(S_1, S_2, \ldots, S_n\right)\right) = \sum_{i=1}^{n} q_n\left(S_1, \ldots, S_{i-1}, \Theta\left(S_i\right), S_{i+1}, \ldots, S_n\right)$$

for all $S_1, S_2, \ldots, S_n \in \mathcal{S}$, then Θ is additive. Moreover, if $\Theta(iI)$ is self- adjoint, then Θ is an additive $*$-derivation.

Proof. Utilizing [24, Lemma 2.2], where it is demonstrated for any $\mathcal{N}$ fulfilling criterion (∇) and ($\triangle$) is valid. Consequently, by invoking Theorem 2.1, it is deduced that the map Θ, previously mentioned, is additive $*$-derivation in a framework to factor VNA. □

A ring $\mathcal{R}$ is called prime if $\mathcal{J}\mathcal{K} \neq 0$ for any nonzero ideals $\mathcal{J}, \mathcal{K} \subseteq \mathcal{R}$, and semiprime if it contains no nonzero ideal whose square is zero. In this situation, $\mathcal{N}$ is called prime algebra.

Corollary 3.3. *Suppose $\mathcal{N}$ is a prime $*$-algebra with unit say $\mathfrak{Z}$ that contains non trivial projection P. Now if Θ from $\mathcal{N}$ to $\mathcal{N}$ fulfills the condition*

$$\Theta\left(q_n\left(S_1, S_2, \ldots, S_n\right)\right) = \sum_{i=1}^{n} q_n\left(S_1, \ldots, S_{i-1}, \Theta\left(S_i\right), S_{i+1}, \ldots, S_n\right)$$

for all $S_1, S_2, \ldots, S_n \in \mathcal{S}$, then Θ is additive. Moreover, if $\Theta(iI)$ is self- adjoint, then Θ is an additive $*$-derivation.

Proof. It is straightforward that $\mathcal{N}$ satisfies (∇) and ($\triangle$). Then from Theorem 2.1, Θ is an additive $*$-derivation. □

4. Open Problems

A natural direction for future research is to explore whether the key conclusions of our study (Theorems 2.1 and related Lemmas) extend to broader classes of algebraic structures, particularly non-associative algebras such as alternative algebras and W^*-algebras.

In the context of alternative rings, Ferreira and Ferreira established the following characterization of prime rings [6, Theorem 1.1]

Theorem 4.1. *Let R be a 3-torsion-free alternative ring. Then R is a prime ring if and only if*

$$aR.b = 0 \quad or \quad a.Rb = 0 \quad \Rightarrow \quad a = 0 \text{ or } b = 0, \quad \forall a, b \in R.$$

It is well known that the 3-torsion-free condition is unnecessary in the case of associative rings. This raises an interesting open question:

Can the main results of our work be extended to non-associative settings, particularly to alternative algebras and other structured algebras, without additional torsion-free assumptions?

Investigating this problem could lead to new insights into the structural properties of non-associative algebras and their prime ideals, potentially uncovering deeper connections between associative and non-associative algebraic systems.

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