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Some new characterizations of bi-dagger matrices

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Abstract. The concept of the bi-dagger matrix was introduced by Hartwig and Spindelböck [8]. In this paper, we provide some new characterizations of bi-dagger matrices. We prove that the index of a bi-dagger matrix is less than or equal to 2 and that a matrix is bi-dagger if and only if it is i-EP, and its index is less than or equal to 2. Specifically, a matrix is bi-dagger if and only if it commutes with its B-T inverse. Finally, we consider Problem 5 in [8] and establish conditions under which a bi-dagger matrix implies bi-normality.

1. Introduction

In this paper, we use the following notations. Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ complex matrices; I_n denote the identity matrix of order n; A^*, $\mathcal{R}(A)$ and $\mathrm{rk}(A)$ represent the conjugate transpose, range space (or column space) and rank of $A \in \mathbb{C}^{m\times n}$, respectively. For any $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ of A is the unique solution to the following Penrose equations [22]:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

Given a square matrix $A \in \mathbb{C}^{n\times n}$, the index of A (denoted by $\mathrm{Ind}(A)$) is the smallest positive integer k such that $\mathrm{rk}\left(A^{k+1}\right) = \mathrm{rk}\left(A^k\right)$. The Drazin inverse A^D of A is the unique solution to the following equations [22]:

$$(1^k)\ XA^{k+1} = A^k,\ (2)\ XAX = X,\ (5)\ XA = AX. \tag{1}$$

For the special case of $\mathrm{Ind}(A) = 1$, the unique solution of (1) is called the group inverse of A and denoted as $A^\#$.

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Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = 1$, it can be proved (see [1]) that there is a unique matrix $X \in \mathbb{C}^{n\times n}$ such that

$$AX = AA^{+},\ \mathcal{R}(X) \subseteq \mathcal{R}(A). \tag{2}$$

The matrix X is called the core inverse of A. A matrix A is said to be core invertible if there exists a matrix X that is the core inverse of A, and we denote $X = A^{\#}$.

Subsequently, the notion is extended to any square matrix. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, Baksalary and Trenkler [2] introduced the B-T inverse of A:

$$A^{\diamond} = \left(A^2A^{\dagger}\right)^{\dagger}. \tag{3}$$

Obviously, when $k = 1$, the B-T inverse coincides with the core inverse [2].

Generalized inverses are one of the main tools for studying special matrices. A matrix $A \in \mathbb{C}^{n\times n}$ is EP if $AA^{\dagger} = A^{\dagger}A$, see [17]. It is easy to check that if A is EP then $\mathrm{Ind}(A) = 1$. Various established characterizations for EP matrices and operators can be seen in [4, 11, 15, 16, 23, 24]. As extensions of EP matrices, i-EP and k-EP are introduced in [12–14, 21].

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, it is i-EP if A^k is EP, and is k-EP if $A^kA^{\dagger} = A^{\dagger}A^k$. In particular, a matrix $A \in \mathbb{C}^{n\times n}$ is bi-dagger if $\left(A^2\right)^{\dagger} = \left(A^{\dagger}\right)^2$, is star-dagger if $A^*A^{\dagger} = A^{\dagger}A^*$ and is bi-normal if $AA^*A^*A = A^*AAA^*$, see [8].

In [8], Hartwig and Spindelböck considered the relationships among bi-dagger matrix, bi-normal matrix, bi-EP matrix and star-dagger, etc. In [12], Malik et al. discussed the relationship between bi-dagger matrix and k-EP matrix. Baksalary and Trenkler [3, Theorem 3.4] obtained that A is bi-dagger if and only if it is EP, when the Moore-Penrose inverse $A^{\dagger}$ of A is idempotent. Tian [18, Theorem 4.2] provided some characterizations of bi-dagger matrices, and proved that for a matrix A with $\mathrm{Ind}(A) = 1$, A is bi-dagger if and only if it is EP. Ferreyra et al. [6, Theorem 6.6] proved that for a matrix A with $\mathrm{Ind}(A) = 2$, A is bi-dagger if and only if it is i-EP. More discussions about bi-dagger matrices can be found in [5, 9, 14].

Although the literature discussed the characterization of bi-dagger matrices for matrix indices equal to one and two, a fundamental problem remains unsolved: What is the range of indices for which a matrix can be bi-dagger? In Section 3, we determine the range of indices for the bi-dagger matrix and provide two characterizations of the bi-dagger matrix using the B-T inverse and the i-EP matrix, respectively.

The conclusions in [8] are concise in form, rich in connotation, and have a profound impact. For example, Meenakshi and Rajian [14] applied bi-dagger and star-dagger to provide a necessary and sufficient condition for the product of two positive semidefinite matrices to be normal. It is important to emphasize that Hartwig and Spindelböck listed seven open problems in [8]. This series of profound and interesting questions has garnered widespread attention.

In this paper, we focus on the fifth problem in [8]:

Problem 1.1. *When does bi-dagger imply bi-normal?*

Baksalary and Trenkler [3, Theorem 3.4] considered the problem and concluded that A is bi-dagger if and only if it is bi-normal when the Moore-Penrose inverse $A^{\dagger}$ of A is idempotent. Groß[7, Section 2, Lemma 1] proved that A is bi-dagger if and only if it is bi-normal when the index of A is one. However, as far as we understand, this problem remains not completely resolved.

We will structure the paper as follows: In Section 2, we will present some preliminary results. Section 3 will cover properties of bi-dagger matrices. In Section 4, we will address Problem 1.1. Finally, we will conclude in Section 5.

2. Preliminaries

In this paper, we let $A \in \mathbb{C}^{n\times n}$, and the singular value decomposition (for short SVD) of A be

$$A = U\begin{bmatrix}\Sigma_A & 0\\ 0 & 0\end{bmatrix}V^*, \tag{4}$$

where $U, V \in \mathbb{C}^{n\times n}$ are unitary matrices, $\Sigma_A \in \mathbb{C}^{r\times r}$ is a diagonal positive definite matrix, and the rank of A is r. Denote

$$U = \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix},\ V^* = \begin{bmatrix} V^*_{A_1} \\ V^*_{A_2} \end{bmatrix},\ \Sigma_A = \begin{bmatrix} \Sigma_{A_1} & 0 \\ 0 & \Sigma_{A_2} \end{bmatrix}, \tag{5}$$

then, by using (4) and (5), the matrix A can be represented in the form of a partitioned matrix as follows

$$A = \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix} \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^*_{A_1} \\ V^*_{A_2} \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix} \begin{bmatrix} \Sigma_{A_1} & 0 & 0 \\ 0 & \Sigma_{A_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V^*_{A_1} \\ V^*_{A_2} \end{bmatrix}, \tag{7}$$

where $U_{A_1} \in \mathbb{C}^{n\times r}$, $U_{A_2} \in \mathbb{C}^{n\times(n-r)}$, $V^*_{A_1} \in \mathbb{C}^{r\times n}$, $V^*_{A_2} \in \mathbb{C}^{(n-r)\times n}$, $\Sigma_{A_1} \in \mathbb{C}^{r_1\times r_1}$, $\Sigma_{A_2} \in \mathbb{C}^{r_2\times r_2}$, $r_1 = \mathrm{rk}\left(A^2\right)$ and $r_1 + r_2 = r$. Using (6), the reduced SVD can be represented in the form

$$A = U_{A_1}\Sigma_A V^*_{A_1}.$$

Lemma 2.1 ([10]). *Let $A, B \in \mathbb{C}^{n\times n}$. The following conditions are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$,

(2) *there exist essential reduced SVD decompositions $A = U_{A_1}\Sigma_A V^*_{A_1}$ and $B = U_{B_1}\Sigma_B V^*_{B_1}$ such that*

$$V^*_{A_1} U_{B_1} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

where Q is a unitary matrix or $V^*_{A_1} U_{B_1} = 0$.

By using (4), the Hartwig-Spindelböck decomposition [2] of a square matrix with the rank r can be represented in the form

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{9}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \mathrm{diag}\left(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t}\right)$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = I_r \tag{10}$$

and $K = U_{A_1} V^*_{A_1}$. Then, the B-T inverse of A is of the form [2]

$$A^\diamond = U \begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{11}$$

For $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, the core-EP decomposition of A is of the form, see [20]

$$A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \tag{12}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary , T is nonsingular and N is nilpotent of index k. When $\mathrm{Ind}(A) = 1$, it is obvious that $N = 0$, and the core inverse of A is of the form

$$A^{\#\!\!\!\bigcirc} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{13}$$

and the group inverse of A is of the form, see [1]

$$A^{\#} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*. \tag{14}$$

Applying the core-EP decomposition (12), Wang and Liu [21] gave a characterization of the i-EP matrix.

Lemma 2.2 ([21]). *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then A is i-EP if and only if there exists a unitary matrix U, such that*

$$A = U\begin{bmatrix} T & 0 \\ 0 & N \end{bmatrix}U^*, \tag{15}$$

where T is non-singular, and N is nilpotent with $\mathrm{Ind}(N) = k$.

Lemma 2.3 ([14]). *Let A and B be two positive semidefinite matrices (for short psd). Then the following are equivalent:*

(1) *AB is normal;*

(2) *AB is psd;*

(3) *AB is bi-dagger and star-dagger.*

3. Properties and characterizations of bi-dagger matrices

In this section, we get the range of index for a bi-dagger matrix, and give characterizations of bi-dagger matrices applying generalized inverses and special matrices.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ and $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$. Then $\mathrm{Ind}(A) \le 2$.*

Proof. Let $A \in \mathbb{C}^{n\times n}$, $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$, the SVD of A be as in (4), and U and V be partitioned as in (5). Then applying (6) and Lemma 2.1, we have

$$A = \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix}\begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_{A_1}^* U_{A_1} & V_{A_1}^* U_{A_2} \\ V_{A_2}^* U_{A_1} & V_{A_2}^* U_{A_2} \end{bmatrix}\begin{bmatrix} U_{A_1}^* \\ U_{A_2}^* \end{bmatrix}, \tag{16}$$

where $V_{A_1}^* U_{A_1}$ is of the form (8) or $V_{A_1}^* U_{A_1} = 0$.

When $V_{A_1}^* U_{A_1} = 0$, from (16), we obtain

$$\begin{aligned} A &= \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix}\begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & V_{A_1}^* U_{A_2} \\ V_{A_2}^* U_{A_1} & V_{A_2}^* U_{A_2} \end{bmatrix}\begin{bmatrix} U_{A_1}^* \\ U_{A_2}^* \end{bmatrix} \\ &= \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix}\begin{bmatrix} 0 & \Sigma_A V_{A_1}^* U_{A_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U_{A_1}^* \\ U_{A_2}^* \end{bmatrix}. \end{aligned} \tag{17}$$

By using (17), it is easy to verify that $\mathrm{Ind}(A) \le 2$.

When $V_{A_1}^* U_{A_1} \ne 0$, it has the form (8). Then we have

$$V^*U = \begin{bmatrix} V_{A_1}^* \\ V_{A_2}^* \end{bmatrix}\begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix} = \begin{bmatrix} Q & 0 & Q_1 \\ 0 & 0 & Q_2 \\ Q_3 & Q_4 & Q_5 \end{bmatrix}, \tag{18}$$

where Q is unitary.

Since V^*U is unitary, we have

$$\begin{bmatrix} Q & 0 & Q_1 \\ 0 & 0 & Q_2 \\ Q_3 & Q_4 & Q_5 \end{bmatrix}\begin{bmatrix} Q^* & 0 & Q_3^* \\ 0 & 0 & Q_4^* \\ Q_1^* & Q_2^* & Q_5^* \end{bmatrix} = I_n.$$

It follows that $QQ^* + Q_1Q_1^* = I_{r_1}$ and $Q_2Q_2^* = I_{r_2}$. Since Q is unitary, then $Q_1 = 0$ and Q_2 is full row rank. Similarly, we have $Q_3 = 0$ and Q_4 is full column rank. By using (7) and (18), it follow that

$$\begin{aligned} A &= \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix}\begin{bmatrix} \Sigma_{A_1} & 0 & 0 \\ 0 & \Sigma_{A_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & Q_2 \\ 0 & Q_4 & Q_5 \end{bmatrix}\begin{bmatrix} U_{A_1}^* \\ U_{A_2}^* \end{bmatrix} \\ &= \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix}\begin{bmatrix} \Sigma_{A_1}Q & 0 & 0 \\ 0 & 0 & \Sigma_{A_2}Q_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} U_{A_1}^* \\ U_{A_2}^* \end{bmatrix}. \end{aligned} \tag{19}$$

By using (19), it is easy to verify that $\mathrm{Ind}(A) \leq 2$. $\square$

In [6, 18], when $\mathrm{Ind}(A) \leq 2$, we see that $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$ if and only if A is i-EP. According to Lemma 2.2 and Theorem 3.1, we can obtain the following Theorem 3.2.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$, then $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$ if and only if A is i-EP and* $\mathrm{Ind}(A) \leq 2$, *i.e. there exists a unitary U such that*

$$A = U\begin{bmatrix} T & 0 \\ 0 & N \end{bmatrix}U^*, \tag{20}$$

where T is non-singular, and N is nilpotent with $\mathrm{Ind}(N) \leq 2$.

A problem can be extended from Theorem 3.2: when $k > 2$, is it true that $\left(A^k\right)^\dagger = \left(A^\dagger\right)^k$ if and only if A is i-EP and $\mathrm{Ind}(A) \leq k$? From the following examples, we see that it is not true.

Example 3.3. *Let*

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to get that $\mathrm{Ind}(A) = 3$, $A^3 = 0$ *and* $\left(A^3\right)^\dagger = 0$. *Therefore, A is i-EP. Furthermore, we have*

$$A^\dagger = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{3}{2} & \frac{3}{2} & 0 \end{bmatrix}, \left(A^\dagger\right)^3 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}.$$

Thus, $\left(A^3\right)^\dagger \neq \left(A^\dagger\right)^3$.

Example 3.4. *Let*

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that $\operatorname{Ind}(B) = 4$, $B^4 = 0$ *and* $\left(B^4\right)^\dagger = 0$. *Therefore, B is i-EP. Furthermore, we have*

$$B^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \left(B^\dagger\right)^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ -2 & 3 & -1 & 0 \end{bmatrix}.$$

Thus, $\left(B^4\right)^\dagger \neq \left(B^\dagger\right)^4$.

As known, a matrix $A \in \mathbb{C}^{n\times n}$ is EP if it commutes with its Moore-Penrose inverse and is normal if it commutes with its conjugate transpose. Therefore, we explore whether we can similarly utilize the commutative relationship of matrices to characterize the bi-dagger matrix.

In the following theorem, we will provide a characterization of bi-dagger matrices using the commutative relationship between A and $A^\diamond$.

Theorem 3.5. *Let* $A \in \mathbb{C}^{n\times n}$. *Then* $AA^\diamond = A^\diamond A$ *if and only if* $\left(A^\dagger\right)^2 = \left(A^2\right)^\dagger$.

Proof. Let the Hartwig-Spindelböck decomposition [2] of $A \in \mathbb{C}^{n\times n}$ be as in (9), and the B-T inverse of A be of the form (11). Then

$$AA^\diamond = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} (\Sigma K)(\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^*,$$
$$A^\diamond A = U\begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} (\Sigma K)^\dagger \Sigma K & (\Sigma K)^\dagger \Sigma L \\ 0 & 0 \end{bmatrix}U^*.$$

From $AA^\diamond = A^\diamond A$, it follows that

$$(\Sigma K)(\Sigma K)^\dagger - (\Sigma K)^\dagger \Sigma K = 0,\ (\Sigma K)^\dagger \Sigma L = 0. \tag{21}$$

When $\operatorname{Ind}(A) = 1$, it is obvious that ΣK is nonsingular. Since $(\Sigma K)^\dagger \Sigma L = 0$, we have $L = 0$. It follows from (9) and Lemma 2.2 that A is EP. Therefore, $\left(A^\dagger\right)^2 = \left(A^2\right)^\dagger$.

When $\operatorname{Ind}(A) = 2$, from (21) we get that ΣK is an EP matrix. According to Lemma 2.2, there exists a unitary matrix U_1, such that

$$\Sigma K = U_1\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}U_1^*, \tag{22}$$

where T_1 is invertible.

Write $\Sigma L = U_1\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, where $L_1 \in \mathbb{C}^{\operatorname{rk}(T_1)\times(n-\operatorname{rk}(A))}$. Since $(\Sigma K)^\dagger \Sigma L = U_1\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U_1^* U_1\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = U_1\begin{bmatrix} T_1^{-1}L_1 \\ 0 \end{bmatrix} = 0$, we get $L_1 = 0$. Then, by using (22) and $\Sigma L = U_1\begin{bmatrix} 0 \\ L_2 \end{bmatrix}$, the matrix A can be represented in the form of a partitioned matrix as follows

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} T_1 & 0 & 0 \\ 0 & 0 & L_2 \\ 0 & 0 & 0 \end{bmatrix}\left(U\begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}\right)^*. \tag{23}$$

Denote $\widetilde{U} = U\begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}$. By using (23), we get

$$A^2 = \widetilde{U}\begin{bmatrix} T_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\widetilde{U}^*,\ A^\dagger = \widetilde{U}\begin{bmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & L_2^\dagger & 0 \end{bmatrix}\widetilde{U}^*.$$

It follows that

$$\left(A^2\right)^\dagger = \left(A^\dagger\right)^2 = \widetilde{U}\begin{bmatrix} T_1^{-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\widetilde{U}^*.$$

On the contrary, suppose that $\left(A^\dagger\right)^2 = \left(A^2\right)^\dagger$.

When $V_{A_1}^* U_{A_1} = 0$, by applying Lemma 2.1, (3) and (17), we have $AA^\Diamond = A^\Diamond A = 0$.

When $V_{A_1}^* U_{A_1} \neq 0$, by using (11) and (19), we have

$$A^\Diamond = \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix}\begin{bmatrix} (\Sigma_{A_1}Q)^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} U_{A_1}^* \\ U_{A_2}^* \end{bmatrix}.$$

Therefore, $AA^\Diamond = A^\Diamond A$. □

By using properties of generalized inverses, we present another characterization of bi-dagger matrices in the following theorem.

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$. Then $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$ if and only if $\mathrm{Ind}(A^2) = 1$ and $\left(A^2\right)^\# = (A^2)^{\textcircled{\#}}$.*

Proof. Suppose that $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$, then A is of the form (20). We can obtain $\mathrm{Ind}(A^2) = 1$, and easily check that

$$\left(A^2\right)^\# = (A^2)^{\textcircled{\#}} = U\begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{24}$$

Conversely, let $\mathrm{Ind}\left(A^2\right) = 1$ and $\left(A^2\right)^\# = \left(A^2\right)^{\textcircled{\#}}$, then $\mathrm{rk}\left(A^2\right) = \mathrm{rk}\left(A^4\right)$. Since $\mathrm{rk}\left(A^2\right) \geq \mathrm{rk}\left(A^3\right) \geq \mathrm{rk}\left(A^4\right)$, we have $\mathrm{rk}\left(A^2\right) = \mathrm{rk}\left(A^3\right)$ i.e. $\mathrm{Ind}(A) \leq 2$. It follows from (12) that

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*,$$

where T is non-singular, and N is nilpotent with $\mathrm{Ind}(N) \leq 2$. Applying (13) and (14) gives

$$\left(A^2\right)^\# = U\begin{bmatrix} T^{-2} & T^{-4}(TS+SN) \\ 0 & 0 \end{bmatrix}U^*, \ (A^2)^{\textcircled{\#}} = U\begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Since $\left(A^2\right)^\# = (A^2)^{\textcircled{\#}}$, then $TS + SN = 0$. It follows from $N^2 = 0$ that $S = 0$. By applying Theorem 3.2, we have $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$. □

4. Conditions under which bi-dagger implies bi-normal

In Section 3, we get some characterizations of the bi-dagger matrix. Based on those results, we obtain several conditions under which bi-dagger implies bi-normal in this section.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ is a bi-dagger matrix. Then the following are equivalent:*

(1) *AA^*A^*A is normal;*

(2) *AA^*A^*A is psd;*

(3) *AA^*A^*A is star-dagger;*

(4) *AA^*A^*A is Hermitian;*

(5) *A is bi-normal;*

(6) *A^*AAA^* is star-dagger.*

Proof. Since A is bi-dagger, by applying Theorem 3.2, we obtain that

$$AA^*A^*A = U\begin{bmatrix} TT^*T^*T & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{25}$$

According to Theorem 3.2, we know that AA^*A^*A is also a bi-dagger matrix. Thus, by applying Lemma 2.3, we have that the conditions (1), (2) and (3) are equivalent.

Furthermore, let AA^*A^*A be Hermitian, then AA^*A^*A is normal. If AA^*A^*A is psd, then AA^*A^*A is Hermitian. Since the conditions (1) and (2) are equivalent, it follows that the conditions (1), (2) and (4) are equivalent.

Since A is bi-normal if and only if AA^* commutes with A^*A, then the conditions (4) and (5) are equivalent.

Since AA^*A^*A is Hermitian, we can obtain that A^*AAA^* is also Hermitian. Therefore A^*AAA^* is star-dagger if and only if AA^*A^*A is Hermitian, that is, the conditions (4) and (6) are equivalent. □

According to Theorem 3.2, we know that if A is bi-dagger then its Drazin inverse is EP. In the following theorem, we apply the properties of Drazin inverse and the core-nilpotent decomposition to give some equivalent conditions that A is bi-normal, when A is bi-dagger.

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, then the core-nilpotent decomposition of A can be represented in the form, see [22]

$$A = C_A + N_A,$$

where $C_A = A^DA^2$ is called the core part of A and N_A is nilpotent with $\mathrm{Ind}(N_A) = k$.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ is bi-dagger. Then the following are equivalent:*

(1) *A is bi-normal;*

(2) *A^D is bi-normal;*

(3) *C_A is bi-normal.*

Proof. Since A is bi-dagger, by applying Theorem 3.1, we have $\mathrm{Ind}(A) \le 2$. And let A be of the form (20).

Suppose that A is bi-normal. By using Theorem 3.2, we have

$$AA^*A^*A = U\begin{bmatrix} TT^*T^*T & 0 \\ 0 & 0 \end{bmatrix}U^* = A^*AAA^* = U\begin{bmatrix} T^*TTT^* & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{26}$$

Then $TT^*T^*T = T^*TTT^*$ and

$$T^{-1}\left(T^{-1}\right)^*\left(T^{-1}\right)^*T^{-1} = \left(T^{-1}\right)^*T^{-1}T^{-1}\left(T^{-1}\right)^*.$$

It follows that

$$\begin{aligned} A^D\left(A^D\right)^*\left(A^D\right)^*A^D &= U\begin{bmatrix} T^{-1}\left(T^{-1}\right)^*\left(T^{-1}\right)^*T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} \left(T^{-1}\right)^*T^{-1}T^{-1}\left(T^{-1}\right)^* & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= \left(A^D\right)^*A^DA^D\left(A^D\right)^*. \end{aligned}$$

Applying Theorem 4.1 gives that A^D is bi-normal. Similarly, we can prove that if A^D is bi-normal then A is bi-normal. Therefore, conditions (1) and (2) are equivalent.

Let A be bi-normal. By using (26), we have $TT^*T^*T = T^*TTT^*$. According to Theorem 3.2, we can obtain

$$C_AC_A^*C_A^*C_A = U\begin{bmatrix} TT^*T^*T & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*TTT^* & 0 \\ 0 & 0 \end{bmatrix}U^* = C_A^*C_AC_AC_A^*.$$

Then C_A is bi-normal. Similarly, we can prove that if C_A is bi-normal then A is bi-normal. Therefore, conditions (1) and (3) are equivalent. □

5. Conclusions

In this paper, we have determined the range of indices for bi-dagger matrices and established the relationship between bi-dagger matrices and i-EP matrices. Additionally, we have provided various characterizations of bi-dagger matrices by leveraging the properties and characteristics of the B-T inverse, core inverse, and group inverse. Finally, based on these findings, we address Problem 1.1 raised by Hartwig and Spindelböck [8], proposing conditions under which bi-dagger matrices imply bi-normality.

References

[1] O.M. Baksalary, G. Trenkler, *Core inverse of matrices*, Linear Multilinear Algebra **58**(6) (2010), 681–697.
[2] O.M. Baksalary, G. Trenkler, *On a generalized core inverse*, Appl. Math. Comput. **236** (2014), 450–457.
[3] O.M. Baksalary, G. Trenkler, *On matrices whose Moore-Penrose inverse is idempotent*, Linear Multilinear Algebra **70**(11) (2022), 2014–2026.
[4] O.M. Baksalary, G. Trenkler, *Characterizations of EP, normal and Hermitian matrices*, Linear Multilinear Algebra **56** (2006), 299–304.
[5] D.E. Ferreyra, E.F. Levis, G. Maharana, V. Orquera, *New characterizations of the diamond partial order*, arXiv preprint arXiv:2401.11494, 2024.
[6] D.E. Ferreyra, F.E. Levis, S.B. Malik, A.N. Priori, *On star-dagger matrices and the core-EP decomposition*, J. Comput. Appl. Math. **438** (2024), 115558.
[7] J. Groß, *On singular periodic matrices*. In: Schipp, B. Kräer, W. (eds) Statistical Inference, Econometric Analysis and Matrix Algebra. Physica-Verlag HD, 2009, pp. 325–332.
[8] R.E. Hartwig, K. Spindelbcök, *Matrices for which A^* and $A^\dagger$ commute*, Linear Multilinear Algebra **14**(3) (1984), 241–256.
[9] M. Jalaeian, M. Mohammadzadeh Karizaki, M. Hassani, *Conditions that the product of operators is an EP operator in Hilbert C^*-module*, Linear Multilinear Algebra **68**(10) (2020), 1990–2004.
[10] O. Kędzierski, *Characterization of matrices satisfying the reverse order law for the Moore-Penrose pseudoinverse*, Linear Multilinear Algebra 2025, 1-20. https://doi.org/10.1080/03081087.2025.2469158
[11] T. Li, Y. Gao, Y. Ke, *A note on EP element and Hermitian element in rings*, Filomat **39**(3) (2025), 799–807.
[12] S.B. Malik, L. Rueda, N. Thome, *The class of m-EP and m-normal matrices*, Linear Multilinear Algebra **64**(11) (2016), 2119–2132.
[13] A.R. Meenakshi, S. Krishnamoorthy, *On k-EP matrices*, Linear Algebra Appl. **269**(1-3) (1998), 219–232.
[14] A.R. Meenakshi, C. Rajian, *On a product of positive semidefinite matrices*, Linear Algebra Appl. **295**(1-3) (1999), 3–6.
[15] D. Mosić, D.S. Djordjević, *EP elements in Banach algebras*, Banach J. Math. Anal. **5**(2) (2011), 25–32.
[16] D. Mosić, D.S. Djordjević, *New characterizations of EP, generalized normal and generalized Hermitian elements in rings*, Appl. Math. Comput. **218**(12) (2012), 6702–6710.
[17] M.H. Pear, *On generalized inverses of matrices*, Math. Proc. Cambridge Philos. Soc. **62**(4) (1966), 673–677.
[18] Y. Tian, *Equalities for mixed operations of Moore-Penrose and group inverses of a matrix*, Aequationes Math. **98** (2024), 1–23.
[19] Y. Tian, H. Wang, *Characterizations of EP matrices and weighted-EP matrices*, Linear Algebra Appl. **434**(5) (2011), 1295–1318.
[20] H. Wang, *Core-EP decomposition and its applications*, Linear Algebra Appl. **508** (2016), 289–300.
[21] H. Wang, X. Liu, *The weak group matrix*, Aequationes Math. **93** (2019), 1261–1273.
[22] G. Wang, Y. Wei, S. Qiao, *Generalized Inverses: Theory and Computations*. Springer, Singapore, 2018.
[23] X. Wang, J. Wei, *EP matrix and the solution of matrix equation*, Filomat (2024), 2747–2753.
[24] S. Xu, J.L. Chen, J. Benítez, *EP elements in rings with involution*, Bull. Malays. Math. Sci. Soc. **42** (2019), 3409–3426.