Filomat 39:13 (2025), 4477–4487
https://doi.org/10.2298/FIL2513477S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a class of concrete general system of difference equations

Mesure Şen[a], Yasin Yazlik[b], Merve Kara[c,*]

[a] *Instutite of Science, Nevşehir Hacı Bektaş Veli University, 50300, Nevşehir, Turkey*
[b] *Department of Mathematics, Faculty of Arts and Sciences, Nevşehir Hacı Bektaş Veli University, 50300, Nevşehir, Turkey*
[c] *Department of Mathematics, Kamil Özdağ Science Faculty, Karamanoğlu Mehmetbey University, 70200 Karaman, Turkey*

Abstract. It is demonstrated that the following class of difference equations system

$$\begin{cases} u_{n+1} = \Phi^{-1}\left(\Psi\left(v_n\right)\frac{a_1\Phi(u_n)+b_1\Psi(v_{n-1})}{c_1\Phi(u_n)+d_1\Psi(v_{n-1})}\right), \\ v_{n+1} = \Psi^{-1}\left(\Phi\left(u_n\right)\frac{a_2\Psi(v_n)+b_2\Phi(u_{n-1})}{c_2\Psi(v_n)+d_2\Phi(u_{n-1})}\right), \end{cases} \quad n \in \mathbb{N}_0,$$

where the initial conditions u_{-i}, v_{-i}, for $i \in \{0,1\}$, are real numbers, the parameters $c_j^2 + d_j^2 \neq 0$, a_j, b_j, c_j, d_j, for $j \in \{1,2\}$, are real numbers, Φ and Ψ are continuous and strictly monotone functions such that $\Phi(\mathbb{R}) = \mathbb{R}$, $\Psi(\mathbb{R}) = \mathbb{R}$, $\Phi(0) = 0$, $\Psi(0) = 0$, can be solved in all cases.

1. Introduction

First of all, recall the notation $\eta = \overline{\zeta, \xi}$ means that $\{\eta \in \mathbb{Z} : \zeta \leq \eta \leq \xi\}$ if $\zeta, \xi \in \mathbb{Z}$, $\zeta \leq \xi$. Further, the set of natural, nonnegative integer, integer and real number are indicated by the notation of $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, respectively.

It is important to know if exist solutions of system of difference equations. Firstly, to give the solutions of system of difference equations, the type of difference equation system must be determined such as linear, non-linear, Riccati, exponential and fuzzy (see e.g. [9, 19, 20] and reference therein). After determining the type of difference equations system, the method to be used must be stated. For example, one of the methods that can be used to solve system of non-linear difference equations and non-linear difference equations, is the change of variables. There are some authors, who use this method [1, 10–17, 23–29] in literature. But, some authors still solve systems of non-linear difference equations, by induction [4–8].

Recently, the following difference equation

$$x_{n+1} = g^{-1}\left(g\left(x_n\right)\frac{\alpha g\left(x_n\right) + \beta g\left(x_{n-1}\right)}{\gamma g\left(x_n\right) + \delta g\left(x_{n-1}\right)}\right),\ n \in \mathbb{N}_0, \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 39A10; Secondary 39A20, 39A23.
Keywords. Difference equations system, solvable in closed-form.
Received: 17 December 2024; Revised: 11 February 2025; Accepted: 22 February 2025
Communicated by Maria Alessandra Ragusa
* Corresponding author: Merve Kara
Email addresses: 1995mesuresen@gmail.com (Mesure Şen), yyazlik@nevsehir.edu.tr (Yasin Yazlik), mervekara@kmu.edu.tr (Merve Kara)
ORCID iDs: https://orcid.org/0009-0006-2463-5256 (Mesure Şen), https://orcid.org/0000-0001-6369-540X (Yasin Yazlik), https://orcid.org/0000-0001-8081-0254 (Merve Kara)

where the parameters $\alpha, \beta, \gamma, \delta$ and the initial values x_{-s}, for $s \in \{0, 1\}$ are real numbers, $\gamma^2 + \delta^2 \neq 0$, g is a strictly monotone and continuous function, $g(\mathbb{R}) = \mathbb{R}$, $g(0) = 0$ is solved by using transformation in [21].

Lately, Stević et al., investigate the following difference equations

$$x_{n+1} = \Phi^{-1}\left(\Phi(x_{n-1})\frac{\alpha\Phi(x_{n-2}) + \beta\Phi(x_{n-4})}{\gamma\Phi(x_{n-2}) + \delta\Phi(x_{n-4})}\right),\ n \in \mathbb{N}_0, \tag{2}$$

where the initial values x_{-p}, for $p = \overline{0, 4}$ and the parameters α, β, γ and δ are real numbers in [22].

In an earlier paper, Kara obtained closed-form solutions of the following general difference equations

$$x_{n+1} = h^{-1}\left(h(x_n)\frac{Ah(x_{n-1}) + Bh(x_{n-2})}{Ch(x_{n-1}) + Dh(x_{n-2})}\right),\ n \in \mathbb{N}_0, \tag{3}$$

where the parameters A, B, C, D and the initial values $x_{-\Phi}$, for $\Phi = \overline{0, 2}$ are real numbers, $A^2+B^2 \neq 0 \neq C^2+D^2$, h is a strictly monotone and continuous function, $h(\mathbb{R}) = \mathbb{R}$, $h(0) = 0$ in [18].

A natural problem is to extend a two-dimensional relative of equation (1) that can be solved in closed-form. In this paper, we will consider such a system. More precisely, we demonstrate the various subclasses of nonlinear difference equation systems of the form. In other words, we are interested in the following general two-dimensional form of equation (1)

$$\begin{cases} u_{n+1} = \Phi^{-1}\left(\Psi(v_n)\frac{a_1\Phi(u_n)+b_1\Psi(v_{n-1})}{c_1\Phi(u_n)+d_1\Psi(v_{n-1})}\right), \\ v_{n+1} = \Psi^{-1}\left(\Phi(u_n)\frac{a_2\Psi(v_n)+b_2\Phi(u_{n-1})}{c_2\Psi(v_n)+d_2\Phi(u_{n-1})}\right), \end{cases} \quad n \in \mathbb{N}_0, \tag{4}$$

where the initial conditions u_t, v_t, for $t \in \{-1, 0\}$ are real numbers, the parameters a_k, b_k, c_k, d_k, for $k \in \{1, 2\}$ are real numbers, Φ and Ψ are continuous and strictly monotone functions, $\Phi(\mathbb{R}) = \mathbb{R}$, $\Psi(\mathbb{R}) = \mathbb{R}$, $\Phi(0) = 0$, $\Psi(0) = 0$. Further, we obtain the solutions of system (4) in closed-form according to states of parameters. In the last case, we will use suitable substitutions on variables and reduce to second-order linear difference equations.

Recurrence relations and difference equations are ancient topics whose rigorous analytical study was largely initiated at the start of the 18th century by de-Moivre who also came up with the phrase "recurrence relation" in [2]. The general solution in closed-form in terms of the parameters η, ζ and the initial conditions z_i, $i \in \{0, 1\}$, to the following linear difference equation of second-order

$$z_{n+2} = \eta z_{n+1} + \zeta z_n,\ n \in \mathbb{N}_0, \tag{5}$$

where $\zeta \neq 0$, was obtained by de-Moivre in [2], as follows:

$$z_n = \frac{(z_1 - \lambda_2 z_0)\lambda_1^n - (z_1 - \lambda_1 z_0)\lambda_2^n}{\lambda_1 - \lambda_2},\ n \in \mathbb{N}_0, \tag{6}$$

when $\zeta \neq 0$ and $\eta^2 + 4\zeta \neq 0$, where $\lambda_{1,2} = \frac{\eta \pm \sqrt{\eta^2+4\zeta}}{2}$ are the roots of the characteristic equation $\lambda^2 - \eta\lambda - \zeta = 0$,

$$z_n = ((z_1 - \lambda z_0)n + \lambda z_0)\lambda^{n-1},\ n \in \mathbb{N}_0, \tag{7}$$

when $\zeta \neq 0$ and $\eta^2 + 4\zeta = 0$, where $\lambda_{1,2} = \lambda = \frac{\eta}{2}$ are the roots of mentioned characteristic equation.

We will use the following very well-known result, which was given by Chapter 1 and page 3-4, in [3].

Lemma 1.1. *Consider the linear difference equation*

$$w_{rn+j} = a_n w_{r(n-1)+j} + b_n,\ n \in \mathbb{N}_0,$$

where a_n and b_n are real number sequences and $j \in \{0, 1, \dots, r-1\}$. Then, the general solution of variable coefficients linear difference equation is given by the following formula

$$w_{rn+j} = \left(\prod_{j=0}^{n} a_j\right) w_{j-r} + \sum_{k=0}^{n} \left(\prod_{j=k+1}^{n} a_j\right) b_k,$$

where the next standard conventions $\prod_{i=j}^{l} \gamma_i = 1$ and $\sum_{i=j}^{l} \eta_i = 0$, for all $l < j$, are utilized here. Moreover if a_n and b_n are constants, that is, $a_n = a$ and $b_n = b$, then the general solution to constant coefficient linear difference equation is given by the following formula

$$w_{rn+j} = \begin{cases} a^{n+1} w_{j-r} + \frac{a^{n+1}-1}{a-1} b, & a \neq 1, \\ w_{j-r} + (n+1)\, b, & a = 1, \end{cases} \quad n \in \mathbb{N}_0.$$

2. Closed-Form Solutions of System (4)

The main results of this study are established and proved in this section.

Theorem 2.1. *Suppose that a_j, b_j, c_j, $d_j \in \mathbb{R}$, for $j \in \{1, 2\}$, such that $c_j^2 + d_j^2 \neq 0$ and the functions Φ and Ψ are continuous and strictly monotone such that $\Phi(\mathbb{R}) = \mathbb{R}$, $\Psi(\mathbb{R}) = \mathbb{R}$, $\Phi(0) = 0$, $\Psi(0) = 0$. Then, the general system* (4) *is solvable in closed-form.*

Proof. If at least one of the initial values u_{-i} or v_{-i}, for $i \in \{0, 1\}$, is equal to zero, then the solution of system (4) is not defined. Moreover, assume that $u_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. Then, from system (4) we have $v_{n_0+1} = 0$. These facts along with (4) imply that v_{n_0+2} is not defined. Similarly, suppose that $v_{n_1} = 0$ for some $n_1 \in \mathbb{N}_0$. Then, from system (4) we have $u_{n_1+1} = 0$ from which along with (4) imply that u_{n_1+2} is not defined. Hence, for every well-defined solution of system (4), we have

$$u_n v_n \neq 0, \; n \geq -1. \tag{8}$$

if and only if $u_{-i} v_{-i} \neq 0$, for $i \in \{0, 1\}$ Firstly, since $\Phi(\mathbb{R}) = \mathbb{R}$, $\Psi(\mathbb{R}) = \mathbb{R}$, $\Phi(0) = 0$, $\Psi(0) = 0$ and $\Phi, \Psi : \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone functions, Φ and Ψ are one to one functions. Further, the only root of the functions Φ and Ψ is 0. So, these functions are homomorphism on $\mathbb{R}$. Taking this property of the functions into consideration, the solutions of system (4) according to the states of the parameters will be examined as follows:
Case 1. $a_j d_j = b_j c_j$, for $j \in \{1, 2\}$: In this case, we have 6 sub-cases.

Subcase 1.1. $a_j = 0 = b_j$, $c_j d_j \neq 0$, for $j \in \{1, 2\}$: In this case, system (4) becomes

$$u_{n+1} = \Phi^{-1}(0), \; v_{n+1} = \Psi^{-1}(0), \; n \in \mathbb{N}_0.$$

By using the properties of functions Φ and Ψ in the last equations, we get the solution of system (4) as follows

$$u_n = 0, \; v_n = 0, \; n \in \mathbb{N}. \tag{9}$$

Subcase 1.2. $a_j = 0$, $b_j d_j \neq 0$, for $j \in \{1, 2\}$: In this case, with these conditions, we straight away get $c_j = 0$, for $j \in \{1, 2\}$, and hereby, $d_j \neq 0$, for $j \in \{1, 2\}$, which implies

$$u_{n+1} = \Phi^{-1}\left(\frac{b_1}{d_1} \Psi(v_n)\right), \; v_{n+1} = \Psi^{-1}\left(\frac{b_2}{d_2} \Phi(u_n)\right), \; n \in \mathbb{N}_0. \tag{10}$$

Also, from (10), we easily obtain

$$\Phi(u_{n+1}) = \frac{b_1}{d_1} \Psi(v_n), \; \Psi(v_{n+1}) = \frac{b_2}{d_2} \Phi(u_n), \; n \in \mathbb{N}_0, \tag{11}$$

from which it follows that

$$\Phi(u_{n+1}) = \frac{b_1 b_2}{d_1 d_2}\Phi(u_{n-1}),\ \Psi(v_{n+1}) = \frac{b_1 b_2}{d_1 d_2}\Psi(v_{n-1}),\ n \in \mathbb{N}. \tag{12}$$

Since the equations in (12) are solvable, we define new variables as following forms

$$z_n = \Phi(u_n),\ t_n = \Psi(v_n),\ n \in \mathbb{N}_0. \tag{13}$$

By substituting the new variables to equations in (12), we obtain the second-order linear difference equations as follows:

$$z_{n+1} = \frac{b_1 b_2}{d_1 d_2} z_{n-1},\ t_{n+1} = \frac{b_1 b_2}{d_1 d_2} t_{n-1},\ n \in \mathbb{N}. \tag{14}$$

By using Lemma 1.1 for $r = 2$, we can write the general solution of the equations in (14) as follows

$$z_{2n+i} = \left(\frac{b_1 b_2}{d_1 d_2}\right)^n z_i,\ t_{2n+i} = \left(\frac{b_1 b_2}{d_1 d_2}\right)^n t_i,\ n \in \mathbb{N}_0, \tag{15}$$

for $i \in \{0, 1\}$. Further, from (13) and the solutions in (15), the general solutions of system (10) can be written by

$$u_{2n+i} = \Phi^{-1}\left(\left(\frac{b_1 b_2}{d_1 d_2}\right)^n \Phi(u_i)\right),\ v_{2n+i} = \Psi^{-1}\left(\left(\frac{b_1 b_2}{d_1 d_2}\right)^n \Psi(v_i)\right),\ n \in \mathbb{N}_0, \tag{16}$$

for $i \in \{0, 1\}$.

Subcase 1.3. $b_j = 0, a_j c_j \neq 0$, for $j \in \{1, 2\}$: In this case, with these conditions, we immediately have $d_j = 0$, for $j \in \{1, 2\}$, and consequently, $c_j \neq 0$, for $j \in \{1, 2\}$, which implies

$$u_{n+1} = \Phi^{-1}\left(\frac{a_1}{c_1}\Psi(v_n)\right),\ v_{n+1} = \Psi^{-1}\left(\frac{a_2}{c_2}\Phi(u_n)\right),\ n \in \mathbb{N}_0. \tag{17}$$

Further, from (17), we get

$$\Phi(u_{n+1}) = \frac{a_1}{c_1}\Psi(v_n),\ \Psi(v_{n+1}) = \frac{a_2}{c_2}\Phi(u_n),\ n \in \mathbb{N}_0. \tag{18}$$

from which it follows that

$$\Phi(u_{n+1}) = \frac{a_1 a_2}{c_1 c_2}\Phi(u_{n-1}),\ \Psi(v_{n+1}) = \frac{a_1 a_2}{c_1 c_2}\Psi(v_{n-1}),\ n \in \mathbb{N}. \tag{19}$$

By using transforms in (13), then we get the second-order linear difference equations as follows:

$$z_{n+1} = \frac{a_1 a_2}{c_1 c_2} z_{n-1},\ t_{n+1} = \frac{a_1 a_2}{c_1 c_2} t_{n-1},\ n \in \mathbb{N}. \tag{20}$$

By using Lemma 1.1 for $r = 2$, we can write the general solution of equations in (20) as follows

$$z_{2n+i} = \left(\frac{a_1 a_2}{c_1 c_2}\right)^n z_i,\ t_{2n+i} = \left(\frac{a_1 a_2}{c_1 c_2}\right)^n t_i,\ n \in \mathbb{N}_0, \tag{21}$$

for $i \in \{0, 1\}$. From (13) and the solutions in (21), the general solution of system (17) can be written by

$$u_{2n+i} = \Phi^{-1}\left(\left(\frac{a_1 a_2}{c_1 c_2}\right)^n \Phi(u_i)\right),\ v_{2n+i} = \Psi^{-1}\left(\left(\frac{a_1 a_2}{c_1 c_2}\right)^n \Psi(v_i)\right),\ n \in \mathbb{N}_0, \tag{22}$$

for $i \in \{0, 1\}$.

Subcase 1.4. $d_j = 0$, for $j \in \{1, 2\}$: In this case, with this condition, we straight away get $b_j = 0$, for $j \in \{1, 2\}$, and hereby, $c_j \neq 0$, for $j \in \{1, 2\}$. Then, there are two cases to be considered. These cases are either $a_j = 0$ or $a_j \neq 0$ for $j \in \{1, 2\}$. These two cases were investigated in Subcase 1 and Subcase 3, respectively.

Subcase 1.5. $c_j = 0$, for $j \in \{1, 2\}$: In this case, with this condition, we immediately have $a_j = 0$, for $j \in \{1, 2\}$, and consequently, $d_j \neq 0$, for $j \in \{1, 2\}$. Then, there are two cases to be considered. These cases are either $b_j = 0$ or $b_j \neq 0$ for $j \in \{1, 2\}$. These two cases were investigated in Sub-case 1 and Sub-case 2, respectively.

Subcase 1.6. $a_j b_j c_j d_j \neq 0$, for $j \in \{1, 2\}$: In this case, with this condition, we straight away get $a_j = \frac{b_j c_j}{d_j}$ for $j \in \{1, 2\}$. Then system (4) reduces to system (10), whose solution is given by formulas (16).

Case 2. $a_j d_j \neq b_j c_j$ for $j \in \{1, 2\}$: In this case, from (8) and the monotonicity of Φ and Ψ, we have the following inequalities

$$\Phi(u_n) \neq 0,\ \Psi(v_n) \neq 0,\ n \geq -1. \tag{23}$$

Then, from (23), system (4) can be written in the following form

$$\frac{\Phi(u_{n+1})}{\Psi(v_n)} = \frac{a_1 \frac{\Phi(u_n)}{\Psi(v_{n-1})} + b_1}{c_1 \frac{\Phi(u_n)}{\Psi(v_{n-1})} + d_1},\ \frac{\Psi(v_{n+1})}{\Phi(u_n)} = \frac{a_2 \frac{\Psi(v_n)}{\Phi(u_{n-1})} + b_2}{c_2 \frac{\Psi(v_n)}{\Phi(u_{n-1})} + d_2},\ n \in \mathbb{N}_0. \tag{24}$$

By using the following change of variables

$$z_n = \frac{\Phi(u_n)}{\Psi(v_{n-1})},\ w_n = \frac{\Psi(v_n)}{\Phi(u_{n-1})},\ n \in \mathbb{N}_0, \tag{25}$$

we get the following Riccati difference equations as follows:

$$z_{n+1} = \frac{a_1 z_n + b_1}{c_1 z_n + d_1},\ w_{n+1} = \frac{a_2 w_n + b_2}{c_2 w_n + d_2},\ n \in \mathbb{N}_0. \tag{26}$$

In here, there are two cases to be considered for system (26).

Subcase 2.1. $c_j = 0$ for $j \in \{1, 2\}$: In this case, system (26) is presented by

$$z_{n+1} = \frac{a_1}{d_1} z_n + \frac{b_1}{d_1},\ w_{n+1} = \frac{a_2}{d_2} w_n + \frac{b_2}{d_2},\ n \in \mathbb{N}_0. \tag{27}$$

Now the subcases $\frac{a_j}{d_j} = 1$ and $\frac{a_j}{d_j} \neq 1$ for $j \in \{1, 2\}$ will be considered separately.

Subsubcase 2.1.1. $a_j = d_j$ for $j \in \{1, 2\}$: In this case, by using Lemma 1.1, if $\frac{a_1}{d_1} = 1$ and $\frac{a_2}{d_2} = 1$, in (27), the solutions of equations in (27) can be written in the following form

$$z_n = z_0 + \frac{b_1}{d_1} n,\ w_n = w_0 + \frac{b_2}{d_2} n,\ n \in \mathbb{N}_0, \tag{28}$$

from which along with (25), it follows that

$$\Phi(u_n) = \left(\frac{\Phi(u_0)}{\Psi(v_{-1})} + \frac{b_1}{d_1} n \right) \Psi(v_{n-1}),\ \Psi(v_n) = \left(\frac{\Psi(v_0)}{\Phi(u_{-1})} + \frac{b_2}{d_2} n \right) \Phi(u_{n-1}),\ n \in \mathbb{N}_0. \tag{29}$$

By substituting the first equation in (29) into the second equation in (29) and the second one in (29) into the first one in (29), we can easily get

$$\begin{cases}\Phi(u_n)= & \left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}n\right)\left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}(n-1)\right)\Phi(u_{n-2}),\\ \Psi(v_n)= & \left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}n\right)\left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}(n-1)\right)\Psi(v_{n-2}),\end{cases} n\in\mathbb{N}. \tag{30}$$

On the other hand, we get the general solution of system (4) as follows

$$\begin{cases}u_{2n+i}= & \Phi^{-1}\left(\Phi(u_i)\prod_{j=1}^{n}\left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}(2j+i)\right)\left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}(2j+i-1)\right)\right),\\ v_{2n+i}= & \Psi^{-1}\left(\Psi(v_i)\prod_{j=1}^{n}\left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}(2j+i)\right)\left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}(2j+i-1)\right)\right),\end{cases} n\in\mathbb{N}_0, \tag{31}$$

for $i\in\{-1,0\}$.

<u>*Subsubcase 2.1.2.*</u> $a_1=d_1$ and $a_2\neq d_2$: In this case, by using Lemma 1.1, if $\frac{a_1}{d_1}=1$ and $\frac{a_2}{d_2}\neq 1$, in (27), we can write the solutions of equations in (27) as follows

$$z_n=z_0+\frac{b_1}{d_1}n,\ w_n=\left(\frac{a_2}{d_2}\right)^n w_0+b_2\frac{\left(\frac{a_2}{d_2}\right)^n-1}{a_2-d_2},\ n\in\mathbb{N}_0. \tag{32}$$

Clearly, from (25), we get

$$\begin{cases}\Phi(u_n)= & \left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}n\right)\Psi(v_{n-1}),\\ \Psi(v_n)= & \left(\frac{a_2}{d_2}\right)^n\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^n-1}{a_2-d_2}\Phi(u_{n-1}),\end{cases} n\in\mathbb{N}_0. \tag{33}$$

By substituting the first equation in (33) into the second equation in (33) and the second one in (33) into the first one in (33), we get that in this case

$$\begin{cases}\Phi(u_n)= & \left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}n\right)\left(\left(\frac{a_2}{d_2}\right)^{n-1}\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^{n-1}-1}{a_2-d_2}\right)\Phi(u_{n-2}),\\ \Psi(v_n)= & \left(\left(\frac{a_2}{d_2}\right)^n\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^n-1}{a_2-d_2}\right)\left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}(n-1)\right)\Psi(v_{n-2}),\end{cases} n\in\mathbb{N}. \tag{34}$$

and consequently

$$\begin{cases}u_{2n+i}= & \Phi^{-1}\left(\Phi(u_i)\prod_{j=1}^{n}\left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}(2j+i)\right)\left(\left(\frac{a_2}{d_2}\right)^{2j+i-1}\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^{2j+i-1}-1}{a_2-d_2}\right)\right),\\ v_{2n+i}= & \Psi^{-1}\left(\Psi(v_i)\prod_{j=1}^{n}\left(\left(\frac{a_2}{d_2}\right)^{2j+i}\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^{2j+i}-1}{a_2-d_2}\right)\left(\frac{\Phi(u_0)}{\Psi(v_{-1})}+\frac{b_1}{d_1}(2j+i-1)\right)\right),\end{cases} n\in\mathbb{N}_0, \tag{35}$$

for $i\in\{-1,0\}$.

<u>*Subsubcase 2.1.3.*</u> $a_1\neq d_1$ and $a_2=d_2$: In this case, by using Lemma 1.1, if $\frac{a_1}{d_1}\neq 1$ and $\frac{a_2}{d_2}=1$, in (27), the general solutions to equations in (27) can be written as follows

$$z_n=\left(\frac{a_1}{d_1}\right)^n z_0+b_1\frac{\left(\frac{a_1}{d_1}\right)^n-1}{a_1-d_1},\ w_n=w_0+\frac{b_2}{d_2}n,\ n\in\mathbb{N}_0. \tag{36}$$

From (25), we obtain

$$\begin{cases}\Phi(u_n)= & \left(\frac{a_1}{d_1}\right)^n\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^n-1}{a_1-d_1}\Psi(v_{n-1}),\\ \Psi(v_n)= & \left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}n\right)\Phi(u_{n-1}),\end{cases} n\in\mathbb{N}_0. \tag{37}$$

By substituting the first equation in (37) into the second equation in (37) and the second one in (37) into the first one in (37), we obtain that in this case

$$\begin{cases}\Phi(u_n)= & \left(\left(\frac{a_1}{d_1}\right)^n\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^n-1}{a_1-d_1}\right)\left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}(n-1)\right)\Phi(u_{n-2}),\\ \Psi(v_n)= & \left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}n\right)\left(\left(\frac{a_1}{d_1}\right)^{n-1}\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^{n-1}-1}{a_1-d_1}\right)\Psi(v_{n-2}),\end{cases}\quad n\in\mathbb{N}. \tag{38}$$

and consequently

$$\begin{cases}u_{2n+i}= & \Phi^{-1}\left(\Phi(u_i)\prod_{j=1}^{n}\left(\left(\frac{a_1}{d_1}\right)^{2j+i}\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^{2j+i}-1}{a_1-d_1}\right)\left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}(2j+i-1)\right)\right),\\ v_{2n+i}= & \Psi^{-1}\left(\Psi(v_i)\prod_{j=1}^{n}\left(\frac{\Psi(v_0)}{\Phi(u_{-1})}+\frac{b_2}{d_2}(2j+i)\right)\left(\left(\frac{a_1}{d_1}\right)^{2j+i-1}\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^{2j+i-1}-1}{a_1-d_1}\right)\right),\end{cases}\quad n\in\mathbb{N}_0, \tag{39}$$

for $i\in\{-1,0\}$.

Subsubcase 2.1.4. $a_j\neq d_j$ for $j\in\{1,2\}$: In this case, by using Lemma 1.1, if $\frac{a_1}{d_1}\neq 1$ and $\frac{a_2}{d_2}\neq 1$, in (27), we have write the solutions of equations in (27) as follows

$$z_n=\left(\frac{a_1}{d_1}\right)^n z_0+b_1\frac{\left(\frac{a_1}{d_1}\right)^n-1}{a_1-d_1},\ w_n=\left(\frac{a_2}{d_2}\right)^n w_0+b_2\frac{\left(\frac{a_2}{d_2}\right)^n-1}{a_2-d_2},\ n\in\mathbb{N}_0. \tag{40}$$

Taking into account (25), we get

$$\begin{cases}\Phi(u_n)= & \left(\frac{a_1}{d_1}\right)^n\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^n-1}{a_1-d_1}\Psi(v_{n-1}),\\ \Psi(v_n)= & \left(\frac{a_2}{d_2}\right)^n\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^n-1}{a_2-d_2}\Phi(u_{n-1}),\end{cases}\quad n\in\mathbb{N}_0. \tag{41}$$

By substituting the first equation in (41) into the second equation in (41) and the second one in (41) into the first one in (41), we have

$$\begin{cases}\Phi(u_n)= & \left(\left(\frac{a_1}{d_1}\right)^n\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^n-1}{a_1-d_1}\right)\left(\left(\frac{a_2}{d_2}\right)^{n-1}\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^{n-1}-1}{a_2-d_2}\right)\Phi(u_{n-2}),\\ \Psi(v_n)= & \left(\left(\frac{a_2}{d_2}\right)^n\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^n-1}{a_2-d_2}\right)\left(\left(\frac{a_1}{d_1}\right)^{n-1}\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^{n-1}-1}{a_1-d_1}\right)\Psi(v_{n-2}),\end{cases}\quad n\in\mathbb{N}, \tag{42}$$

from which it follows that

$$\begin{cases}u_{2n+i}= & \Phi^{-1}\left(\Phi(u_i)\prod_{j=1}^{n}\left(\left(\frac{a_1}{d_1}\right)^{2j+i}\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^{2j+i}-1}{a_1-d_1}\right)\left(\left(\frac{a_2}{d_2}\right)^{2j+i-1}\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^{2j+i-1}-1}{a_2-d_2}\right)\right),\\ v_{2n+i}= & \Psi^{-1}\left(\Psi(v_i)\prod_{j=1}^{n}\left(\left(\frac{a_2}{d_2}\right)^{2j+i}\frac{\Psi(v_0)}{\Phi(u_{-1})}+b_2\frac{\left(\frac{a_2}{d_2}\right)^{2j+i}-1}{a_2-d_2}\right)\left(\left(\frac{a_1}{d_1}\right)^{2j+i-1}\frac{\Phi(u_0)}{\Psi(v_{-1})}+b_1\frac{\left(\frac{a_1}{d_1}\right)^{2j+i-1}-1}{a_1-d_1}\right)\right),\end{cases} \tag{43}$$

for $n\in\mathbb{N}_0$, $i\in\{-1,0\}$.

Subcase 2.2. $c_j\neq 0$ for $j\in\{1,2\}$: In this case, by employing the following substitution

$$c_1z_n+d_1=\frac{p_{n+1}}{p_n},\ n\in\mathbb{N}_0. \tag{44}$$

to the first equation in (26), we have the following second-order constant coefficients linear difference equation

$$p_{n+2}-(a_1+d_1)\,p_{n+1}-(b_1c_1-a_1d_1)\,p_n=0, n\in\mathbb{N}_0. \tag{45}$$

The characteristic equation for (45) can be written as follows

$$\lambda^2-(a_1+d_1)\,\lambda-(b_1c_1-a_1d_1)=0,$$

where $\lambda_1=\frac{a_1+d_1+\sqrt{(a_1+d_1)^2+4(b_1c_1-a_1d_1)}}{2}$, $\lambda_2=\frac{a_1+d_1-\sqrt{(a_1+d_1)^2+4(b_1c_1-a_1d_1)}}{2}$,
if $\Delta_1=(a_1+d_1)^2+4\,(b_1c_1-a_1d_1)\neq 0$, and $\lambda_{1,2}=\frac{a_1+d_1}{2}$, if $\Delta_1=(a_1+d_1)^2+4\,(b_1c_1-a_1d_1)=0$. Then, the solution of equation (45) with the initial values p_0, p_1 is

$$p_n=\frac{(p_1-\lambda_2p_0)\,\lambda_1^n-(p_1-\lambda_1p_0)\,\lambda_2^n}{\lambda_1-\lambda_2},\ n\in\mathbb{N}_0, \tag{46}$$

if $(a_1+d_1)^2+4\,(b_1c_1-a_1d_1)\neq 0$, and

$$p_n=((p_1-\lambda_1p_0)\,n+\lambda_1p_0)\,\lambda_1^{n-1},\ n\in\mathbb{N}_0, \tag{47}$$

if $(a_1+d_1)^2+4\,(b_1c_1-a_1d_1)=0$. By using (44) and $\frac{p_1}{p_0}=c_1z_0+d_1$, the solution of the first equation in (26) can be written by

$$z_n=\frac{1}{c_1}\frac{(c_1z_0+d_1-\lambda_2)\,\lambda_1^{n+1}-(c_1z_0+d_1-\lambda_1)\,\lambda_2^{n+1}}{(c_1z_0+d_1-\lambda_2)\,\lambda_1^{n}-(c_1z_0+d_1-\lambda_1)\,\lambda_2^{n}}-\frac{d_1}{c_1},\ n\in\mathbb{N}_0, \tag{48}$$

if $(a_1+d_1)^2+4\,(b_1c_1-a_1d_1)\neq 0$, and

$$z_n=\frac{1}{c_1}\frac{((c_1z_0+d_1-\lambda_1)\,(n+1)+\lambda_1)\,\lambda_1^{n}}{((c_1z_0+d_1-\lambda_1)\,n+\lambda_1)\,\lambda_1^{n-1}}-\frac{d_1}{c_1},\ n\in\mathbb{N}_0, \tag{49}$$

if $(a_1+d_1)^2+4\,(b_1c_1-a_1d_1)=0$.

Similarly, by using the following change of variable

$$c_2w_n+d_2=\frac{q_{n+1}}{q_n},\ n\in\mathbb{N}_0, \tag{50}$$

to the second equation in (26), we obtain the following second-order constant coefficients linear difference equation

$$q_{n+2}-(a_2+d_2)\,q_{n+1}-(b_2c_2-a_2d_2)\,q_n=0, n\in\mathbb{N}_0. \tag{51}$$

The characteristic equation for (51) is

$$\lambda^2-(a_2+d_2)\,\lambda-(b_2c_2-a_2d_2)=0,$$

where $\lambda_3=\frac{a_2+d_2+\sqrt{(a_2+d_2)^2+4(b_2c_2-a_2d_2)}}{2}$, $\lambda_4=\frac{a_2+d_2-\sqrt{(a_2+d_2)^2+4(b_2c_2-a_2d_2)}}{2}$,
if $\Delta_2=(a_2+d_2)^2+4\,(b_2c_2-a_2d_2)\neq 0$, and $\lambda_{3,4}=\frac{a_2+d_2}{2}$ if $\Delta_2=(a_2+d_2)^2+4\,(b_2c_2-a_2d_2)=0$. In addition, the solution of equation (51) with the initial values q_0, q_1 is

$$q_n=\frac{(q_1-\lambda_4q_0)\,\lambda_3^n-(q_1-\lambda_3q_0)\,\lambda_4^n}{\lambda_3-\lambda_4},\ n\in\mathbb{N}_0, \tag{52}$$

if $(a_2+d_2)^2+4(b_2c_2-a_2d_2)\neq 0$, and

$$q_n=((q_1-\lambda_3 q_0)n+\lambda_3 q_0)\lambda_3^{n-1},\ n\in\mathbb{N}_0, \tag{53}$$

if $(a_2+d_2)^2+4(b_2c_2-a_2d_2)=0$. By using (50) and $\frac{q_1}{q_0}=c_2w_0+d_2$, the solution of the second equation in (26) can be written by

$$w_n=\frac{1}{c_2}\frac{(c_2w_0+d_2-\lambda_4)\lambda_3^{n+1}-(c_2w_0+d_2-\lambda_3)\lambda_4^{n+1}}{(c_2w_0+d_2-\lambda_4)\lambda_3^{n}-(c_2w_0+d_2-\lambda_3)\lambda_4^{n}}-\frac{d_2}{c_2},\ n\in\mathbb{N}_0, \tag{54}$$

if $(a_2+d_2)^2+4(b_2c_2-a_2d_2)\neq 0$, and

$$w_n=\frac{1}{c_2}\frac{((c_2w_0+d_2-\lambda_3)(n+1)+\lambda_3)\lambda_3^{n}}{((c_2w_0+d_2-\lambda_3)n+\lambda_3)\lambda_3^{n-1}}-\frac{d_2}{c_2},\ n\in\mathbb{N}_0, \tag{55}$$

if $(a_2+d_2)^2+4(b_2c_2-a_2d_2)=0$.

From the relations (48), (49), (54) and (55), we observe that there are fundamentally two different formulas for solutions of system (4), depending on the states of $\Delta_1\Delta_2\neq 0$ and $\Delta_1=\Delta_2=0$.

Subsubcase 2.2.1. $\Delta_1\Delta_2\neq 0$: In this case, by using (25), (48) and (54), we obtain

$$\begin{cases}\Phi(u_n)&=\left(\frac{1}{c_1}\frac{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{n+1}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{n+1}}{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{n}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{n}}-\frac{d_1}{c_1}\right)\Psi(v_{n-1}),\\ \Psi(v_n)&=\left(\frac{1}{c_2}\frac{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{n+1}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{n+1}}{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{n}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{n}}-\frac{d_2}{c_2}\right)\Phi(u_{n-1}),\end{cases}\quad n\in\mathbb{N}_0. \tag{56}$$

By substituting the first equation in (56) into the second equation in (56) and the second one in (56) into the first one in (56), we have

$$\begin{cases}\Phi(u_n)&=\left(\frac{1}{c_1}\frac{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{n+1}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{n+1}}{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{n}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{n}}-\frac{d_1}{c_1}\right)\\ &\quad\times\left(\frac{1}{c_2}\frac{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{n}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{n}}{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{n-1}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{n-1}}-\frac{d_2}{c_2}\right)\Phi(u_{n-2}),\\ \Psi(v_n)&=\left(\frac{1}{c_2}\frac{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{n+1}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{n+1}}{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{n}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{n}}-\frac{d_2}{c_2}\right)\\ &\quad\times\left(\frac{1}{c_1}\frac{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{n}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{n}}{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{n-1}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{n-1}}-\frac{d_1}{c_1}\right)\Psi(v_{n-2}),\end{cases}\quad n\in\mathbb{N}, \tag{57}$$

and consequently

$$\begin{cases}u_{2n+i}=\Phi^{-1}\Bigg[\Phi(u_i)\prod_{j=1}^{n}&\left(\frac{1}{c_1}\frac{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{2j+i+1}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{2j+i+1}}{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{2j+i}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{2j+i}}-\frac{d_1}{c_1}\right)\\ &\times\left(\frac{1}{c_2}\frac{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{2j+i}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{2j+i}}{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{2j+i-1}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{2j+i-1}}-\frac{d_2}{c_2}\right)\Bigg],\\ v_{2n+i}=\Psi^{-1}\Bigg[\Psi(v_i)\prod_{j=1}^{n}&\left(\frac{1}{c_2}\frac{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{2j+i+1}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{2j+i+1}}{\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_4\right)\lambda_3^{2j+i}-\left(c_2\frac{\Psi(v_0)}{\Phi(u_{-1})}+d_2-\lambda_3\right)\lambda_4^{2j+i}}-\frac{d_2}{c_2}\right)\\ &\times\left(\frac{1}{c_1}\frac{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{2j+i}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{2j+i}}{\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_2\right)\lambda_1^{2j+i-1}-\left(c_1\frac{\Phi(u_0)}{\Psi(v_{-1})}+d_1-\lambda_1\right)\lambda_2^{2j+i-1}}-\frac{d_1}{c_1}\right)\Bigg],\end{cases} \tag{58}$$

for $n \in \mathbb{N}_0$ and $i \in \{-1, 0\}$.

Subsubcase 2.2.2. $\Delta_1 = \Delta_2 = 0$: In this case, from (25), (49) and (55), we obtain

$$\begin{cases} \Phi(u_n) &= \left(\frac{1}{c_1} \frac{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)(n+1) + \lambda_1\right)\lambda_1^n}{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)n + \lambda_1\right)\lambda_1^{n-1}} - \frac{d_1}{c_1} \right) \Psi(v_{n-1}), \\ \Psi(v_n) &= \left(\frac{1}{c_2} \frac{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)(n+1) + \lambda_3\right)\lambda_3^n}{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)n + \lambda_3\right)\lambda_3^{n-1}} - \frac{d_2}{c_2} \right) \Phi(u_{n-1}), \end{cases} \quad n \in \mathbb{N}_0. \tag{59}$$

By substituting the first equation in (59) into the second equation in (59) and the second equation in (59) into the first equation in (59), we obtain

$$\begin{cases} \Phi(u_n) = \left(\frac{1}{c_1} \frac{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)(n+1) + \lambda_1\right)\lambda_1^n}{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)n + \lambda_1\right)\lambda_1^{n-1}} - \frac{d_1}{c_1} \right) \\ \qquad \times \left(\frac{1}{c_2} \frac{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)n + \lambda_3\right)\lambda_3^{n-1}}{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)(n-1) + \lambda_3\right)\lambda_3^{n-2}} - \frac{d_2}{c_2} \right) \Phi(u_{n-2}), \\ \Psi(v_n) = \left(\frac{1}{c_2} \frac{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)(n+1) + \lambda_3\right)\lambda_3^n}{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)n + \lambda_3\right)\lambda_3^{n-1}} - \frac{d_2}{c_2} \right) \\ \qquad \times \left(\frac{1}{c_1} \frac{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)n + \lambda_1\right)\lambda_1^{n-1}}{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)(n-1) + \lambda_1\right)\lambda_1^{n-2}} - \frac{d_1}{c_1} \right) \Psi(v_{n-2}), \end{cases} \quad n \in \mathbb{N}. \tag{60}$$

Then, the general solutions of system (4) are

$$\begin{cases} u_{2n+i} = \Phi^{-1}\Bigg[\Phi(u_i) \prod_{j=1}^{n} \left(\frac{1}{c_1} \frac{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)(2j+i+1) + \lambda_1\right)\lambda_1^{2j+i}}{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)(2j+i) + \lambda_1\right)\lambda_1^{2j+i-1}} - \frac{d_1}{c_1} \right) \\ \qquad \times \left(\frac{1}{c_2} \frac{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)(2j+i) + \lambda_3\right)\lambda_3^{2j+i-1}}{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)(2j+i-1) + \lambda_3\right)\lambda_3^{2j+i-2}} - \frac{d_2}{c_2} \right)\Bigg], \\ v_{2n+i} = \Psi^{-1}\Bigg[\Psi(v_i) \prod_{j=1}^{n} \left(\frac{1}{c_2} \frac{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)(2j+i+1) + \lambda_3\right)\lambda_3^{2j+i}}{\left(\left(c_2 \frac{\Psi(v_0)}{\Phi(u_{-1})} + d_2 - \lambda_3\right)(2j+i) + \lambda_3\right)\lambda_3^{2j+i-1}} - \frac{d_2}{c_2} \right) \\ \qquad \times \left(\frac{1}{c_1} \frac{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)(2j+i) + \lambda_1\right)\lambda_1^{2j+i-1}}{\left(\left(c_1 \frac{\Phi(u_0)}{\Psi(v_{-1})} + d_1 - \lambda_1\right)(2j+i-1) + \lambda_1\right)\lambda_1^{2j+i-2}} - \frac{d_1}{c_1} \right)\Bigg], \end{cases} \tag{61}$$

for $n \in \mathbb{N}_0$ and $i \in \{-1, 0\}$. □

Corollary 2.2. *Consider system* (4) *with the initial conditions u_{-t}, v_{-t}, for $t \in \{0, 1\}$ and the parameters $c_j^2 + d_j^2 \neq 0$ a_j, b_j, c_j, d_j, for $j \in \{1, 2\}$, which are real numbers. Then the following statements are true.*

a) If $a_j d_j = b_j c_j$, $a_j = b_j = 0$ and $c_j d_j \neq 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (9).

b) If $a_j d_j = b_j c_j$, $a_j = 0$ and $b_j d_j \neq 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (16).

c) If $a_j d_j = b_j c_j$, $b_j = 0$ and $a_j c_j \neq 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (22).

d) If $a_j d_j = b_j c_j$, $d_j = 0$ and $a_j = 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (9).

e) If $a_j d_j = b_j c_j$, $d_j = 0$ and $a_j \neq 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (22).

f) If $a_j d_j = b_j c_j$, $c_j = 0$ and $b_j = 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (9).

g) If $a_j d_j = b_j c_j$, $c_j = 0$ and $b_j \neq 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (16).

h) If $a_jd_j = b_jc_j$ and $a_jb_jc_jd_j \neq 0$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (16).

i) If $a_jd_j \neq b_jc_j$, $c_j = 0$ and $a_j = d_j$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (31).

j) If $a_jd_j \neq b_jc_j$, $c_j = 0$, for $j \in \{1, 2\}$, $a_1 = d_1$ and $a_2 \neq d_2$, then the general solution to system (4) *is given by* (35).

k) If $a_jd_j \neq b_jc_j$, $c_j = 0$, for $j \in \{1, 2\}$, $a_1 \neq d_1$ and $a_2 = d_2$, then the general solution to system (4) *is given by* (39).

l) If $a_jd_j \neq b_jc_j$, $c_j = 0$ and $a_j \neq d_j$ for $j \in \{1, 2\}$, then the general solution to system (4) *is given by* (43).

m) If $a_jd_j \neq b_jc_j$, $c_j \neq 0$, for $j \in \{1, 2\}$ and $\left((a_1 + d_1)^2 + 4(b_1c_1 - a_1d_1)\right)\left((a_2 + d_2)^2 + 4(b_2c_2 - a_2d_2)\right) \neq 0$ then the general solution to system (4) *is given by* (58).

n) If $a_jd_j \neq b_jc_j$, $c_j \neq 0$, for $j \in \{1, 2\}$ and $(a_1 + d_1)^2 + 4(b_1c_1 - a_1d_1) = 0 = (a_2 + d_2)^2 + 4(b_2c_2 - a_2d_2)$, then the general solution to system (4) *is given by* (61).

References

[1] R. Abo-Zeid, *Global behavior of a higher order rational difference equation*, Filomat **30(12)** (2016), 3265–3276.
[2] A. de-Moivre, *The Doctrine of Chances*, 3^{nd} edition, In Landmark Writings in Western Mathematics, London, 1756.
[3] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 2005.
[4] E. M. Elsayed, *Qualitative behavior of a rational recursive sequence*, Indag. Math. **19(2)** (2008), 189–201.
[5] E. M. Elsayed, *Qualitative behavior of difference equation of order two*, Math. Comput. Model **50** (2009), 1130–1141.
[6] E. M. Elsayed, *Qualitative properties for a fourth order rational difference equation*, Acta. Appl. Math. **110(2)** (2010), 589–604.
[7] E. M. Elsayed, F. Alzahrani, I. Abbas, N. H. Alotaibi, *Dynamical behavior and solution of nonlinear difference equation via Fibonacci sequence*, J. Appl. Anal. Comput. **10** (2020), 282–296.
[8] E. M. Elsayed, B. S. Aloufi, O. Moaaz, *The behavior and structures of solution of fifth-order rational recursive sequence*, Symmetry **14** (2022), 641.
[9] S. Etemad, M. A. Ragusa, S. Rezapour, A. Zada, *Existence property of solutions for multi-order q-difference FBVPs based on condensing operators and end-point technique*, Fixed Point Theory, **25(1)** (2024), 115–1421.
[10] A. Ghezal, *Note on a rational system of* $(4k + 4)-$*order difference equations: periodic solution and convergence*, J. Appl. Math. Comput. **69(2)** (2023), 2207–2215.
[11] M. Gumus, R. Abo-Zeid, *Qualitative study of a third order rational system of difference equations*, Math. Morav. **25(1)** (2021), 81–97.
[12] Y. Halim, N. Touafek, J. F. T. Rabago, *Dynamic behavior of a second-order nonlinear rational difference equation*, Turkish. J. Math. **39(6)** (2015), 1004–1018.
[13] H. Hamioud, I. Dekkar, N. Touafek, *Solvability of a third-order system of nonlinear difference equations via a generalized fibonacci sequence*, Miskolc Math. Notes. **25** (2024), 271-285.
[14] M. Kara, Y. Yazlik, D. T. Tollu, *Solvability of a system of higher order nonlinear difference equations*, Hacet. J. Math. Stat. **49(5)** (2020), 1566–1593.
[15] M. Kara, Y. Yazlik, *On a solvable three-dimensional system of difference equations*, Filomat **34(4)** (2020), 1167–1186.
[16] M. Kara, Y. Yazlik, *Solvable three-dimensional system of higher-order nonlinear difference equations*, Filomat **36(10)** (2022), 3453–3473.
[17] M. Kara, Y. Yazlik, *On a solvable system of rational difference equations of higher order*, Turkish. J. Math. **46** (2022), 587–611.
[18] M. Kara, *On a general non-linear difference equation of third-order*, Turk. J. Math. Comput. Sci. **16(1)** (2024), 126–136.
[19] M. Kara, Y. Yazlik, N. Touafek, *On a difference equation whose solution is related to Fibonacci numbers*, Filomat, **38(20)** (2024), 7199–7207.
[20] Y. Meng, X. R. Du, H. H. Pang, *Iterative positive solutions to a coupled Riemann-Liouville fractional q-difference system with the Caputo fractional q-derivative boundary conditions*, J. Funct. Spaces **2023(1)** (2023), art.n.5264831, 1–16.
[21] S. Stević, B. Iričanin, W. Kosmala, Z. Smarda, *On a nonlinear second order difference equation*, J. Inequal. Appl. **88** (2022), 1–11.
[22] S. Stević, B. Iričanin, W. Kosmala, *On a family of nonlinear difference equations of the fifth order solvable in closed-form*, AIMS Math. **8(10)** (2023), 22662–22674.
[23] N. Taskara, D. T. Tollu, N. Touafek, Y. Yazlik, *A solvable system of difference equations*, Commun. Korean. Math. Soc. **35(1)** (2020), 301–319.
[24] D. T. Tollu, Y. Yazlik, N. Taskara, *Behavior of positive solutions of a difference equation*, J. Appl. Math. Inform. **35** (2017), 217–230.
[25] D. T. Tollu, Y. Yazlik, N. Taskara, *On a solvable nonlinear difference equation of higher order*, Turkish J. Math. **42** (2018), 1765–1778.
[26] N. Touafek, *On a general system of difference equations defined by homogeneous functions*, Math. Slovaca. **71** (2021), 697–720.
[27] I. Yalcinkaya, H. El-Metwally, M. Bayram, D. T. Tollu, *On the dynamics of a higher-order fuzzy difference equation with rational terms*, Soft Comput. **27(15)** (2023), 10469–10479.
[28] Y. Yazlik, D. T. Tollu, N. Taskara, *On the solutions of difference equation systems with Padovan numbers*, Appl. Math. **4(12A)** (2013) 1–15.
[29] Y. Yazlik, M. Kara, *On a solvable system of difference equations of higher-order with period two coefficients*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68(2)** (2019), 1675–1693.