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Pointwise bi-slant doubly warped product submanifolds in para-Kaehler manifolds

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Abstract. In this article, we consider pointwise slant and pointwise bi-slant submanifolds whose ambient spaces are para-Kaehler manifolds. We prove that there exist pointwise bi-slant $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ non-trivial doubly warped product type 1-2 submanifolds whose ambient spaces are para-Kaehler manifolds by constructing examples. We get a characterization and some theorems. Then, we obtain an inequality and we get some results by using the inequality.

1. Introduction

Slant submanifolds in para-Hermitian manifold were studied by P.Alegre and A.Carriazo[1]. B.-Y. Chen and O.J. Garay introduced pointwise slant submanifolds in [10]. Also, pointwise slant submanifolds were studied by F. Etayo under the name quasi-slant submanifolds [13]. Pointwise slant submanifolds of different construction on Riemannian and semi-Riemannian manifold are studied by many geometers in [3, 4, 6, 7, 21, 22].

B.A. Rozenfeld defined para-Kaehler manifolds [25]. Rozenfeld compared Kaehler definition in the complex case with Rashevskij's description and founded the analogy between para-Kaehler and Kaehler ones. Then, P.K.Rashevski studied properties of para-Kaehler manifolds in 1948 [23].

The concept of warped products emerged in the physical and mathematical subjects before 1969. For example, Kruchkovich used semi-reducible structure which is utilized for warped product in 1957[18]. It has been successfully used in general theory of relativity, string theory and black holes. On the other hand, warped product manifolds were indicated and worked by R.L. Bishop and B.O'Neill[9]. Warped product CR-submanifolds whose ambient spaces are Kaehler manifolds was studied by B.Y. Chen at the beginning of this century[11].

Later the concept of warped products has been an important topic of study in geometry [2, 5, 8, 16, 19, 24, 29, 30]. Using Chen's [10] and Sahin's [26, 27] articles, we studied in detail the doubly warped pruduct cases of pointwise bi-slant submanifolds whose ambient spaces are para-Kaehler manifolds.

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B. Sahin introduced pointwise semi-slant submanifolds whose ambient spaces are Kaehler manifolds [27]. In contrast to Kaehler manifolds not having warped products of semi-slant submanifolds [26], he showed that there do exist warped product pointwise semi-slant submanifolds and studied them in detail [27].

S. Sular and C. Özgür [28] and A. Olteanu [20] studied doubly warped product submanifolds and they obtained geometric inequality in Riemannian structures. Doubly warped products are generalization of singly warped products[31]. Non-trivial doubly warped product submanifolds are doubly warped product submanifolds's special case. Because, warping functions k_2 and k_1 are non-constant functions in the form $\mathcal{K} ={}_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$. In this study, we studied pointwise bi-slant submanifolds whose ambient spaces are para-Kaehler manifolds. Utilizing this concept, we research the geometry of non-trivial doubly warped product pointwise bi-slant submanifolds, known as doubly warped product submanifolds's special situation whose ambient spaces are para-Kaehler manifolds by constructing examples and we determine an inequality.

This article is organized as follows. In section 2, we give preliminaries and definitions utilized for this article. In section 3, we define pointwise bi-slant submanifolds of a para-Kaehler manifold and we also check their properties. In section 4, we introduce pointwise bi-slant non-trivial doubly warped product submanifolds whose ambient spaces are para-Kaehler manifolds supported with examples. In section 5, we determine an inequality for mixed totally geodesic pointwise bi-slant non-trivial doubly warped product submanifolds whose ambient spaces are para-Kaehler manifolds.

2. Preliminaries

Let $\bar{\mathcal{K}}$ be a $2\bar{\mathrm{n}}$ -dimensional semi-Riemannian structure. If it is provided with $(\mathcal{P}, \breve{g})$, that $\mathcal{P}$ is a $(1,1)$ tensor, $\breve{g}$ is to expression semi-Riemannian metric.

$$\mathcal{P}^2 \mathcal{X}_a = \mathcal{X}_a, \quad \breve{g}(P\mathcal{X}_a, P\mathcal{Y}_b) = -\breve{g}(\mathcal{X}_a, \mathcal{Y}_b) \tag{1}$$

for any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{K}}$, it is named a para-Hermitian structure. Besides that, it is called to be para-Kaehler manifold, if it satisfies $\bar{\nabla}\mathcal{P} = 0$ identically[17].

Let currently $\mathcal{K}$ be a submanifold of $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$. The Gauss and Weingarten formulas are given by

$$\bar{\nabla}_{\mathcal{X}_\mathrm{a}} \mathcal{Y}_\mathrm{b} = \nabla_{\mathcal{X}_\mathrm{a}} \mathcal{Y}_\mathrm{b} + h_1(\mathcal{X}_\mathrm{a}, \mathcal{Y}_\mathrm{b}), \tag{2}$$

$$\bar{\nabla}_{\mathcal{X}_\mathrm{a}} \mathcal{V}_c = -A_{\mathcal{V}_c} \mathcal{X}_\mathrm{a} + \nabla^{\perp}_{\mathcal{X}_\mathrm{a}} \mathcal{V}_c. \tag{3}$$

For $\mathcal{X}_\mathrm{a}, \mathcal{Y}_\mathrm{b} \in \Gamma(\mathcal{TK})$ and $\mathcal{V}_c \in \Gamma(\mathcal{TK}^{\perp})$, that h_1 is the second fundamental form of $\mathcal{K}$, $A_{\mathcal{V}_c}$ is the Weingarten tensor with respect to $\mathcal{V}_c$ and $\nabla^{\perp}$ is the normal connection. $A_{\mathcal{V}_c}$ and h_1 are related by

$$\breve{g}_1(A_{\mathcal{V}_c} \mathcal{X}_a, \mathcal{Y}_b) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{V}_c), \tag{4}$$

here $\breve{g}_1$ also denotes the induced semi-Riemannian metric on $\mathcal{K}$.

For all tangent vector field $\mathcal{X}_a$, we denote

$$\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a + S\mathcal{X}_a, \tag{5}$$

that $R\mathcal{X}_a$ is the tangential part of $\mathcal{P}\mathcal{X}_a$ and $S\mathcal{X}_a$ is the normal part of $\mathcal{P}\mathcal{X}_a$.

For all normal vector field $\mathcal{V}_c$,

$$\mathcal{P}\mathcal{V}_c = r\mathcal{V}_c + s\mathcal{V}_c, \tag{6}$$

that $r\mathcal{V}_c$ and $s\mathcal{V}_c$ are the tangential and normal vectors of $\mathcal{P}\mathcal{V}_c$, respectively. The mean curvature vector field is defined by

$$\bar{\mathrm{H}} = \frac{1}{\bar{\mathrm{n}}} trace h_1. \tag{7}$$

Definition 2.1. We call that a submanifold $\mathcal{K}$ of a para-Kaehler manifold $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$ is pointwise slant, if for all timelike or spacelike tangent vector field X_a, the ratio $\breve{g}_1(RX_a, RX_a)/\breve{g}_1(\mathcal{P}X_a, PX_a)$ is non-constant.

We see that a pointwise slant submanifold whose ambient spaces are para-Kaehler manifold is named slant, [10] if its Wirtinger function α is globally constant. We notice that all slant submanifolds are pointwise slant submanifolds.

If $\mathcal{K}$ is a para-complex (para-holomorphic) submanifold, in that case, $\mathcal{P}X_a = RX_a$ and the above ratio is equal to 1. Moreover if $\mathcal{K}$ is totaly real (anti-invariant), then $R = 0$, so $\mathcal{P}X_a = SX_a$ and the above ratio equals 0. Hence, both totally real and para-complex submanifolds are the particular situations of pointwise slant submanifolds. Neither totally real nor para-complex pointwise slant submanifold can be named a proper pointwise slant.

Definition 2.2. Let $\mathcal{K}$ be a proper pointwise slant submanifold whose ambient space is para Hermitian manifold $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$. We call that it is of
type-1 if for any spacelike or timelike vector field X_a, RX_a is timelike or spacelike and $\frac{|RX_a|}{|\mathcal{P}X_a|} > 1$.
type-2 if for any spacelike or timelike vector field X_a, RX_a is timelike or spacelike and $\frac{|RX_a|}{|\mathcal{P}X_a|} < 1$.
Similar to the method of P. Alegre and A. Carriazo used [1], the following theorem and results were obtained.

Theorem 2.3. *Let $\mathcal{K}$ be a pointwise slant submanifold whose ambient space is para-Hermitian manifold $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$. So,*
(a) *$\mathcal{K}$ is pointwise slant submanifold of type-1 if and only if for any spacelike (timelike) vector field X_a, RX_a is timelike (spacelike), also arise a function $\mu \in (1, +\infty)$. Therefore,*

$$R^2 = \mu Id. \tag{8}$$

If θ indicates the slant function of $\mathcal{K}$, $\mu = \cosh^2\theta$.
(b) *$\mathcal{K}$ is pointwise slant submanifold of type-2 if and only if for any spacelike (timelike) vector field X_a, RX_a is timelike (spacelike), also arise a function $\mu \in (0, 1)$. Therefore,*

$$R^2 = \mu Id. \tag{9}$$

If θ indicates the slant function of $\mathcal{K}$, $\mu = \cos^2\theta$.

Proof. Firstly, if $\mathcal{K}$ is the pointwise slant submanifold of type-1 for any spacelike tangent vector field X_a, RX_a is timelike and by the equation of (1), $\mathcal{P}X_a$ is too. Furthermore, they supply $|RX_a|/|\mathcal{P}X_a| > 1$. Therefore, arise the slant function θ. Because of

$$\cosh\theta = \frac{|RX_a|}{|\mathcal{P}X_a|} = \frac{\sqrt{-\breve{g}(RX_a, RX_a))}}{\sqrt{-\breve{g}(\mathcal{P}X_a, \mathcal{P}X_a)}}. \tag{10}$$

Using (1), we have

$$\breve{g}(R^2X_a, X_a) = \cosh^2\theta\breve{g}(X_a, X_a).$$

Thus, we get $R^2X_a = X_aI$. So, from (10), we get $\mu = \cosh^2\theta$.
The same method for any timelike tangent vector field $\mathcal{Z}$, if $R\mathcal{Z}$ and $\mathcal{P}\mathcal{Z}$ are spacelike, in place of (10), we get

$$\cosh\theta = \frac{|R\mathcal{Z}|}{|\mathcal{P}\mathcal{Z}|} = \frac{\sqrt{\breve{g}(R\mathcal{Z}, R\mathcal{Z}))}}{\sqrt{\breve{g}(\mathcal{P}\mathcal{Z}, \mathcal{P}\mathcal{Z})}}.$$

Because of $R^2X_a = \mu X_a$, for any spacelike and timelike X_a it further provides for lightlike vector fields and therefore we get $R^2 = \mu Id$. The converse of (a) is straightforward.
Similarly, we have (b).

Lastly, for both pointwise slant submanifolds of type-*1* and type-2, if $\mathcal{X}_a$ is spacelike, in that case, $\mathcal{P}\mathcal{X}_a$ is timelike. Thus, all pointwise slant submanifold of type-1 and type-2 should be a neutral semi-Riemann structure.
Using (1),(5),(8) and (9), we obtain

Corollary 2.4. *Let $\mathcal{K}$ be a pointwise slant submanifold of a para-Hermitian structure $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$ with the slant function θ. For any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{K})$, we obtain:*
$\mathcal{K}$ is of type-1, if and only if

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cosh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \quad \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = \sinh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{11}$$

$\mathcal{K}$ is of type-2, if and only if

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cos^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \quad \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = -\sin^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{12}$$

Using (1),(5),(6), (8) and (9), we obtain

Corollary 2.5. *Let $\mathcal{K}$ be a pointwise slant submanifold whose ambient space is para-Hermitian manifold $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$. Then, Let $\mathcal{K}$ be a pointwise slant submanifold of a para-Hermitian structure $\bar{\mathcal{K}}$. Therefore $\mathcal{K}$ is a pointwise slant submanifold of*
*(for type-*1*), if and only if*

$$rS\mathcal{X}_a = -\sinh^2\theta\mathcal{X}_a \quad \textit{and} \quad sS\mathcal{X}_a = -RS\mathcal{X}_a, \tag{13}$$

*(for type-*2*), if and only if*

$$rS\mathcal{X}_a = \sin^2\theta\mathcal{X}_a \quad \textit{and} \quad sS\mathcal{X}_a = -RS\mathcal{X}_a. \tag{14}$$

For all timelike (spacelike) vector field $\mathcal{X}_a$.

3. Pointwise bi-slant submanifolds whose ambient spaces are para-Kaehler manifolds

In this part, we introduce and study pointwise bi-slant submanifolds whose ambient spaces are para-Kaehler manifolds.

Definition 3.1. A semi-Riemannian structure $\mathcal{K}$ of a para-Hermitian manifold $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$ is named to pointwise bi-slant submanifold, if two orthogonal distributions D^{θ_1} , D^{θ_2} with $\mathcal{K}$ at the point $q \in \mathcal{K}$ arise. Therefore,
1) $\mathcal{T}\mathcal{K} = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$;
2) $\mathcal{P}\mathcal{D}^{\theta_1} \perp \mathcal{D}^{\theta_2}$ and $\mathcal{P}\mathcal{D}^{\theta_2} \perp \mathcal{D}^{\theta_1}$;
3) $\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}$ are pointwise slant distributions with slant functions θ_1^a and θ_2^b. Then, we say the corner $\{\theta_1^a, \theta_2^b\}$ of the slant functions is named the bi-slant submanifold. A pointwise bi-slant submanifold $\mathcal{K}$ is named proper if its bi-slant function satisfies $\theta_1^a, \theta_2^b \neq 0, \frac{\pi}{2}$, also θ_1^a, θ_2^b is non-constant on $\mathcal{K}$.

Let $\mathcal{K}$ be a pointwise bi-slant submanifold of a para-Hermitian structure $\bar{\mathcal{K}}$. From the above definition and (6), we get

$$T(D_{\bar{i}}) \subset \mathcal{D}_{\bar{i}}, \quad \bar{i} = 1, 2. \tag{15}$$

For any $\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{K})$ we have

$$\mathcal{X}_a = \mathcal{R}_k\mathcal{X}_a + \mathcal{R}_l\mathcal{X}_a. \tag{16}$$

Where $\mathcal{R}_k$ and $\mathcal{R}_l$ are the projections from $\mathcal{TK}$ on the $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$. For any non-null vector field $\mathcal{X}_a \in \Gamma(\mathcal{TK})$. Applying $\mathcal{R}$ to (16) and using (5), (6) we have

$$\mathcal{R}\mathcal{X}_a = \mathcal{R}\mathcal{R}_k\mathcal{X}_a + \mathcal{R}\mathcal{R}_l\mathcal{X}_a + \mathcal{S}(\mathcal{R}_k\mathcal{X}_a + \mathcal{R}_l\mathcal{X}_a). \tag{17}$$

Thus, we get

$$\mathcal{R}\mathcal{R}_k = \mathcal{R}_1, \quad \mathcal{R}\mathcal{R}_l = \mathcal{R}_2. \tag{18}$$

Using (18) and (16) in (17), we get

$$\mathcal{R}\mathcal{X}_a = \mathcal{R}_1\mathcal{X}_a + \mathcal{R}_2\mathcal{X}_a + \mathcal{S}\mathcal{X}_a. \tag{19}$$

For $\mathcal{X}_a \in (\Gamma\mathcal{K})$ and from (15),(16) we get
For type-1,

$$\mathcal{R}_i^2\mathcal{X}_a = (-\cosh^2\theta_i)\mathcal{X}_a \quad i \in 1,2 \tag{20}$$

and for type-2

$$\mathcal{R}_i^2\mathcal{X}_a = (-\cos^2\theta_i)\mathcal{X}_a \quad i \in 1,2. \tag{21}$$

Lemma 3.2. *Let $\mathcal{K}$ be a pointwise bi-slant type 1-2 submanifold whose ambient space is para-Kaehler manifold $(\bar{\mathcal{K}}, \mathcal{P}, \breve{g})$ with pointwise slant distributions $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ with distinct slant function θ_1^a , θ_2^b Suppose that $\mathcal{K}$ is one of the known two types:1,2.*
***1)** for type-1,*

$$\begin{aligned}(\sinh^2\theta_1 + \sinh^2\theta_2)\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d) &= \breve{g}(A_{\mathcal{S}R_1\mathcal{Y}_b}\mathcal{Z}_d - A_{\mathcal{S}\mathcal{Y}_b}R_2\mathcal{Z}_d, \mathcal{X}_a)\\ &+ \breve{g}(A_{\mathcal{S}R_2\mathcal{Z}_d}\mathcal{Y}_b - A_{\mathcal{S}\mathcal{Z}_d}R_1\mathcal{Y}_b, \mathcal{X}_a).\end{aligned} \tag{22}$$

***2)** for type-2,*

$$\begin{aligned}(\sinh^2\theta_2 + \sinh^2\theta_1)\breve{g}(\nabla_{\mathcal{Z}_d}\mathcal{W}_c, \mathcal{X}_a) &= \breve{g}(A_{\mathcal{S}R_1\mathcal{X}_a}\mathcal{W}_c - A_{\mathcal{S}\mathcal{X}_a}R_2\mathcal{W}_c, \mathcal{Z}_d)\\ &+ \breve{g}(A_{\mathcal{S}R_2\mathcal{W}_c}\mathcal{X}_a - A_{\mathcal{S}\mathcal{W}_c}R_1\mathcal{X}_a, \mathcal{Z}_d).\end{aligned} \tag{23}$$

$\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^{\theta_1})$, $\mathcal{Z}_d, \mathcal{W}_c \in \Gamma(\mathcal{D}^{\theta_2})$.

Proof. For any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^{\theta_1})$ and $\mathcal{Z}_d \in \Gamma(\mathcal{D}^{\theta_2})$, we get

$$\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d) = -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}_d).$$

Utilizing (5) and (6), we get

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d) &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_1\mathcal{Y}_b, \mathcal{P}\mathcal{Z}_d) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}_d)\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_1\mathcal{Y}_b, \mathcal{P}\mathcal{Z}_d) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b, \mathcal{R}_2\mathcal{Z}_d)\\ &- \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b, \mathcal{S}\mathcal{Z}_d)\\ &= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{R}_1\mathcal{Y}_b, \mathcal{Z}_d) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b, \mathcal{R}_2\mathcal{Z}_d)\\ &- \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b, \mathcal{S}\mathcal{Z}_d).\end{aligned}$$

Using (1),(3),(5),(6) and (20), we obtain

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d) &= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_1^2\mathcal{Y}_b, \mathcal{Z}_d) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{R}_1\mathcal{Y}_b, \mathcal{Z}_d) + \breve{g}(A_{\mathcal{S}\mathcal{Y}_b}\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d)\\ &+ \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Z}_d, \mathcal{S}\mathcal{Y}_b) + \cosh^2\theta_1\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d)\\ &+ 2\sinh 2\theta_1\mathcal{X}_a\breve{g}(\mathcal{X}_a, \mathcal{Y}_b) - \breve{g}(A_{\mathcal{S}\mathcal{R}_1\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}_d)\\ &+ \breve{g}(A_{\mathcal{S}\mathcal{Y}_b}\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Z}_d, \mathcal{P}\mathcal{Y}_b) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Z}_d, \mathcal{R}_1\mathcal{Y}_b).\end{aligned}$$

Using (1),(3) with symmetry of the shape operator and orthogonality of the distributions, we get

$$
\begin{aligned}
(\sinh^2\theta_1)\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}_d) &= \breve{g}(A_{\mathcal{S}\mathcal{R}_1\mathcal{Y}_b}\mathcal{Z}_d - A_{\mathcal{S}\mathcal{Y}_b}\mathcal{R}_2\mathcal{Z}_d,\mathcal{X}_a) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a} r\mathcal{S}\mathcal{Z}_d,\mathcal{Y}_b) \\
&+ \breve{g}(\bar{\nabla}_{\mathcal{X}_a} s\mathcal{S}\mathcal{Z}_d,\mathcal{Y}_b) - \breve{g}(A_{\mathcal{S}\mathcal{Z}_d}\mathcal{X}_a,\mathcal{R}_1\mathcal{Y}_b).
\end{aligned}
$$

Using (3),(13) and (14)

$$
\begin{aligned}
(\sinh^2\theta_1+\sinh^2\theta_2)\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}_d) &= \breve{g}(A_{\mathcal{S}\mathcal{R}_1\mathcal{Y}_b}\mathcal{Z}_d - A_{\mathcal{S}\mathcal{Y}_b}\mathcal{R}_2\mathcal{Z}_d,\mathcal{X}_a) \\
&+ \breve{g}(A_{\mathcal{S}\mathcal{R}_2\mathcal{Z}_d}\mathcal{Y}_b,\mathcal{X}_a) - \breve{g}(A_{\mathcal{S}\mathcal{Z}_d}\mathcal{R}_1\mathcal{Y}_b,\mathcal{X}_a)
\end{aligned}
$$

which proves Case (1). By the similary way, the proof of Case (2) is obtained.

Corollary 3.3. *Let $\mathcal{K}$ be a pointwise bi-slant type-1,2 submanifold in para-Kaehler manifold $\bar{\mathcal{K}}$ having pointwise slant distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ with distinct slant function θ_1^a and θ_2^b . Then distribution $\mathcal{D}^{\theta_1}$ defines o totally geodesic foliation if and only if*

$$
\breve{g}(A_{SR_1\mathcal{Y}_b}\mathcal{Z}_d - A_{S\mathcal{Y}_b}R_2\mathcal{Z}_d + A_{SR_2\mathcal{Z}_d}\mathcal{Y}_b - A_{S\mathcal{Z}_d}R_1\mathcal{Y}_b,\mathcal{X}_a) = 0
$$

for any $\mathcal{X}_a,\mathcal{Y}_b\in\Gamma(\mathcal{D}^{\theta_1})$ and $\mathcal{Z}_d\in\Gamma(\mathcal{D}^{\theta_2})$.

Proof. From equation (22), we get the proof.

Corollary 3.4. *Let $\mathcal{K}$ be a pointwise bi-slant type-1,2 submanifold in para-Kaehler manifold $\bar{\mathcal{K}}$ having pointwise slant distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ with distinct slant function θ_1^a and θ_2^b . Then distribution $\mathcal{D}^{\theta_2}$ defines o totally geodesic foliation if and only if*

$$
\breve{g}(A_{SR_1\mathcal{X}_a}\mathcal{W}_c - A_{S\mathcal{X}_a}R_2\mathcal{W}_c + A_{SR_2\mathcal{W}_c}\mathcal{X}_a - A_{S\mathcal{W}_c}R_2\mathcal{X}_a,\mathcal{Z}_d) = 0
$$

for any $\mathcal{X}_a\in\Gamma(\mathcal{D}^{\theta_1})$ and $\mathcal{Z}_d,\mathcal{W}_c\in\Gamma(\mathcal{D}^{\theta_2})$.

Proof. From equation (23), we have the proof.

4. Pointwise bi-slant non-trivial doubly warped product submanifolds whose ambient spaces are para-Kaehler manifolds

Let $(\mathcal{L},\bar{g}_1)$ and $(\mathcal{E},\bar{g}_2)$ be two semi-Riemannian submanifold, $k_1:\mathcal{L}\to(0,\infty)$, $k_2:\mathcal{E}\to(0,\infty)$ and $q:\mathcal{L}\times\mathcal{E}\to\mathcal{L}$, $a:\mathcal{L}\times\mathcal{E}\to\mathcal{E}$ the projection maps given by $q(z,p)=z$ and $a(z,p)=p$ for all $(z,p)\in\mathcal{L}\times\mathcal{E}$. The warped product $\mathcal{K}={}_{k_2}\mathcal{L}\times_{k_1}\mathcal{E}$ is the manifold $\mathcal{L}\times\mathcal{E}$ equipped with the semi-Riemannian constructure such that

$\breve{g}(\mathcal{X}_a,\mathcal{Y}_b)=(k_2\circ a)^2\bar{g}_1(t_*\mathcal{X}_a,t_*\mathcal{Y}_b)+(k_1\circ q)^2\bar{g}_2(t_*\mathcal{X}_a,t_*\mathcal{Y}_b)$

for every $\mathcal{X}_a$ and $\mathcal{Y}_b$ of $\mathcal{K}$ where * describes the tangent map [9]. The functions k_1,k_2 are named the warping functions of the warped product manifold. Especially, warped product manifold $\mathcal{K}$ is called to be non-trivial doubly warped product manifold, if the warping functions are non-constant.

It follows that $\mathcal{L}\times\{p_2\}$ and $\{p_1\}\times\mathcal{E}$ are totally umbilical submanifolds with closed mean curvature vector fields in $({}_{k_2}\mathcal{L}\times_{k_1}\mathcal{E},\breve{g})$ [14], where $p_1\in\mathcal{L}$ and $p_2\in\mathcal{E}$. For more details on doubly warped products, we use articles [12, 14, 31].

Remark 4.1. If we suppose
(i) both $k_1\equiv 1$ and $k_2\equiv 1$, then we get a product manifold.
(ii) either $k_1\equiv 1$ or $k_2\equiv 1$, but not both, then we get a warped product.
(iii) k_1 and k_2 are non-constant warping functions, then we obtain a non-trivial doubly warped product.

Definition 4.2. The doubly warped product of $\mathcal{K} = {}_{k_2}\mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ is named pointwise bi-slant non-trivial doubly warped product type 1-2 submanifolds of slant submanifolds $\mathcal{K}^{\theta_1}$ and $\mathcal{K}^{\theta_2}$ with distinct slant functions θ_1^a and θ_2^b, respectively, in para-Kaehler manifold $\breve{\mathcal{K}}$, where θ_1^a and θ_2^b are non-constant functions (angles), k_1 and k_2 are non-constant warping functions.

In this article, we focused on the third feature of Remark 4.1., which is doubly warped product submanifold's a important situation. Since k_1 and k_2 are non-constant warping functions and θ_1^a and θ_2^b angles are non-constant functions, we found pointwise bi-slant non-trivial doubly warped product type 1-2 submanifolds whose ambient spaces are para-Kaehler manifolds. We get very interesting and original results, theorems and examples.

The covariant derivative formulas for a non-trivial doubly warped product manifolds are expressed by:

$$\nabla_{\mathcal{X}_a}\mathcal{Y}_b = \nabla^{\theta_1}_{\mathcal{X}_a}\mathcal{Y}_b - \breve{g}(\mathcal{X}_a, \mathcal{Y}_b)\nabla(\ln k_2) \tag{24}$$

$$\nabla_{\mathcal{X}_a}\mathcal{V}_c = \nabla_{\mathcal{V}_c}\mathcal{X}_a = \mathcal{X}_a(\ln k_1)\mathcal{V}_c + \mathcal{V}_c(\ln k_2)\mathcal{X}_a \tag{25}$$

$$\nabla_{\mathcal{V}_c}\mathcal{Z}_d = \nabla^{\theta_2}_{\mathcal{V}_c}\mathcal{Z}_d - \breve{g}(\mathcal{V}_c, \mathcal{Z}_d)\nabla(\ln k_1) \tag{26}$$

where ∇ is the Levi-Civita connection on $\mathcal{K}$ and indicate by ∇^{θ_1} and ∇^{θ_2} the Levi-Civita connection of $\bar{g}_1$ and $\bar{g}_2$ respectively, for every $\mathcal{X}_a$, $\mathcal{Y}_b$ vector fields on $\mathcal{L}$ and $\mathcal{V}_c$, $\mathcal{Z}_d$ vector field on $\mathcal{E}$ [12].

Now we write an example with related to the pointwise bi-slant non-trivial doubly warped product submanifolds whose ambient spaces are para-Kaehler manifolds.

Let $\mathcal{K}$ be a semi-Riemannian submanifold of $\bar{\mathrm{R}}^{24}_{12}$ described by the immersion $\psi : \mathcal{K} \to \bar{\mathrm{R}}^{24}_{12}$ with the cartesian coordinates $(\mathtt{x}_1, ..., \mathtt{x}_{24})$ and the almost para-complex structure
$\mathcal{P}(\frac{\partial}{\partial x_{\mathtt{i}}}) = \frac{\partial}{\partial x_{\mathtt{i}+2}}$ $\breve{i} = (1, 2, 5, 6, 9, 10, 13, 14, 17, 18, 21, 22)$ and $\mathcal{P}(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_{j-2}}$ $j = (3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24)$
Let $\bar{\mathrm{R}}^{24}_{12}$ be a semi-Riemannian structure of signature
$(+, +, -, -, +, +, -, -, +, +, -, -, +, -, -, +, +, -, +, -, +, -, -, +)$ with the canonical basis $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial \mathtt{x}_{24}})$.

Example 4.3. Let $\mathcal{K}$ be described by the immersion $\bar{\psi}$ as follows

$$\begin{aligned}\bar{\psi}(m, \mathtt{n}, \mathtt{c}, \mathtt{t}) \quad = \quad & (m \sin \mathtt{c}, m \cos \mathtt{c}, n \sin c, n \cos \mathtt{c}, m \sin \mathtt{t}, m \cos t, n \sin \mathtt{t}, n \cos \mathtt{t}, \\ & x, 2m, y, 2n, \sqrt{2}t, \sqrt{2}c, c, t, \sqrt{3}c, \sqrt{3}t, x, y, mc, nc, nt, mt)\end{aligned}$$

$$\bar{\psi}_{\mathtt{m}} = \sin \mathtt{c}\frac{\partial}{\partial x_1} + \cos \mathtt{c}\frac{\partial}{\partial x_2} + \sin t\frac{\partial}{\partial x_5} + \cos t\frac{\partial}{\partial x_6} + 2\frac{\partial}{\partial x_{10}} + c\frac{\partial}{\partial x_{21}} + \mathtt{t}\frac{\partial}{\partial x_{24}}$$

$$\bar{\psi}_{\mathtt{n}} = \sin \mathtt{c}\frac{\partial}{\partial x_3} + \cos \mathtt{c}\frac{\partial}{\partial x_4} + \sin t\frac{\partial}{\partial x_7} + \cos t\frac{\partial}{\partial x_8} + 2\frac{\partial}{\partial x_{12}} + \mathtt{c}\frac{\partial}{\partial x_{22}} + t\frac{\partial}{\partial x_{23}}$$

$$\begin{aligned}\bar{\psi}_{\mathtt{c}} \quad = \quad & m \cos c\frac{\partial}{\partial x_1} - m \sin \mathtt{c}\frac{\partial}{\partial x_2} + \mathtt{n} \cos c\frac{\partial}{\partial x_3} - \mathtt{n} \sin \mathtt{c}\frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial x_{14}} + \frac{\partial}{\partial x_{15}} \\ & + \sqrt{3}\frac{\partial}{\partial x_{17}} + \mathtt{m}\frac{\partial}{\partial x_{21}} + \mathtt{n}\frac{\partial}{\partial x_{22}}\end{aligned}$$

$$\begin{aligned}\bar{\psi}_{\mathtt{t}} \quad = \quad & m \cos t\frac{\partial}{\partial x_5} - m \sin t\frac{\partial}{\partial x_6} + \mathtt{n} \cos t\frac{\partial}{\partial x_7} - \mathtt{n} \sin t\frac{\partial}{\partial x_8} + \sqrt{2}\frac{\partial}{\partial x_{13}} + \frac{\partial}{\partial x_{16}} \\ & + \sqrt{3}\frac{\partial}{\partial x_{18}} + \mathtt{n}\frac{\partial}{\partial x_{23}} + \mathtt{m}\frac{\partial}{\partial x_{24}}\end{aligned}$$

describes a pointwise bi-slant submanifold $\mathcal{K}$ with type-1,2 in $(\bar{\mathrm{R}}^{24}_{12}, \mathcal{P}, \breve{g})$ para-complex manifold with $\mu_1 = \mathcal{R}_1^2 = (\frac{6+2\mathtt{tc}}{6+\mathtt{t}^2+\mathtt{c}^2})^2$ and $\mu_2 = \mathcal{R}_2^2 = \frac{2}{(\mathtt{m}^2-\mathtt{n}^2)(\mathtt{n}^2-\mathtt{m}^2+3)}$ for $(\mathtt{m} \neq \mathtt{n})$. Actually $D^{\theta_1} = span\{\bar{\psi}_{\mathtt{m}}, \bar{\psi}_n\}$ is pointwise bi-slant distribution with θ_1 slant function and $\mathcal{D}^{\theta_2} = span\{\bar{\psi}_c, \bar{\psi}_{\mathtt{t}}\}$ is pointwise bi-slant distribution with θ_2

slant function.
It is easy to notice that D^{θ_1} and $\mathcal{D}^{\theta_2}$ distributions are integrable. The induced metric tensor $\breve{g}_{\mathcal{K}}$ on $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ is given by
$\breve{g}_{\mathcal{K}} = (6 + \mathtt{t}^2 + \mathtt{c}^2)(dm^2 - dn^2) + (2\mathtt{m}^2 - 2\mathtt{n}^2)(dc^2 + dt^2)$. Thus,
***)** for μ_1, if $2\mathtt{tc} > \mathtt{t}^2 + \mathtt{c}^2$ and for μ_2, if $0 < (\mathtt{m}^2 - \mathtt{n}^2) < 1$ or $3 > (\mathtt{m}^2 - \mathtt{n}^2) > 2$, $\mathcal{K}$ is a pointwise bi-slant non-trivial doubly warped product type-1 submanifold whose ambient space is $\bar{\mathtt{R}}^{24}_{12}$ para-Kaehler manifold with warping functions $k_2 = \sqrt{(6 + \mathtt{t}^2 + \mathtt{c}^2)}$ and $k_1 = \sqrt{(2\mathtt{m}^2 - 2\mathtt{n}^2)}$
***)** for μ_1, if $-6 < 2\mathtt{tc} < \mathtt{t}^2 + \mathtt{c}^2$ and for μ_2, if $1 < (\mathtt{m}^2 - \mathtt{n}^2) < 2$, $\mathcal{K}$ is a pointwise bi-slant non-trivial doubly warped product type-2 submanifold whose ambient space is $\bar{\mathtt{R}}^{24}_{12}$ para-Kaehler manifold with warping functions $k_2 = \sqrt{(6 + \mathtt{t}^2 + \mathtt{c}^2)}$ and $k_1 = \sqrt{(2\mathtt{m}^2 - 2\mathtt{n}^2)}$.

Lemma 4.4. *Let $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ be a pointwise bi-slant non-trivial doubly warped product type 1-2 submanifold whose ambient space is para-Kaehler manifold $\breve{\mathcal{K}}$ with distinct slant functions θ_1^a and θ_2^b. Then (for type-1)*

$$\breve{g}(h_1(\mathcal{X}_a, \mathcal{V}_c), \mathcal{S}\mathcal{R}_2\mathcal{Z}_d) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) = -\sinh 2\theta_2^b \mathcal{X}_a(\theta_2^b)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c) \tag{27}$$

$$\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{V}_c), \mathcal{S}\mathcal{Z}_d) = -2\tanh \theta_2^b \mathcal{X}_a(\theta_2^b)\breve{g}(\mathcal{R}_2\mathcal{Z}_d, \mathcal{V}_c) \tag{28}$$

$\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{K}_1)$ and $\mathcal{V}_c, \mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}_2)$

Proof. For type-1, using (1) (2), (3), (4), (5), (6) and (25), we derive

$$\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{V}_c) = \bar{g}(\nabla_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{V}_c) = (\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c). \tag{29}$$

Also, we get

$$\begin{aligned}
\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{V}_c) &= \breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{P}\mathcal{V}_c)\\
&= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Z}_d, \mathcal{P}\mathcal{V}_c)\\
&= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{S}\mathcal{V}_c) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Z}_d, \mathcal{P}\mathcal{V}_c)\\
&= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{S}\mathcal{V}_c) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}r\mathcal{S}\mathcal{Z}_d, \mathcal{V}_c)\\
&- \breve{g}(\bar{\nabla}_{\mathcal{X}_a}s\mathcal{S}\mathcal{Z}_d, \mathcal{V}_c) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{R}_2\mathcal{Z}_d, \mathcal{V}_c).
\end{aligned}$$

Using (1) (2), (3), (4), (5), (6), (13),(14), (20) and (25), we derive

$$\begin{aligned}
\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{V}_c) &= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c) + \bar{g}(h_1(\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{V}_c)\\
&- \sinh^2\theta_2^b\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{V}_c) - \sinh 2\theta_2^b\mathcal{X}_a(\theta_2^b)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\
&- \breve{g}(h_1(\mathcal{X}_a, \mathcal{V}_c), \mathcal{S}\mathcal{R}_2\mathcal{Z}_d).
\end{aligned}$$

Using (20) and (25)

$$\begin{aligned}
(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c) &= \cosh^2\theta_2^b(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\
&+ \breve{g}(h_1(\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{V}_c)\\
&- \sinh^2\theta_2^b(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\
&- \sinh 2\theta_2^b\mathcal{X}_a(\theta_2^b)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\
&- \breve{g}(h_1(\mathcal{X}_a, \mathcal{V}_c), \mathcal{S}\mathcal{R}_2\mathcal{Z}_d).
\end{aligned} \tag{30}$$

We get (27) and interchanging $\mathcal{Z}_d$ and $\mathcal{R}_2\mathcal{Z}_d$ in equation (27), we get (28)
The proof is completed. Also for type-2, we use a similar method.

Lemma 4.5. *Let $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ be a pointwise bi-slant non-trivial doubly warped product type 1-2 submanifold whose ambient space is para-Kaehler manifold $\breve{\mathcal{K}}$ with distinct slant functions θ_1^a and θ_2^b. Then (for type-1)*

$$\begin{aligned}
&\breve{g}(h_1(\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{V}_c), \mathcal{S}\mathcal{R}_2\mathcal{Z}_d) = 2\cosh^2\theta_2^b\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\
&(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)
\end{aligned} \tag{31}$$

for $\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$ *and* $\mathcal{V}_c, \mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$.

Proof. Using (1) (2), (3), (4), (5) and (6), we get

$$\begin{aligned}\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) &= \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{X}_a, \mathcal{S}\mathcal{V}_c)\\ &= \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{V}_c) - \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{X}_a, \mathcal{R}_2\mathcal{V}_c)\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{X}_a, \mathcal{V}_c) - \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{X}_a, \mathcal{R}_2\mathcal{V}_c)\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{R}_1\mathcal{X}_a, \mathcal{V}_c) - \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{S}\mathcal{X}_a, \mathcal{V}_c) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c)\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{R}_1\mathcal{X}_a}\mathcal{Z}_d, \mathcal{V}_c) - \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{S}\mathcal{X}_a, \mathcal{V}_c) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c).\end{aligned}$$

Using (25)

$$\begin{aligned}\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) &= -(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\ &+ \breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a)\\ &- (\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c).\end{aligned} \tag{32}$$

By polarization, we get

$$\begin{aligned}\breve{g}(h_1(\mathcal{X}_a, \mathcal{V}_c), \mathcal{S}\mathcal{Z}_d) &= -(\mathcal{R}_1\mathcal{X}_a(lnk_1) + (lnk_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\ &+ \breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a)\\ &+ (\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c).\end{aligned} \tag{33}$$

Substracting (33) from (32)

$$\begin{aligned}\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{V}_c), \mathcal{S}\mathcal{Z}_d) &= -2\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c)\\ &\quad(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a).\end{aligned} \tag{34}$$

Interchanging $\mathcal{Z}_d$ by $\mathcal{R}_2\mathcal{Z}_d$ in (34), We have (31)
Also for type-2, we use a similar method.

Theorem 4.6. *There exists a proper pointwise bi-slant non-trivial doubly warped product type 1-2 submanifold* $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ *of a para-Kaehler manifold* $\breve{\mathcal{K}}$ *with distinct slant functions* θ_1^a, θ_2^b, *if and only if*

$$\tanh\theta_2^b\mathcal{X}_a(\theta_2^b) \neq 0$$

for $\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$ *and* $\mathcal{V}_c, \mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$.

Proof. Using (27) and (31), we obtain

$$-2\cosh^2\theta_2^b\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a) = -\sinh 2\theta_2^b\mathcal{X}_a(\theta_2^b)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c). \tag{35}$$

Since $\mathcal{K}$ is proper $\theta_2^b \neq \frac{\pi}{2}$ and hence from (35), we get
$[(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a) - \tanh\theta_2^b\mathcal{X}_a(\theta_2^b)]\breve{g}(\mathcal{Z}_d, \mathcal{V}_c) = 0$, which implies that
$(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a) = \tanh(\theta_2^b)\mathcal{X}_a(\theta_2^b)$.
The proof is completed.

A pointwise bi-slant non-trivial doubly warped product type1-2 submanifold $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ whose ambient space is para-Kaehler manifold $\breve{\mathcal{K}}$ is mixed totaly geodesic if $h_1(\mathcal{X}_a, \mathcal{Z}_d) = 0$. For any $\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$ and $\mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$.

Theorem 4.7. *Let* $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ *be a pointwise bi-slant non-trivial doubly warped product type 1-2 submanifold whose ambient space is para-Kaehler manifold* $\breve{\mathcal{K}}$ *with distinct slant functions* θ_1^a *and* θ_2^b. $\mathcal{K}$ *is a mixed totally geodesic doubly warped product submanifold, then one of the following two situations appears:*

(i) *either* $\theta_2^b = \frac{\pi}{2}$, *i.e.,* $\mathcal{K}$ *is a doubly warped product pointwise submanifold of* ${}_{k_2}\mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\perp}$, *where* $\mathcal{K}^{\perp}$ *is a anti-invariant submanifold* $\breve{\mathcal{K}}$.
(ii) *or* $(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a) = 0$ *and* k_1, k_2 *are constant.*

Corollary 4.8. *For a proper pointwise mixed geodesic bi-slant non-trivial doubly warped product type 1-2 submanifold* $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ *whose ambient space is para-Kaehler manifold* $\breve{\mathcal{K}}$. *Then* $\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$, $\mathcal{V}_c, \mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$ *and* $(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a) = 0$. *So,* k_1 *and* k_2 *are constant.*

Proof. If $\mathcal{K}$ be mixed totally geodesic, from (35) we have
$2\cosh^2\theta_2^b \breve{g}(\mathcal{Z}_d, \mathcal{V}_c)(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a) = 0$. From which we get either $\cosh^2\theta_2^b = 0$ i.e., $\theta_2^b = \frac{\pi}{2}$ or $(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a) = 0$. İn this way, proof is completed.

Lemma 4.9. *Let* $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ *be a pointwise bi-slant non-trivial doubly warped product type 1-2 submanifold whose ambient space is para-Kaehler manifold* $\breve{\mathcal{K}}$ *with distinct slant functions* θ_1^a *and* θ_2^b. *Then we obtain*

$$\breve{g}(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{Z}_d) = -\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) \tag{36}$$

$$\begin{aligned}
&\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{V}_c)\\
&= -(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\
&+(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c)
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{R}_1\mathcal{X}_a) - \breve{g}(h_1(\mathcal{R}_1\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) =\\
&-(\cosh^2\theta_1^a\mathcal{X}_a(\ln k_1) + (\ln k_2)\cosh^2\theta_1^a\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\
&+(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c)
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\breve{g}(h_1(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c), \mathcal{S}\mathcal{X}_a) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{R}_2\mathcal{V}_c)\\
&= -(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c)\\
&+\cosh^2\theta_2^b(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{R}_1\mathcal{X}_a) - \breve{g}(h_1(\mathcal{R}_1\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{V}_c)\\
&-\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{R}_2\mathcal{V}_c) + \breve{g}(h_1(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c), \mathcal{S}\mathcal{X}_a)\\
&= (+\cosh^2\theta_2^b - \cosh^2\theta_1^a)(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c).
\end{aligned} \tag{40}$$

For any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$ *and* $\mathcal{V}_c, \mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$.

Proof. (for type-1) for any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$ and $\mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$, using (1),(2),(3) and (5) we obtain

$$\begin{aligned}
\breve{g}(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{Z}_d) &= \bar{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{S}\mathcal{Z}_d)\\
&= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}_d) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{R}_2\mathcal{Z}_d)\\
&= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{Z}_d) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{Y}_b).
\end{aligned}$$

Using (19)

$$\begin{aligned}
\breve{g}(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{Z}_d) &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_1\mathcal{Y}_b, \mathcal{Z}_d) - \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b, \mathcal{Z}_d) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{Y}_b)\\
&= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_1\mathcal{Y}_b, \mathcal{Z}_d) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) + \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d, \mathcal{Y}_b).
\end{aligned}$$

Using (25)

$$\begin{aligned}
\breve{g}(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{Z}_d) &= -(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{R}_1\mathcal{Y}_b, \mathcal{Z}_d)\\
&\quad -\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) +\\
&\quad (\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{R}_2\mathcal{Z}_d, \mathcal{Y}_b).
\end{aligned} \tag{41}$$

We get (36). Also for any $\mathcal{X}_a \in \Gamma(\mathcal{TK}^{\theta_1})$ and $\mathcal{V}_c, \mathcal{Z}_d \in \Gamma(\mathcal{TK}^{\theta_2})$

$$\begin{aligned}\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a) &= -\breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{V}_c, \mathcal{P}\mathcal{X}_a) + \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{V}_c, \mathcal{R}_1\mathcal{X}_a)\\ &= \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{V}_c, \mathcal{X}_a) - \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{R}_1\mathcal{X}_a, \mathcal{V}_c).\end{aligned}$$

Utilizing (19) and (1)

$$\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a) = \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{R}_2\mathcal{V}_c, \mathcal{X}_a) + \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{S}\mathcal{V}_c, \mathcal{X}_a) - \breve{g}(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{R}_1\mathcal{X}_a, \mathcal{V}_c). \tag{42}$$

Using (3) and (25) in (42), we get (37). The relations (38) and (39) can be derived from (37) by replacing $\mathcal{X}_a$ by $\mathcal{R}_1\mathcal{X}_a$ and $\mathcal{V}_c$ by $\mathcal{R}_2\mathcal{V}_c$, respectively. Adding (39) with (38), we get (40).
Now, interchanging $\mathcal{V}_c$ by $\mathcal{R}_2\mathcal{V}_c$ in (38), we get

$$\begin{aligned}&\breve{g}(h_1(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c), \mathcal{S}\mathcal{R}_1\mathcal{X}_a) - \breve{g}(h_1(\mathcal{R}_1\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{R}_2\mathcal{V}_c) =\\ &-\cosh^2\theta_1^a(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c)\\ &+\cosh^2\theta_2^b(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c).\end{aligned} \tag{43}$$

If we interchange $\mathcal{Z}_d$ by $\mathcal{R}_2\mathcal{Z}_d$ in (39) and (40) then, we obtain

$$\begin{aligned}&\breve{g}(h_1(\mathcal{R}_2\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) =\\ &-(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{R}_2\mathcal{Z}_d, \mathcal{V}_c)\\ &+\cosh^2\theta_2^b(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\end{aligned} \tag{44}$$

and

$$\begin{aligned}&\breve{g}(h_1(\mathcal{R}_2\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{R}_1\mathcal{X}_a) - \breve{g}(h_1(\mathcal{R}_1\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{V}_c) =\\ &-\cosh^2\theta_1^a(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{R}_2\mathcal{Z}_d, \mathcal{V}_c)\\ &+\cosh^2\theta_2^b(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c).\end{aligned} \tag{45}$$

Interchanging $\mathcal{V}_c$ by $\mathcal{R}_2\mathcal{V}_c$ in (44) and (45), we get

$$\begin{aligned}&\breve{g}(h_1(\mathcal{R}_2\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c), \mathcal{S}\mathcal{X}_a) - \breve{g}(h_1(\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{R}_2\mathcal{V}_c) =\\ &-\cosh^2\theta_2^b(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\ &+\cosh^2\theta_2^b(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c)\end{aligned} \tag{46}$$

and

$$\begin{aligned}&\breve{g}(h_1(\mathcal{R}_2\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c), \mathcal{S}\mathcal{R}_1\mathcal{X}_a) - \breve{g}(h_1(\mathcal{R}_1\mathcal{X}_a, \mathcal{R}_2\mathcal{Z}_d), \mathcal{S}\mathcal{R}_2\mathcal{V}_c) =\\ &-\cosh^2\theta_1^a\cosh^2\theta_2^b(\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\ &+\cosh^2\theta_2^b(\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c).\end{aligned} \tag{47}$$

The proof is completed. Using similar way, we get result for type-2

Hiepko's Theorem. [15] *Let $\mathcal{D}_a$ and $\mathcal{D}_b$ be two orthogonal distribution on a Riemannian structure $\mathcal{K}$. Accept that $\mathcal{D}_a$ and $\mathcal{D}_b$ are involutive. So that $\mathcal{D}_a$ is a totally geodesic foliation and $\mathcal{D}_b$ is a spherical foliation. Moreover $\mathcal{K}$ is locally isometric to a doubly warped product ${}_{k_2}\mathcal{K}^a \times_{k_1} \mathcal{K}^b$, where $\mathcal{K}^a$ and $\mathcal{K}^b$ are integral manifolds of $\mathcal{D}_a$ and $\mathcal{D}_b$.*

Now, we give a characterization with related to the pointwise bi-slant non-trivial doubly warped product type 1-2 submanifolds whose ambient spaces are para-Kaehler manifolds.

Theorem 4.10. *Let $\mathcal{K}$ be a proper pointwise bi-slant type 1-2 submanifold whose ambient space is para-Kaehler manifold $\breve{\mathcal{K}}$ with pointwise slant distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$. Later $\mathcal{K}$ is a non-trivial doubly warped product type*

1-2 submanifold of the form $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$, where $\mathcal{K}^{\theta_1}$ and $\mathcal{K}^{\theta_2}$ are pointwise slant submanifolds with distinct slant functions θ_1^a, θ_2^b. If and only if the shape operator of $\mathcal{K}$ satisfies (Type-1)

$$\mathcal{A}_{\mathcal{S}\mathcal{R}_1\mathcal{X}_a}\mathcal{Z}_d + \mathcal{A}_{\mathcal{S}\mathcal{Z}_d}\mathcal{R}_1\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{R}_2\mathcal{Z}_d}\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{X}_a}\mathcal{R}_2\mathcal{Z}_d = (\cosh^2\theta_2^b - \cosh^2\theta_1^a)(\mathcal{X}_a\breve{\gamma})\mathcal{Z}_d \tag{48}$$

where $\breve{\gamma}$ is a function on $\mathcal{K}$, so that $\mathcal{V}_c(\breve{\gamma}) = 0$, for any $\mathcal{V}_c \in \Gamma(\mathcal{D}^{\theta_2})$, for any $\mathcal{X}_a \in \Gamma(\mathcal{D}^{\theta_1})$ and $\mathcal{Z}_d \in \Gamma(\mathcal{D}^{\theta_2})$.

Proof. Let $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ be a pointwise bi-slant non-trivial doubly warped product type-1,2 submanifold whose ambient space is para-Kaehler manifold $\bar{\mathcal{K}}$.
For $\mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$ and $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$. using (4) and (36), we derive

$$\breve{g}(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) + \breve{g}(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{Z}_d) = 0$$

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{Y}_b}\mathcal{Z}_d + \mathcal{A}_{\mathcal{S}\mathcal{Z}_d}\mathcal{Y}_b, \mathcal{X}_a) = 0. \tag{49}$$

Interchanging $\mathcal{Y}_b$ by $\mathcal{R}_1\mathcal{Y}_b$ in (49), we get

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}_1\mathcal{Y}_b}\mathcal{Z}_d + \mathcal{A}_{\mathcal{S}\mathcal{Z}_d}\mathcal{R}_1\mathcal{Y}_b, \mathcal{X}_a) = 0. \tag{50}$$

Again interchanging $\mathcal{Z}_d$ by $\mathcal{R}_2\mathcal{Z}_d$, (49) yieldes

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{Y}_b}\mathcal{R}_2\mathcal{Z}_d - \mathcal{A}_{\mathcal{S}\mathcal{R}_2\mathcal{Z}_d}\mathcal{Y}_b, \mathcal{X}_a) = 0. \tag{51}$$

Substracting (51) from (50), we have

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}_1\mathcal{Y}_b}\mathcal{Z}_d + \mathcal{A}_{\mathcal{S}\mathcal{Z}_d}\mathcal{R}_1\mathcal{Y}_b - \mathcal{A}_{\mathcal{S}\mathcal{R}_2\mathcal{Z}_d}\mathcal{Y}_b - \mathcal{A}_{\mathcal{S}\mathcal{Y}_b}\mathcal{R}_2\mathcal{Z}_d, \mathcal{X}_a) = 0. \tag{52}$$

From (40) and (52), we get (48)
Conversely, Let $\mathcal{K}$ be a proper pointwise bi-slant type-1 submanifold of $\bar{\mathcal{K}}$. Later for any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^{\theta_1})$, $\mathcal{Z}_d \in \Gamma(\mathcal{D}^{\theta_2})$ and from (22), (48), we get

$$(\sinh^2\theta_1^a - \sinh^2\theta_2^b)\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d) - (\cosh^2\theta_2^b - \cosh^2\theta_1^a)(\mathcal{X}_a\gamma)\breve{g}(\mathcal{X}_a, \mathcal{Z}_d) = 0. \tag{53}$$

Because of $\theta_1^a \neq \theta_2^b$, the leaves of the distribution $\mathcal{D}^{\theta_1}$ are totally geodesic in $\mathcal{K}$. Also, for any $\mathcal{X}_a \in \Gamma(\mathcal{D}^{\theta_1})$, $\mathcal{V}_c, \mathcal{Z}_d \in \Gamma(\mathcal{D}^{\theta_2})$ and from (23) and (48), we get

$$(\sinh^2\theta_2^b - \sinh^2\theta_1^a)\breve{g}(\nabla_{\mathcal{Z}_d}\mathcal{V}_c, \mathcal{X}_a) = (\cosh^2\theta_2^b - \cosh^2\theta_1^a)(\mathcal{X}_a\gamma)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c). \tag{54}$$

Utilizing trigonometric informations on (54), we have

$$\breve{g}(\nabla_{\mathcal{Z}_d}\mathcal{V}_c, \mathcal{X}_a) = -(\mathcal{X}_a\gamma)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c). \tag{55}$$

By polarization, we find

$$\breve{g}(\nabla_{\mathcal{V}_c}\mathcal{Z}_d, \mathcal{X}_a) = -(\mathcal{X}_a\gamma)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c). \tag{56}$$

From (54) and (55), we obtain $\breve{g}([\mathcal{Z}_d, \mathcal{V}_c], \mathcal{X}_a) = 0$. the distribution $\mathcal{D}^{\theta_2}$ is integrable. We think a leaf $\mathcal{K}^{\theta_2}$ of $\mathcal{D}^{\theta_2}$ and h_2 be the second fundamental form of $\mathcal{K}^{\theta_2}$ in $\mathcal{K}$. Later from (55), we get

$$\breve{g}(h_2(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{X}_a) = \breve{g}(\nabla_{\mathcal{Z}_d}\mathcal{V}_c, \mathcal{X}_a) = -(\mathcal{X}_a\gamma)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c). \tag{57}$$

Therefore, we get $h_2(\mathcal{Z}_d, \mathcal{V}_c) = -\nabla_{\bar{\gamma}}\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)$, where $\nabla_{\bar{\gamma}}$ is the gradient of $\bar{\gamma}$, the leaf $\mathcal{K}^{\theta_2}$ is totally umbilicial in $\mathcal{K}$ with mean curvature vector $H_2 = -\nabla_{\bar{\gamma}}$. Because of $\mathcal{V}_c(\bar{\gamma}) = 0$ for any $\mathcal{V}_c \in \Gamma(\mathcal{D}^{\theta_2})$, we can easily get H_2 is parallel corresponding to the normal connection $\mathcal{D}^{\theta_2}$ of $\mathcal{K}^{\theta_2}$ in $\mathcal{K}$. Thus $\mathcal{K}^{\theta_2}$ is an extrinsic sphere in $\mathcal{K}$. From Hiepko's theorem, we deduce that $\mathcal{K}$ is a locally, doubly warped product submanifold. So, the proof is completed.

5. An optimal inequality

In this part, we establish an inequality for the squared norm of the second fundamental form of a mixed totally geodesic non-trivial doubly warped product pointwise bi-slant submanifold.

Let $\mathcal{K} = _{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ be a $(s = 2p + 2q)$-dimensional pointwise bi-slant non-trivial doubly warped product submanifold whose ambient space is $(2m)$-dimensional para-Kaehler manifold $\bar{\mathcal{K}}$. Let be a dimension $d_1 = 2p$ of $\mathcal{K}^{\theta_1}$ and a dimension $d_2 = 2q$ of $\mathcal{K}^{\theta_2}$. We take tangent spaces of $\mathcal{K}^{\theta_1}$ and $\mathcal{K}^{\theta_2}$ by $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$. We create orthonormal frames according to type-1 and type-2. Firstly for type-1, we create the local orthonormal frames of $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$, respectively. Assume that
$\{\mathtt{E}_1, .., \mathtt{E}_p, \mathtt{E}_{p+1} = \sec h\theta_1^a \mathcal{R}_1 \mathtt{E}_1, ..., \mathtt{E}_{2p} = \sec h\theta_1^a \mathcal{R}_1 \mathtt{E}_p\}$ that θ_1^a is nonconstant,
$\{\mathtt{E}_{2p+1} = \mathtt{E}_1^*, ..., \mathtt{E}_{2p+q} = \mathtt{E}_q^*, \mathtt{E}_{2p+q+1} = \mathtt{E}_{q+1}^* = \sec h\theta_2^b \mathcal{R}_2 \mathtt{E}_1^*, ..., \mathtt{E}_{2p+2q} = \mathtt{E}_{2q}^* = \sec h\theta_2^b \mathcal{R}_2 \mathtt{E}_q^*\}$ that θ_2^b is non-constant.
At the moment, we will give orthonormal frames of the local orthonormal frames of $\mathcal{SD}^{\theta_1}$, $\mathcal{SD}^{\theta_2}$. This frames respectively are
$\{\mathtt{E}_{s+1} = \hat{\mathtt{E}}_1 = \csc h\theta_1^a \mathcal{S}\mathtt{E}_1, ..., \mathtt{E}_{n+p} = \hat{\mathtt{E}}_p = \csc h\theta_1^a \mathcal{S}\mathtt{E}_p, \mathtt{E}_{n+p+1} = \hat{\mathtt{E}}_{p+1} = \csc h\theta_1^a$
$sech\theta_1^a \mathcal{S}\mathcal{R}_1 \mathtt{E}_1, ..., \mathtt{E}_{n+2p} = \hat{\mathtt{E}}_{n+2p} = \csc h\theta_1^a \sec h\theta_1^a \mathcal{S}\mathcal{R}_1 \mathtt{E}_p\}$,
$\{\mathtt{E}_{s+2p+1} = \bar{\mathtt{E}}_1 = \csc h\theta_2^b \mathcal{S}\mathtt{E}_1^*, ..., \mathtt{E}_{s+2p+q} = \bar{\mathtt{E}}_q = \csc h\theta_2^b \mathcal{S}\mathtt{E}_q^*, \mathtt{E}_{s+2p+q+1} = \bar{\mathtt{E}}_{q+1} = \csc h\theta_2^b \mathcal{S}\mathcal{R}_2 \mathtt{E}_1^*, ..., \mathtt{E}_{2s} = \bar{\mathtt{E}}_{2q} = \csc h\theta_2^b \sec h\theta_2^b \mathcal{S}\mathcal{R}_2 \mathtt{E}_1^*\}$.
Lets assume that
* on $\mathcal{D}^{\theta_1}$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,p}$, where $p = dim(\mathcal{D}^{\theta_1})$; also, supposed that $\breve{g}(\mathtt{E}_v, \mathtt{E}_v) = 1$,
* on $\mathcal{D}^{\theta_2}$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,q}$, where $q = dim(\mathcal{D}^{\theta_2})$ also $\breve{g}(\hat{\mathtt{E}}_w, \hat{\mathtt{E}}_w) = \mp 1$,
* on $\mathcal{SD}^{\theta_1}$: orthonormal basis $\{S\mathtt{E}_v\}_{v=1,...,p}$, where $p = dim(\mathcal{SD}^{\theta_1})$ also $\breve{g}(\mathcal{S}\mathtt{E}_v, \mathcal{S}\mathtt{E}_v) = -1$,
* on $\mathcal{SD}^{\theta_2}$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,q}$, where $q = dim(\mathcal{SD}^{\theta_2})$ also $\breve{g}(\bar{\mathtt{E}}_w, \bar{\mathtt{E}}_w) = \mp 1$.

Theorem 5.1. *Let $\mathcal{K} = _{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ be an s-dimensional mixed totally geodesic pointwise bi-slant non-trivial doubly warped product submanifold whose ambient space is $(2m)$- dimensional para-Kaehler manifold $\bar{\mathcal{K}}$.*
where $\mathcal{K}^{\theta_1}$, $\mathcal{K}^{\theta_2}$ are proper pointwise slant submanifolds with θ_1^a and θ_2^b are slant angles in $\mathcal{K}$. Also, $\mathcal{K}^{\theta_1}$, $\mathcal{K}^{\theta_2}$ are spacelike. Then (for type-1), we get
***1)** The squared norm of the second fundamental form h_1 of $\mathcal{K}$ supplies*

$$\begin{aligned} \|h_1\|^2 &\leq 2q \csc h^2 \theta_1^a (\cosh^2 \theta_1^a + \cosh^2 \theta_2^b)\{\|\nabla(\ln k_1 + \ln k_2)\|^2 \\ &- \sum_{r=1}^{p}((e_r \ln k_1)^2 + (e_r \ln k_2)^2)\} \end{aligned} \tag{58}$$

where $\nabla(\ln k_1 + \ln k_2)$ defines the gradient of $(\ln k_1 + \ln k_2)$ along $\mathcal{K}^{\theta_2}$, $\mathcal{K}^{\theta_1}$ and θ_1^a , θ_2^b are pointwise slant angles of $\mathcal{K}^{\theta_1}$ and $\mathcal{K}^{\theta_2}$, respectively.
***2)** If the equality sign of (58) holds the same way, then $\mathcal{K}^{\theta_1}$ is totally geodesic and $\mathcal{K}^{\theta_2}$ is totally umbilical in $\bar{\mathcal{K}}$.*

Proof. From $\|h_1\|^2 = \|h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1})\|^2 + 2\|h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})\|^2 + \|h_1(\mathcal{D}^{\theta_2}, \mathcal{D}^{\theta_2})\|^2$. Because of $\mathcal{K}$ is mixed totally geodesic, the middle term of the right-hand side should be zero. In that case, we get

$$\|h_1\|^2 = \sum_{v,w=1}^{s} \breve{g}(h_1(\mathtt{E}_v, \mathtt{E}_w), (h_1(\mathtt{E}_v, \mathtt{E}_w)) = \sum_{r=s+1}^{2m} \sum_{v,w=1}^{2p+2q} \breve{g}(h_1(\mathtt{E}_v, \mathtt{E}_w), \mathtt{E}_r)^2.$$

Now, we use the frames of $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ in above equation, as follows

$$\begin{aligned} \|h_1\|^2 &= \sum_{r=s+1}^{2m} \sum_{v,w=1}^{2p} \breve{g}(h_1(\mathtt{E}_v, \mathtt{E}_w), \mathtt{E}_r)^2 + \sum_{r=s+1}^{2m} \sum_{v=1}^{2p} \sum_{w=1}^{2q} \breve{g}(h_1(\mathtt{E}_v, \mathtt{E}_w), \mathtt{E}_r)^2 \\ &+ \sum_{r=s+1}^{2m} \sum_{v,w=1}^{2q} \breve{g}(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \mathtt{E}_r)^2. \end{aligned} \tag{59}$$

Because of $\mathcal{K}$ is mixed totally geodesic, the second term in the right hand side of (59) becomes zero and this equation can be seperated for the frames of $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ components, as follows

$$\begin{aligned}\|h_1\|^2 &= \sum_{r=s+1}^{n+2q}\sum_{v,w=1}^{2p}\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}_w),\mathtt{E}_r)^2+\sum_{r=s+2q+1}^{2m}\sum_{v,w=1}^{2p}\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}_w),\mathtt{E}_r)^2\\&+\sum_{r=s+1}^{s+2p}\sum_{v,w=1}^{2q}\breve{g}(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\mathtt{E}_r)^2+\sum_{r=s+2p+1}^{2m}\sum_{v,w=1}^{2q}\breve{g}(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\mathtt{E}_r)^2. \qquad (60)\end{aligned}$$

Next by removing the frames $S\mathcal{D}^{\theta_1}$, $S\mathcal{D}^{\theta_2}$ components in (60), we get

$$\begin{aligned}\|h_1\|^2 \;\geq\; &\sum_{\mathtt{r}=1}^{2\mathtt{q}}\sum_{\mathtt{v,w}=1}^{2\mathtt{p}}\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}_w),\bar{\mathtt{E}}_r)^2+\sum_{\mathtt{r}=1}^{2\mathtt{p}}\sum_{\mathtt{v,w}=1}^{2\mathtt{p}}\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}_w),\hat{\mathtt{E}}_r)^2\\+\;&\sum_{r=1}^{2p}\sum_{v,w=1}^{2q}\breve{g}(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\bar{\mathtt{E}}_r)^2+\sum_{r=1}^{2q}\sum_{v,w=1}^{2q}\breve{g}(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\hat{\mathtt{E}}_r)^2. \qquad (61)\end{aligned}$$

Because of we could not find any relation for $\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}_w),\hat{\mathtt{E}}_r)$ for any $v,w = 1,2,...,2p$ and $r = 1,2,...,2p$, $\breve{g}(h_1(E^*_v,E^*_w),\bar{\mathtt{E}}_r)$ for any $v,w,\mathtt{r} = 1,2,...,2\mathtt{q}$, we leave the second and fourth terms. So, we have

$$\|h_1\|^2 \geq \sum_{r=1}^{2\mathtt{q}}\sum_{v,w=1}^{2p}\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}_w),\bar{\mathtt{E}}_r)^2+\sum_{\mathtt{r}=1}^{2\mathtt{p}}\sum_{v,w=1}^{2q}\breve{g}(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\hat{\mathtt{E}}_r)^2. \qquad (62)$$

Because of (36), (62)

$$\|h_1\|^2 \geq \sum_{\mathtt{r}=1}^{2q}\sum_{v,w=1}^{2p}\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}^*_w),\bar{\mathtt{E}}_r)^2+\sum_{r=1}^{2p}\sum_{v,w=1}^{2q}\breve{g}(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\hat{\mathtt{E}}_r)^2. \qquad (63)$$

Because of $\mathcal{K}$ is mixed totally geodesic, we get

$$\breve{g}(h_1(\mathtt{E}_v,\mathtt{E}^*_w),\mathtt{E}_r) = 0. \qquad (64)$$

For every $v,w = 1,...,2p, r = s+1,...,2q$. By virtue of (64), we get from (63) that

$$\|h_1\|^2 \geq \sum_{r=1}^{2p}\sum_{v,w=1}^{2q}\breve{g}(h_1(\mathtt{E}^*_v,\mathtt{E}^*_w),\hat{\mathtt{E}}_r)^2. \qquad (65)$$

Thus by utilizing the orthonormal frame fields of $\mathcal{S}D^{\theta_1}$, SD^{θ_2}, we have

$$
\begin{aligned}
\|h_1\|^2 \geq\ & \csc h^2\theta_1^a \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(E_v^*, E_w^*), \mathcal{S}\mathrm{E}_r)^2 \\
+\ & \csc h^2\theta_1^a \sec h^2\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(\mathcal{R}_2\mathrm{E}_v^*, \mathrm{E}_w^*), \mathcal{S}\mathrm{E}_r)^2 \\
+\ & \csc h^2\theta_1^a \sec h^2\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(\mathrm{E}_v^*, \mathcal{R}_2\mathrm{E}_w^*), \mathcal{S}\mathrm{E}_r)^2 \\
+\ & \csc h^2\theta_1^a \sec h^4\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(\mathcal{R}_2\mathrm{E}_v^*, \mathcal{R}_2\mathrm{E}_w^*), \mathcal{S}\mathrm{E}_r)^2 \\
+\ & \csc h^2\theta_1^a \sec h^2\theta_1^a \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(\mathrm{E}_v^*, \mathrm{E}_w^*), \mathcal{S}\mathcal{R}_1\mathrm{E}_r)^2 \\
+\ & \csc h^2\theta_1^a \sec h^2\theta_1^a \sec h^2\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(\mathcal{R}_2\mathrm{E}_v^*, \mathrm{E}_w^*), \mathcal{S}\mathcal{R}_1\mathrm{E}_r)^2 \\
+\ & \csc h^2\theta_1^a \sec h^2\theta_1^a \sec h^2\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(\mathrm{E}_v^*, \mathcal{R}_2\mathrm{E}_w^*), \mathcal{S}\mathcal{R}_1\mathrm{E}_r)^2 \\
+\ & \csc h^2\theta_1^a \sec h^2\theta_1^a \sec h^4\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} \breve{g}(h_1(\mathcal{R}_2\mathrm{E}_v^*, \mathcal{R}_2\mathrm{E}_w^*), \mathcal{S}\mathcal{R}_1\mathrm{E}_r)^2.
\end{aligned}
$$

Using (37)-(39),(43)-(47) and (64) in the above equation, we have

$$
\begin{aligned}
&\|h_1\|^2 \geq \csc h^2\theta_1^a \sum_{r=1}^{p}\sum_{v,w=1}^{q} (\mathcal{R}_1\mathrm{E}_r(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathrm{E}_r)^2 \breve{g}(\mathrm{E}_v^*, \mathrm{E}_w^*)^2 \\
&+ \csc h^2\theta_1^a \sum_{r=1}^{p}\sum_{v,w=1}^{q} (\mathcal{R}_1\mathrm{E}_r(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathrm{E}_r)^2 \breve{g}(\mathrm{E}_v^*, \mathrm{E}_w^*)^2 \\
&+ \csc h^2\theta_1^a \sec h^2\theta_1^a \cosh^2\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} (\mathcal{R}_1\mathrm{E}_r(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathrm{E}_r)^2 \breve{g}(\mathrm{E}_v^*, \mathrm{E}_w^*)^2 \\
&+ \csc h^2\theta_1^a \sec h^2\theta_1^a \cos h^2\theta_2^b \sum_{r=1}^{p}\sum_{v,w=1}^{q} (\mathcal{R}_1\mathrm{E}_r(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathrm{E}_r)^2 \breve{g}(\mathrm{E}_v^*, \mathrm{E}_w^*)^2 \\
&= 2q \csc h^2\theta_1^a [1 + \sec h^2\theta_1^a \cosh^2\theta_2^b] \sum_{r=1}^{p} (\mathcal{R}_1\mathrm{E}_r(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathrm{E}_r)^2. \qquad (66)
\end{aligned}
$$

At the moment

$$
\begin{aligned}
&\|\nabla(\ln k_1 + \ln k_2)\|^2 = \sum_{r=1}^{2p} (\mathrm{E}_r \ln k_1)^2 + \sum_{r=1}^{2p} (\mathrm{E}_r \ln k_2)^2 \\
&= \sum_{r=1}^{p} (\mathrm{E}_r \ln k_1)^2 + \sum_{r=1}^{p} (\sec h\theta_1^a \mathcal{R}_1\mathrm{E}_r \ln k_1)^2 + \sum_{r=1}^{p} (\mathrm{E}_r \ln k_2)^2 + \sum_{r=1}^{p} (\sec h\theta_1^a \mathcal{R}_1\mathrm{E}_r \ln k_2)^2 \\
&= \sum_{r=1}^{p} (\mathrm{E}_r \ln k_1)^2 + (\mathrm{E}_r \ln k_2)^2 + \sec h^2\theta_1^a \sum_{r=1}^{p} (\mathcal{R}_1\mathrm{E}_r \ln k_1)^2 + (\mathcal{R}_1\mathrm{E}_r \ln k_2)^2
\end{aligned}
$$

From the above equation, we derive

$$\sum_{r=1}^{p}(\mathcal{R}_1\mathrm{E}_r \ln k_1 + \mathcal{R}_1\mathrm{E}_r \ln k_2)^2 = \cosh^2\theta_1^a(\|\nabla \ln k_1 + \ln k_2\|^2 - \sum_{r=1}^{p}(\mathrm{E}_r \ln k_1)^2 + (\mathrm{E}_r \ln k_2)^2). \tag{67}$$

Using (67) in (66), we have (58).

$$h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1}) \subset S\mathcal{D}^{\theta_1} \oplus S\mathcal{D}^{\theta_2} \tag{68}$$

and from the leaving second term in (61), we have $\breve{g}_1(h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1}), \mathcal{S}\mathcal{D}^{\theta_1}) = 0$ which implies that $h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1}) \perp \mathcal{S}\mathcal{D}^{\theta_1}$, i.e.,

$$h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1}) \subset S\mathcal{D}^{\theta_2}. \tag{69}$$

Also from (36) and (64), we find $h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1}) \perp S\mathcal{D}^{\theta_2}$, i.e.,

$$h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1}) \subset S\mathcal{D}^{\theta_1}. \tag{70}$$

From (68)-(70), we have that

$$h_1(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1}) = 0. \tag{71}$$

Since $\mathcal{K}^{\theta_1}$ is totally geodesic in $\mathcal{K}$ [9], from (71), we find that $\mathcal{K}^{\theta_1}$ is totally geodesic in $\bar{\mathcal{K}}$.

$$h_1(\mathcal{D}^{\theta_2}, \mathcal{D}^{\theta_2}) \subset S\mathcal{D}^{\theta_2} \tag{72}$$

Because of leaving fourth term in (61), we get $\breve{g}_1(h_1(\mathcal{D}^{\theta_2}, \mathcal{D}^{\theta_2}), \mathcal{S}\mathcal{D}^{\theta_2}) = 0$ which implies that $h_1(\mathcal{D}^{\theta_2}, \mathcal{D}^{\theta_2}) \perp \mathcal{S}\mathcal{D}^{\theta_2}$, i.e.,

$$h_1(\mathcal{D}^{\theta_2}, \mathcal{D}^{\theta_2}) \subset S\mathcal{D}^{\theta_1} \tag{73}$$

Moreover, utilizing (64) in (71), we find

$$\begin{aligned}\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a) &= (\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\ &+ (\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{R}_2\mathcal{V}_c).\end{aligned} \tag{74}$$

For any $\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_1})$ and $\mathcal{Z}_d, \mathcal{V}_c \in \Gamma(\mathcal{T}\mathcal{K}^{\theta_2})$. By polarization of (74), we get

$$\begin{aligned}\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a) &= (\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{R}_1\mathcal{X}_a)\breve{g}(\mathcal{Z}_d, \mathcal{V}_c)\\ &+ (\mathcal{X}_a(\ln k_1) + (\ln k_2)\mathcal{X}_a)\breve{g}(\mathcal{V}_c, \mathcal{R}_2\mathcal{Z}_d).\end{aligned} \tag{75}$$

Substracting (75)from (74), we have

$$\breve{g}(h_1(\mathcal{Z}_d, \mathcal{V}_c), \mathcal{S}\mathcal{X}_a) = (\mathcal{R}_1\mathcal{X}_a(\ln k_1) + (\ln k_2\mathcal{R}_1\mathcal{X}_a))\breve{g}(\mathcal{Z}_d, \mathcal{V}_c). \tag{76}$$

From (73) , (76) and the fact that $\mathcal{K}^{\theta_2}$ is totally umbilical in $\mathcal{K}$ [9], we find that $\mathcal{K}^{\theta_2}$ is totally umbilical in $\bar{\mathcal{K}}$. The proof is completed.

Remark 5.2. *If $\mathcal{K}^{\theta_1}, \mathcal{K}^{\theta_2}$ manifolds of above theorem is timelike, equation (58) should be modified by*

$$\begin{aligned}\|h_1\|^2 \geq\ & 2q \csc h^2\theta_1^a(\cosh^2\theta_1^a + \cosh^2\theta_2^b)\{\|\nabla(\ln k_1 + \ln k_2)\|^2\\ &- \sum_{r=1}^{p}((e_r \ln k_1)^2 + (e_r \ln k_2)^2)\}.\end{aligned} \tag{77}$$

Similarly, for proper pointwise slant submanifolds $\mathcal{K}^{\theta_1}$, $\mathcal{K}^{\theta_2}$ (type-2), we achieve

Theorem 5.3. *Let $\mathcal{K} =_{k_2} \mathcal{K}^{\theta_1} \times_{k_1} \mathcal{K}^{\theta_2}$ be an s-dimensional mixed totally geodesic pointwise bi-slant non-trivial doubly warped product submanifold whose ambient space is $(2m)$ dimensional para-Kaehler manifold $\bar{\mathcal{K}}$. where, $\mathcal{K}^{\theta_1}$, $\mathcal{K}^{\theta_2}$ are pointwise slant submanifolds with θ_1^a and θ_2^b are slant angles in $\mathcal{K}$. Also, $\mathcal{K}^{\theta_1}$, $\mathcal{K}^{\theta_2}$ are spacelike and timelike, respectively. Then, (for type-2) The squared norm of the second fundamental form of $\mathcal{N}_x$ supplies:*

$$\begin{aligned} \|h_1\|^2 &\leq 2q\csc^2\theta_1^a(\cos^2\theta_1^a + \cos^2\theta_2^b)\{\|\nabla(\ln k_1 + \ln k_2)\|^2 \\ &- \sum_{r=1}^{p}((e_r \ln k_1)^2 + (e_r \ln k_2)^2)\} \end{aligned} \tag{78}$$

(respectivelly

$$\begin{aligned} \|h_1\|^2 &\geq 2q\csc^2\theta_1^a(\cos^2\theta_1^a + \cos^2\theta_2^b)\{\|\nabla(\ln k_1 + \ln k_2)\|^2 \\ &- \sum_{r=1}^{p}((e_r \ln k_1)^2 + (e_r \ln k_2)^2)\}). \end{aligned} \tag{79}$$

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