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Remarks regarding some special matrices

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Abstract. In this paper, by using matix representation for quaternions and octonions, we provide a procedure to obtain some example of $k-$potent matrices of order 4 or 8, over the real field or over the field $\mathbb{Z}_p$, with p a prime number.

1. Introduction

In this paper, in the study of $k-$potent elements we extend the results obtained in [1] to generalised quaternion algebras and generalised octonion algebras. As an application, by using the matrix representations for quaternions and octonions, we give examples of classes of matrices which are $k-$potents and a procedure to find such examples.

The paper is organised as follows: in the second paragraph we present matrix representations and their properties which can be used to provide examples of $k-$potent matrices and in the part three we study $k-$potents elements in quaternion algebras and octonion algebras over the real field. The paper end with conclusions and an idea for a further research.

2. Matrix representation

In this paper, the field K is considered with characteristic different from two. In the following, we will consider the quaternion algebra over an arbitrary field K.

For two elements $a, b \in K$, we define a generalized quaternion algebra, denoted by $\mathbb{H}(\alpha,\beta) = \left(\frac{a,b}{K}\right)$, with basis $\{1, f_1, f_2, f_3\}$ and multiplication given in the following table:

$\cdot$	1	f_1	f_2	f_3
1	1	f_1	f_2	f_3
f_1	f_1	a	f_3	af_2
f_2	f_2	$-f_3$	b	$-bf_1$
f_3	f_3	$-af_2$	bf_1	$-ab$

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If $q \in \mathbb{H}(a,b)$, $q = q_0 + q_1 f_1 + q_2 f_2 + q_3 f_3$, then

$$\overline{q} = q_0 - q_1 f_1 - q_2 f_2 - q_3 f_3$$

is called the *conjugate* of the element q. For $q \in \mathbb{H}(a,b)$, we consider the following elements:

$$\mathbf{t}(q) = q + \overline{q} \in K$$

and

$$\mathbf{n}(q) = q\overline{q} = q_0^2 - aq_1^2 - bq_2^2 + abq_3^2 \in K,$$

called the *trace*, respectively, the *norm* of the element $q \in \mathbb{H}(a,b)$. It follows that

$$(q + \overline{q})\, q = q^2 + \overline{q}q = q^2 + \mathbf{n}(q) \cdot 1$$

and

$$q^2 - \mathbf{t}(q)\, q + \mathbf{n}(q) = 0, \forall q \in \mathbb{H}(a,b),$$

therefore the generalized quaternion algebra is a *quadratic algebra*.

If, for $x \in \mathbb{H}(a,b)$, the relation $\mathbf{n}(x) = 0$ implies $x = 0$, then the algebra $\mathbb{H}(a,b)$ is a *division* algebra. A quaternion non-division algebra is called a *split* algebra.

Using the above notations, we remark that $\mathbb{H}(-1,-1) = \left(\frac{-1,-1}{\mathbb{R}}\right)$ is a division algebra.

A generalized octonion algebra over an arbitrary field K, with $charK \neq 2$, is an algebra of dimension 8, denoted $\mathbb{O}(a,b,c)$, with basis $\{1, f_1, ..., f_7\}$ and multiplication given in the following table:

$\cdot$	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
1	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	f_1	α	f_3	af_2	f_5	af_4	$-f_7$	$-af_6$
f_2	f_2	$-f_3$	b	$-bf_1$	f_6	f_7	bf_4	bf_5
f_3	f_3	$-af_2$	bf_1	$-ab$	f_7	af_6	$-bf_5$	$-abf_4$
f_4	f_4	$-f_5$	$-f_6$	$-f_7$	c	$-cf_1$	$-cf_2$	$-cf_3$
f_5	f_5	$-af_4$	$-f_7$	$-af_6$	cf_1	$-ac$	cf_3	acf_2
f_6	f_6	f_7	$-bf_4$	bf_5	cf_2	$-cf_3$	$-bc$	$-bcf_1$
f_7	f_7	af_6	$-bf_5$	abf_4	cf_3	$-acf_2$	bcf_1	abc

The algebra $\mathbb{O}(a,b,c)$ is a non-commutative and a non-associative algebra, but it is *alternative*, *flexible* and *power-associative*.

If $x \in \mathbb{O}(a,b,c)$, $x = x_0 + x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5 + x_6 f_6 + x_7 f_7$, then $\overline{x} = x_0 - x_1 f_1 - x_2 f_2 - x_3 f_3 - x_4 f_4 - x_5 f_5 - x_6 f_6 - x_7 f_7$ is called the *conjugate* of the element x. For $x \in \mathbb{O}(a,b,c)$, we define the elements:

$$\mathbf{t}(x) = x + \overline{x} \in K$$

and

$$\mathbf{n}(x) = x\overline{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 - cx_4^2 + acx_5^2 + bcx_6^2 - abcx_7^2 \in K.$$

These elements are called the *trace*, respectively, the *norm* of the element $x \in \mathbb{O}(a,b,c)$. It follows that

$$(x + \overline{x})\, x = x^2 + \overline{x}x = x^2 + \mathbf{n}(x) \cdot 1$$

and

$$x^2 - \mathbf{t}(x)x + \mathbf{n}(x) = 0, \forall x \in x \in \mathbb{O}(a,b,c),$$

therefore the generalized octonion algebra is a *quadratic* algebra.

If, for $x \in \mathbb{O}(a,b,c)$, the relation $\mathbf{n}(x) = 0$ implies $x = 0$, then the algebra $\mathbb{O}(a,b,c)$ is a *division* algebra (see [3] and [4]).

If we take $a = b = c = -1$, $K = \mathbb{R}$, then we obtain $\mathbb{H}(-1,-1)$, the quaternion division algebra, usually denoted by $\mathbb{H}$ and octonion division algebra $\mathbb{O}(-1,-1,-1)$, usually denoted by $\mathbb{O}$. For example, by taking $a = -1$ and $b = 1$, $K = \mathbb{R}$, we obtain a split quaternion algebra. In the following, we will denote $\mathbb{H}_K = \left(\frac{-1,-1}{K}\right)$ and $\mathbb{O}_K = \left(\frac{-1,-1,-1}{K}\right)$. For other details regarding properties of quaternions over an arbitrary field, the reader is referred to **[4], [6], [2],** p. 431-449, etc.

We know that a finite-dimensional associative algebra A over an arbitrary field K is algebraically isomorphic to a subalgebra of a matrix algebra over the same field K. Therefore, each element $a \in A$ has a matrix representation. That means, there is a map $f : A \to \mathcal{M}_n(K)$ such that $f(x) = M_x \in \mathcal{M}_n(K)$, where $dimA = n$. For an arbitrary quaternion algebra $\mathbb{H}(a,b)$, the map

$$\varphi : \mathbb{H}(a,b) \to \mathcal{M}_4(K)$$

$$\varphi(q) = \begin{pmatrix} q_0 & aq_1 & bq_2 & -abq_3 \\ q_1 & q_0 & bq_3 & -bq_2 \\ q_2 & -aq_3 & q_0 & aq_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix}, q \in \mathbb{H}(a,b), \tag{1}$$

is called *the left representation* and the map

$$\rho : \mathbb{H}(a,b) \to \mathcal{M}_4(K)$$

$$\rho(q) = \begin{pmatrix} q_0 & aq_1 & bq_2 & -abq_3 \\ q_1 & q_0 & -bq_3 & bq_2 \\ q_2 & aq_3 & q_0 & -aq_1 \\ q_3 & q_2 & -q_1 & q_0 \end{pmatrix}, q \in \mathbb{H}(a,b), \tag{2}$$

where $q = q_0 + q_1 f_1 + q_2 f_2 + q_3 f_3$, is called *the right representation* (see [5]).

In the same paper [5], were defined, for real octonions, two representations maps, left and right representations, by using the maps φ and ρ, defined on real quaternions. These maps can be defined for all octonion algebras $\mathbb{O}(a,b,c)$ over an arbitrary field K, namely

$$\Phi : \mathbb{O}(a,b,c) \to \mathcal{M}_8(K),$$

$$\Phi(x) = \begin{pmatrix} \varphi(x') & -\rho(x'')E_4 \\ \varphi(x'')E_4 & \rho(x') \end{pmatrix},$$

the left representation, where the octonion x can be written under the form $x = x' + x'' f$, with $x', x'' \in \mathbb{H}(a,b)$, by using the Cayley-Dickson process. In the same way, we define the right representation

$$\Psi : \mathbb{O}(a,b,c) \to \mathcal{M}_8(K),$$

$$\Psi(x) = \begin{pmatrix} \rho(x') & -\varphi\left(\overline{x''}\right) \\ \varphi(x') & \rho(\overline{x'}) \end{pmatrix},$$

where $\overline{x'}, \overline{x''}$ are the conjugates of the quaternions x' and x'' and $E_4 = diag(1,-1,-1,-1)$.

For $x \in \mathbb{O}(a,b,c)$, $x = x_0 + x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5 + x_6 f_6 + x_7 f_7$ with $\{1, f_1, ..., f_7\}$ the base in $\mathbb{O}(a,b,c)$, we have

$$\Phi(x) = \begin{pmatrix} x_0 & ax_1 & bx_2 & -abx_3 & cx_4 & -acx_5 & -bcx_6 & abcx_7 \\ x_1 & x_0 & bx_3 & -bx_2 & cx_5 & -cx_4 & bcx_7 & -bcx_6 \\ x_2 & -ax_3 & x_0 & ax_1 & cx_6 & -acx_7 & -cx_4 & acx_5 \\ x_3 & -x_2 & x_1 & x_0 & cx_7 & -cx_6 & cx_5 & -cx_4 \\ x_4 & -ax_5 & -bx_6 & abx_7 & x_0 & ax_1 & bx_2 & -abx_3 \\ x_5 & -x_4 & -bx_7 & bx_6 & x_1 & x_0 & -bx_3 & bx_2 \\ x_6 & ax_7 & -x_4 & -ax_5 & x_2 & ax_3 & x_0 & -ax_1 \\ x_7 & x_6 & -x_5 & -x_4 & x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \quad (3)$$

and

$$\Psi(x) = \begin{pmatrix} x_0 & ax_1 & bx_2 & -abx_3 & cx_4 & -acx_5 & -bcx_6 & abcx_7 \\ x_1 & x_0 & -bx_3 & bx_2 & -cx_5 & cx_4 & -bcx_7 & bcx_6 \\ x_2 & ax_3 & x_0 & -ax_1 & -cx_6 & acx_7 & cx_4 & -acx_5 \\ x_3 & x_2 & -x_1 & x_0 & -cx_7 & cx_6 & -cx_5 & cx_4 \\ x_4 & ax_5 & bx_6 & -abx_7 & x_0 & -ax_1 & -bx_2 & abx_3 \\ x_5 & x_4 & bx_7 & -bx_6 & -x_1 & x_0 & bx_3 & -bx_2 \\ x_6 & -ax_7 & x_4 & ax_5 & -x_2 & -ax_3 & x_0 & ax_1 \\ x_7 & -x_6 & x_5 & x_4 & -x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \quad (4)$$

Proposition 1. ([5], Lemma 1.2) *With the above notations, for $\varepsilon \in \{\varphi, \rho\}$, we have:*
i) $\varepsilon(x+y) = \varepsilon(x) + \varepsilon(y)$,
ii) $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$,
iii) $\varepsilon(\lambda x) = \lambda\varepsilon(x), \lambda \in K, \varepsilon(1) = I_4$
iv) $\varepsilon(\overline{x}) = \varepsilon^T(x)$
v) $\varepsilon\left(x^{-1}\right) = \varepsilon^{-1}(x)$
vi) $\varepsilon(x) = \varepsilon(y)$ if and only if $x = y$, with $x, y \in \mathbb{H}$.

We remark that the above properties are proved in the real case for division algebra $\mathbb{H}$, but these are true in the general case, for an arbitrary field with characteristic different from 2. Properties i), iii), iv) and v) are also satisfied for the maps Φ and Ψ over reals. Moreover, the following properties were proved over the real field.

Proposition 2. ([5],Theorem 2.5, Theorem 2.9, Theorem 2,10, Theorem 2.11)
With the above notations, for $\varepsilon \in \{\Phi, \Psi\}$, we have
i) $\varepsilon(x+y) = \varepsilon(x) + \varepsilon(y)$,
ii) $\varepsilon\left(x^2\right) = \varepsilon(x)^2$
iii) $\varepsilon(xyx) = \varepsilon(x)\varepsilon(y)\varepsilon(x)$,
iv) $\varepsilon(\lambda x) = \lambda\varepsilon(x), \lambda \in K, \varepsilon(1) = I_8$
v) $\varepsilon(\overline{x}) = \varepsilon^T(x)$
vi) $\varepsilon(x) = \varepsilon(y)$ if and only if $x = y$, where $x, y \in \mathbb{O}$.

Properties ii) and iii) from the above proposition were proved by using alternativity and the the Moufang identities. But, since Moufang identities are true in any octonion algebra over a field of characteristic not two (see **[4]**), these properties are also true for $K = \mathbb{Z}_p$, p a prime number, $p \neq 2$.

Proposition 3. *With the notations from the above proposition, we have that*

$$\varepsilon(x^n) = \varepsilon^n(x),$$

for $x \in \mathbb{O}(a,b,c)$ over an arbitrary field K and n a positive integer.

Proof. We use induction. From condition ii) from the above proposition, taking $x = y$, we have $\varepsilon\left(x^3\right) = \varepsilon^3(x)$. From condition iii), for $y = x^2$, we have $\varepsilon\left(x^4\right) = \varepsilon^4(x)$ and so on.

In the paper [1], we studied some properties of $k-$potent elements over algebras obtained by the Cayley-Dickson process.

Definition 4.

i) The element x in the ring R is called *nilpotent* if there is a positive integer n such that $x^n = 0$. If the number n is the smallest with this property, then it is called the *nilpotency index*.

ii) The element x in the ring R is called a *k-potent* element, for $k > 1$, a positive integer, if k is the smallest number such that $x^k = x$. The number k is called the *k-potency index*. For $k = 2$, we have idempotent elements, for $k = 3$, we have tripotent elements, and so on.

Remark 5. From the above definition and Proposition 3, if $x \in \mathbb{H}(a, b)$ or $x \in \mathbb{O}(a, b, c)$ is a $k-$potent element, it results that the matrices $\varepsilon(x)$, for $\varepsilon \in \{\varphi, \rho\}$ or $\varepsilon \in \{\Phi, \Psi\}$ are $k-$potent matrices over the field K.

Example 6. i) By using some results obtained in [1], we consider quaternions over the field $K = \mathbb{Z}_5$ and the element $x = 2 + 3i + j + 3k$ which is a 5−potent element over $\mathbb{H}_{\mathbb{Z}_5}$. Indeed, $x = 2 + 3\gamma, \gamma = i + 2j + k$, with $\gamma^2 = -1$ and $\mathbf{n}_x = 1$. Therefore, $x^2 = 2\gamma$ and $x^4 = 1$. The matrices

$$\varphi(x) = \begin{pmatrix} 2 & -3 & -1 & -3 \\ 3 & 2 & -3 & 1 \\ 1 & 3 & 2 & -3 \\ 3 & -1 & 3 & 2 \end{pmatrix} \text{ and } \rho(x) = \begin{pmatrix} 2 & -3 & -1 & -3 \\ 3 & 2 & 3 & -1 \\ 1 & -3 & 2 & 3 \\ 3 & 1 & -3 & 2 \end{pmatrix}$$

are 5−potent matrices, that means $\varphi^4(z) = \rho^4(z) = I_4$.

ii) With the same arguments as above, if we consider octonions over the field $K = \mathbb{Z}_{13}$ and the element $x \in \mathbb{O}_{\mathbb{Z}_{13}}$, $x = 3 + 2f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7$, we obtain that x is a 13−potent element. The matrices

$$\Phi(x) = \begin{pmatrix} 3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 3 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 3 & -2 & -1 & -1 & 1 & 1 \\ 1 & -1 & 2 & 3 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 3 & -2 & -1 & -1 \\ 1 & -1 & 1 & -1 & 2 & 3 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 3 & 2 \\ 1 & 1 & -1 & -1 & 1 & 1 & -2 & 3 \end{pmatrix}$$

and

$$\Psi(x) = \begin{pmatrix} 3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 3 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 3 & 2 & 1 & 1 & -1 & -1 \\ 1 & 1 & -2 & 3 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 3 & 2 & 1 & 1 \\ 1 & 1 & -1 & 1 & -2 & 3 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 3 & -2 \\ 1 & -1 & 1 & 1 & -1 & -1 & 2 & 3 \end{pmatrix}$$

are also 13-potent matrices.

3. k-potents elements in quaternion algebras and octonion algebras over the real field

In the following, we will consider $\mathbb{H}$ and $\mathbb{O}$ the real division quaternion algebra and the real division octonion algebra. Let $\mathbb{A} \in \{\mathbb{H}, \mathbb{O}\}$. If $x \in \mathbb{A}$. $x \neq 0$, is a $k-$potent element, therefore $x^k = x$. Since $\mathbb{A}$ is a division algebra, we have that x is an invertible element, therefore $n_x \neq 0$ and $x^{k-1} = 1$. It is clear from here that a $k-$potent element is a solution of the equation

$$x^{k-1} = 1. \qquad (5)$$

In the following, we will provide solutions of this equation. From relation (5), we obtain that $n_x^{k-1} = 1$. Since n_x is a positive real number, we obtain that $n_x = 1$. Let $x \in \mathbb{H}$, $x = x_0 + x_1 f_1 + x_2 f_2 + x_3 f_3$, with $n_x = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, and $x_0, x_1, x_2, x_3 \in (-1, 1)$. We denote by

$$\cos\alpha = x_0, \sin\alpha = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

$$\theta = \frac{x_1 f_1 + x_2 f_2 + x_3 f_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \in \mathbb{H}, \text{ with } \theta^2 = -1.$$

and the element x can be write under the form

$$x = \cos\alpha + \theta \sin\alpha.$$

It is clear that $x^2 = (\cos\alpha + \theta\sin\alpha)^2 = \cos^2\alpha - \sin^2\theta + (2\cos\alpha\sin\alpha)\theta = \cos 2\alpha + \theta \sin 2\alpha$. By using induction, we obtain that

$$x^n = \cos n\alpha + \theta \sin n\alpha.$$

Therefore, from the above relation, the element $x \in \mathbb{H}$ satisfing condition $x^{k-1} = 1$ has the form

$$x = \cos\frac{2\pi}{k-1} + \theta \sin\frac{2\pi}{k-1}.$$

Example 7. i) We consider $x = \frac{1}{2} + \frac{1}{2} f_1 + \frac{1}{2} f_2 + \frac{1}{2} f_3 = \cos\frac{\pi}{3} + \theta\sin\frac{\pi}{3}$, where $\theta = \frac{f_1 + f_2 + f_3}{\sqrt{3}}$. We obtain that $x^6 = 1$, then $x^7 = x$ and x is a $7-$potent element. Therefore, the matrices

$$\varphi(x) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ and } \rho(x) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

are $7-$potent matrices.

ii) For $x = -\frac{1}{2} + \frac{1}{2} f_1 - \frac{1}{2} f_2 + \frac{1}{2} f_3 = \cos\frac{2\pi}{3} + \theta\sin\frac{2\pi}{3}$, where $\theta = \frac{f_1 - f_2 + f_3}{\sqrt{3}}$. We obtain that $x^3 = 1$, then $x^4 = x$ and x is $4-$potent element. The matrices

$$\varphi(q) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \text{ and } \rho(q) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

are $4-$potent matrices.

iii) For $x = \frac{1}{2} f_1 - \frac{1}{2} f_2 + \frac{\sqrt{2}}{2} f_3 = \cos\frac{\pi}{2} + \theta\sin\frac{\pi}{2}$, where $\theta = \frac{f_1 - f_2 + \sqrt{2} f_3}{2}$. We obtain that $x^4 = 1$, then $x^5 = x$ and x is $5-$potent element. The matrices

$$\varphi(q) = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \text{ and } \rho(q) = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & 0 & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

are $5-$potent matrices.

Remark 8. In the following, we consider the quaternion split algebra $\mathbb{H}(a,b)$ or the octonion split algebra $\mathbb{O}(a,b,c)$ over $K=\mathbb{R}$. The result from [1], Proposition 1 is also true for these algebras and can be proved by a straightforward calculation. Indeed, if an element $x\in\mathbb{H}(a,b)$ or $x\in\mathbb{O}(a,b,c)$ is $k-$potent with $n_x=0$ and $t_x\neq 0$, it results that: $x^2=t_xx\Rightarrow x^3=t_xx^2=t_x^2x$, therefore $x=x^k=t_x^{k-1}x$ and $t_x^{k-1}=1$, then $t_x^k=t_x$ is k-potent over $\mathbb{R}$. We get $t_x=1$ or $t_x=-1$ and $x_0=\frac{t_x}{2}$. We have that $t_x^2=t_x$ or $t_x^3=t_x$, therefore there are only idempotent and tripotent elements in split algebras $\mathbb{H}(a,b)$ or $\mathbb{O}(a,b,c)$ over $\mathbb{R}$.

Example 9.

i) We consider the split quaternion algebra $\mathbb{H}(1,1)$ and the quaternion $q=\frac{1}{2}(1+f_1+f_2+f_3)$, with $\mathbf{n}(q)=0$. We have that the matrices

$$\varphi(q)=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{pmatrix}\text{ and }\rho(q)=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\end{pmatrix},$$

are idempotent matrices. For $w=\frac{1}{2}(-1+f_1+f_2+f_3)$, the matrices

$$\varphi(w)=\begin{pmatrix}-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{pmatrix}\text{ and }\rho(w)=\begin{pmatrix}-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\end{pmatrix}$$

are tripotent matrices.

For $z=\frac{1}{2}\left(f_1+f_2+\sqrt{2}f_3\right)$, the matrices

$$\varphi(z)=\begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}&-\frac{\sqrt{2}}{2}\\ \frac{1}{2}&0&\frac{\sqrt{2}}{2}&-\frac{1}{2}\\ \frac{1}{2}&-\frac{\sqrt{2}}{2}&0&\frac{1}{2}\\ \frac{\sqrt{2}}{2}&-\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}\text{ and }\rho(z)=\begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}&-\frac{\sqrt{2}}{2}\\ \frac{1}{2}&0&-\frac{\sqrt{2}}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{\sqrt{2}}{2}&0&-\frac{1}{2}\\ \frac{\sqrt{2}}{2}&\frac{1}{2}&-\frac{1}{2}&0\end{pmatrix}$$

are nilpotent matrices with 2 as a nilpotency index.

ii) If we consider the split quaternion algebra $\mathbb{H}(2,3)$ and the quaternion $q_1=\frac{1}{2}\left(1+f_1+f_2+\frac{\sqrt{6}}{3}f_3\right)$, with $\mathbf{n}(q_1)=0$,we have that the matrices

$$\varphi(q_1)=\begin{pmatrix}\frac{1}{2}&1&\frac{3}{2}&-\sqrt{6}\\ \frac{1}{2}&\frac{1}{2}&\frac{\sqrt{6}}{2}&-\frac{3}{2}\\ \frac{1}{2}&-\frac{\sqrt{6}}{3}&\frac{1}{2}&1\\ \frac{\sqrt{6}}{6}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{pmatrix}\text{ and }\rho(q_1)=\begin{pmatrix}\frac{1}{2}&1&\frac{3}{2}&-\sqrt{6}\\ \frac{1}{2}&\frac{1}{2}&-\frac{\sqrt{6}}{2}&\frac{3}{2}\\ \frac{1}{2}&\frac{\sqrt{6}}{3}&\frac{1}{2}&-1\\ \frac{\sqrt{6}}{6}&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

are idempotent.

If we take $q_2=\frac{1}{2}\left(-1+f_1+f_2+\frac{\sqrt{6}}{3}f_3\right)$, with $\mathbf{n}(q_2)=0$,we have that the matrices

$$\varphi(q_1)=\begin{pmatrix}-\frac{1}{2}&1&\frac{3}{2}&-\sqrt{6}\\ \frac{1}{2}&-\frac{1}{2}&\frac{\sqrt{6}}{2}&-\frac{3}{2}\\ \frac{1}{2}&-\frac{\sqrt{6}}{3}&-\frac{1}{2}&1\\ \frac{\sqrt{6}}{6}&-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{pmatrix}\text{ and }\rho(q_1)=\begin{pmatrix}-\frac{1}{2}&1&\frac{3}{2}&-\sqrt{6}\\ \frac{1}{2}&-\frac{1}{2}&-\frac{\sqrt{6}}{2}&\frac{3}{2}\\ \frac{1}{2}&\frac{\sqrt{6}}{3}&-\frac{1}{2}&-1\\ \frac{\sqrt{6}}{6}&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\end{pmatrix}$$

are tripotent.

If we take $q_3 = \frac{1}{2}\left(1 + \sqrt{2}f_1 + f_2 + f_3\right)$, the matrices

$$\varphi(q_3) = \begin{pmatrix} \frac{1}{2} & \sqrt{2} & \frac{3}{2} & -3 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & \sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix} \text{and } \rho(q_3) = \begin{pmatrix} \frac{1}{2} & \sqrt{2} & \frac{3}{2} & -3 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & -\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}$$

are idempotent. For $q_4 = \frac{1}{2}\left(-1 + \sqrt{2}f_1 + f_2 + f_3\right)$, we obtain the following tripotent matrices

$$\varphi(q_4) = \begin{pmatrix} -\frac{1}{2} & \sqrt{2} & \frac{3}{2} & -3 \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -1 & -\frac{1}{2} & \sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \end{pmatrix} \text{and } \rho(q_4) = \begin{pmatrix} -\frac{1}{2} & \sqrt{2} & \frac{3}{2} & -3 \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} & -\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Conclusions. The $k-$potent matrices have many applications in manny fields of research as for example combinatorics and graph theoty, control theory, etc. For this reason, we considered that a procedure to obtain some example of $k-$potent matrices of order 4 or 8, over the real field or over the field $\mathbb{Z}_p$, with p a prime number, is very usefull. The connections with quaternions and octonins give us such examples. For a further research, we will study the possibility to obtain new procedures which allow us to obtain new classes and examples of $k-$potent matrices.

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References

[1] C. Flaut, A. Baias, *Some Remarks Regarding Special Elements in Algebras Obtained by the Cayley–Dickson Process over $\mathbb{Z}_p$*, Axioms **13** (2024), no. 6, 351.
[2] C. Flaut, Š. Hošková-Mayerová, D. Flaut, *Models and Theories in Social Systems*, Springer Nature, 2019.
[3] R. D. Schafer, *On the algebras formed by the Cayley-Dickson process*, Amer. J. Math. **76** (1954), 435–446.
[4] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.
[5] Y. Tian, *Matrix Representations of Octonions and their Applications*, 2000.
[6] J. Voight, *Quaternion Algebras*, Springer Nature Switzerland AG, 2021. ISBN: 978-3-030-56692-0.