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On the local existence of solutions to an abstract quasilinear second order Kirchhoff equation with a nonlinear inhomogeneous term

Abdelhamid Hallouz[a,c,*], Svetlin Georgiev Georgiev[b], Braiki Mohamed Hocine[a,d], Zakaria Chedjara[e]

[a]*Department of Mathematics, Ibn Khaldoun University, Tiaret-Algeria*
[b]*Department of Mathematics, Sorbonne University, Paris, France*
[c]*Laboratory of Informatics and Mathematics, University of Tiaret, Algeria*
[d]*Laboratory of Analysis and Control of PDEs, Djillali Liabes University, Algeria*
[e]*Faculty of Applied Sciences, Ibn Khaldoun University, Tiaret-Algeria*

Abstract. The purpose of this paper is the study of the local existence of solutions to an abstract quasi-linear Kirchhoff equation with a nonlinear in-homogeneous term submitted to an internal viscous damping of fractional type. We establish the local existence using a method introduced by Kato[26] combined with the multiplier method and some fixed point argument.

1. Introduction

We are concerned in this paper by the study of the local existence of the following system

$$\begin{cases} \Psi''(t) + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A}\Psi(t) + \gamma\partial_t^{\alpha,\eta}\Psi(t) = f(\Psi(t)), & \text{in } [0, L], \\ \Psi(0) = \Psi_0, \ \Psi'(0) = \Psi_1, \end{cases} \tag{1}$$

where $\gamma > 0$, $\eta > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo fractional derivative of order, $0 < \alpha < 1$, with respect to the time variable t see [1, 7, 8, 14, 15]. It is given by the following formula

$$\partial_t^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} \frac{d}{d\tau} f(\tau) d\tau, \qquad \eta \geq 0.$$

Where $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1} e^{-t} dt.$

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* Corresponding author: Abdelhamid Hallouz
Email addresses: abdelhamidelpt@gmail.com (Abdelhamid Hallouz), svetlingeorgiev1@gmail.com (Svetlin Georgiev Georgiev), hocimath7@gmail.com (Braiki Mohamed Hocine), zakaria.chedjara@gmail.com (Zakaria Chedjara)
ORCID iDs: https://orcid.org/0009-0007-1272-032X (Abdelhamid Hallouz), https://orcid.org/0000-0001-8015-4226 (Svetlin Georgiev Georgiev), https://orcid.org/0009-0003-2104-7908 (Braiki Mohamed Hocine), https://orcid.org/0009-0004-8284-2152 (Zakaria Chedjara)

1.1. Literature

In recent years, special attention has been focused on fractional derivatives, both in their interpretation and as a non-local dissipation; for more details, see [2, 3, 12, 21, 22, 34, 37] and the references therein.

In [32, 33] B. Mbodje investigates the asymptotic behavior of solutions to the wave equation with a boundary viscoelastic damper of the fractional derivative type. He showed that the system is well-posed in the sense of a semi-group. He also proved that the associated semi-group is not exponentially stable, but only strongly.

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on} \quad (0,1)\times(0,+\infty), \\ u_x(0,t) = 0 & \text{in} \quad (0,+\infty), \\ u_x(1,t) = -\gamma\partial_t^{\beta,\eta}u(1,t) & \text{in} \quad (0,+\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in} \quad (0,1). \end{cases}$$

A great deal of attention was paid to the multi-dimensional case of this system, proving its result and improving it with polynomial or generic decay, as details may be found here[3, 9].

In [4] K. Ammari et al. give an extensive attention to the following system

$$\begin{cases} u'' + \mathcal{A}u + \mathcal{B}^*\mathcal{B}\partial_t^{\eta,\alpha}u = 0, \quad \text{in} \ (0,+\infty), \\ u(0) = u_0,\ u'(0) = u_1. \end{cases} \tag{2}$$

Where as before $\partial_t^{\eta,\alpha}$ denote the generalized Caputo fractional derivative. This system is a particular case of system (1) when $\phi \equiv 1$, $\mathcal{B}^*\mathcal{B} \equiv \gamma$ and $f \equiv 0$. The well-posedness is proved using semi-group theory, the authors remarked that this kind of viscous damping push the system to loss its exponential decay to zero, and provided an optimal polynomial decay rate of the solutions.
The following abstract system has been extensively studied in the literature

$$u'' + \phi(\|\mathcal{A}^{\frac{1}{2}}u\|^2)\mathcal{A}u + V(t) = U(t). \tag{3}$$

Using a method developed by Kato[26], the authors of [24] examined the local existence and blow-up of solutions in finite time when $V \equiv 0$ and $U(t) = f(u(t))$. Using the same methodology as previously, he introduced a viscose damping $\delta u'(t)$ to the system in[23]. The results were similar with a slight variation. The specific instance in which $V \equiv 0$ and $U \equiv 0$ are presented in [6]. System (3) with various dissipations and specific perturbations was the subject of extensive literature; for this, we can refer to [6, 16–20, 23, 25, 25, 26, 30] and the references therein. We can quote [5] and the references therein, as well as the extensive literature devoted to the study of the linear model of the system (3).

This work is organized as follows: In Section 2, we provide the necessary notations and hypotheses for the study. In Section 3, we transform system (1) as an augmented model to simplify the study. In section 4, we deal with the local existence of solutions to system (1). In Section 5, we present an example of application to illustrate our study. In Section 6, we recap this study, give some remarks and open problems.

2. Hypotheses and preliminaries

In this section, we prepare some hypotheses that will be needed in the proof of our result. Let $\mathcal{H}$ be a real Hilbert space, with inner product $< .,. >_{\mathcal{H}}$ and norm $\|.\|_{\mathcal{H}}$. Let $\mathcal{A}$ be a non-negative self-adjoint linear operator in $\mathcal{H}$, with domain $\mathcal{D}(\mathcal{A}) = \mathcal{V}$ endowed with the graph norm of $\mathcal{A}$, denoted $\|.\|_{\mathcal{V}}$, i.e., $\|u\|^2_{\mathcal{V}} = \|u\|_{\mathcal{H}} + \|\mathcal{A}u\|_{\mathcal{H}}$, is a real Hilbert space and its injection in $\mathcal{H}$ is continuous. The same is true for $\mathcal{A}^{\frac{1}{2}}$ with domain $\mathcal{W} = \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, also endowed with the graph norm $\|u\|^2_{\mathcal{W}} = \|u\|_{\mathcal{H}} + \|\mathcal{A}^{\frac{1}{2}}u\|_{\mathcal{H}}$ and for this graph $\mathcal{V}$ is dense in $\mathcal{W}$ (See [31]). We introduce the space $\widetilde{\mathcal{V}} := L^2(\mathbb{R}\times[0,T];\mathcal{H})$ and for $\theta, \widetilde{\theta} \in \widetilde{\mathcal{V}}$ we take its norm by

$$\|\theta\|^2_{\widetilde{\mathcal{V}}} := \int_{-\infty}^{+\infty} \|\theta(\tau,t)\|^2_{\mathcal{H}}\, d\tau.$$

and its inner-product is given by

$$< \theta, \widetilde{\theta} >_{\widetilde{\mathcal{V}}} = \int_{-\infty}^{+\infty} < \theta(\tau, t), \widetilde{\theta}(\tau, t) >_{\mathcal{H}} d\tau.$$

Assume that there exists $m_0 > 0$ and ϕ satisfies

$$\phi \in C^1[0, \infty) \quad \text{and} \quad \phi(s) \geq m_0 > 0. \tag{4}$$

Let f be a non-linear operator with domain $\mathcal{D}(f) = \{u \in \mathcal{H} | f(u) \in \mathcal{H}\}$, we assume further that

$$\mathcal{W} \subset \mathcal{D}(f), \quad f(0) = 0 \quad \text{and} \quad f(u) \in \mathcal{W} \quad \text{for any} \quad u \in \mathcal{V}. \tag{5}$$

For each $\nu > 0$, there exists $L_\nu > 0$ such that if

$$\begin{aligned} & \|\mathcal{A}^{\frac{1}{2}} u\|_{\mathcal{W}} \leq \nu, \quad \text{and} \quad \|\mathcal{A}^{\frac{1}{2}} v\|_{\mathcal{W}} \leq \nu, \quad \text{for any} \quad u, v \in \mathcal{V}, \quad \text{then} \\ & \|\mathcal{A}^{\frac{1}{2}} f(u)\|_{\mathcal{H}} \leq L_\nu, \\ & \|f(u) - f(v)\|_{\mathcal{H}} \leq L_\nu \|\mathcal{A}^{\frac{1}{2}} u - \mathcal{A}^{\frac{1}{2}} v\|_{\mathcal{H}}, \end{aligned} \tag{6}$$

$$|\tau|\theta \in L^2(\mathbb{R}, \mathcal{H}) \quad \text{Set} \quad \widetilde{\zeta} = \|\tau\theta\|_{\widetilde{\mathcal{V}}}. \tag{7}$$

Remark 2.1. *We can deduce from equations (5) and (6) that*

$$\|f(u)\|_{\mathcal{H}} \leq \nu L_\nu, \quad \text{for} \quad u \in \mathcal{V}, \quad \text{with} \quad \|\mathcal{A}^{\frac{1}{2}} v\|_{\mathcal{W}} \leq \nu. \tag{8}$$

Before treating the existence of system (1), we see it useful to define what we mean by a solution.

Definition 2.2. *A function $\mathcal{U} = (\Psi, \theta)^T : [0, L] \to \mathcal{H} \times \widetilde{\mathcal{V}}$ is called a solution to system (25) on $[0, L]$ if*

1. *$\Psi \in C([0, L]; \mathcal{V}) \cap C^1([0, L]; \mathcal{W}) \cap C^2([0, L]; \mathcal{H})$ and $\theta \in C^1([0, L]; \widetilde{\mathcal{V}})$.*
2. *$\mathcal{U}$ satisfies the first two equations in (25).*
3. *$\Psi(0) = \Psi_0,\ \Psi'(0) = \Psi_1,\ \theta(0) = \theta_0$.*

We recall the definition also of the Yosida approximation of $\mathcal{A}$ and its properties.

Lemma 2.3. **[13]** *Let $\mathcal{A}$ be a non-negative self-adjoint operator in $\mathcal{H}$, set $\mathcal{J}_\lambda = (\mathcal{I} + \lambda\mathcal{A})^{-1}$ for $\lambda > 0$ be a resolvent of $\mathcal{A}$, and $\mathcal{A}_\lambda = \mathcal{A}\mathcal{J}_\lambda$ be the Yosida approximation of $\mathcal{A}$. Then*

1. *$\|\mathcal{J}_\lambda\|_{\mathcal{H}} \leq 1$ and $\mathcal{J}_\lambda w \xrightarrow[\lambda \to 0]{} w$ in $\mathcal{H}$.*
2. *$\|\mathcal{A}_\lambda w\|_{\mathcal{H}} \leq \|\mathcal{A} w\|_{\mathcal{H}}$ and $\mathcal{A}_\lambda w \xrightarrow[\lambda \to 0]{} \mathcal{A} w$ in $\mathcal{H}$ for $w \in \mathcal{V}$.*

Lastly, we will briefly review Kato's theory, which is essential to the existence proof. We consult **[26, 27]** for further information. The Hilbert spaces X, Y have the norms $\|.\|_X$ and $\|.\|_Y$, respectively. For the linear equation, we are interested in the abstract Cauchy system in X.

$$\begin{cases} \dfrac{d}{dt} u(t) + Q(t) u(t) = f(t), \quad \text{On} \quad [0, L], \\ u(0) = u_0. \end{cases} \tag{9}$$

We denote by $\mathcal{B}(X, Y)$ the set of all bounded linear operators endowed with its norm. $\|.\|_{\mathcal{B}}$. By $\mathcal{G}(X, M, \beta)$ the set of all operators $\mathcal{A}$ in X such that $-\mathcal{A}$ generates a C_0-semi-group $(e^{-t\mathcal{A}})$ with

$$\|e^{-t\mathcal{A}}\|_X \leq M e^{\beta t}, \quad t \in [0, \infty).$$

We write $\mathcal{G}(X) = \cup\{\mathcal{G}(X, M, \beta) | M > 0, \beta \in \mathbb{R}\}$.

Definition 2.4. [28] *Let $(\mathcal{A}(t))_{0\leq t\leq L}$ be a family of operators in $\mathcal{G}(X)$. $(\mathcal{A}(t))_{0\leq t\leq L}$ is said to be stable with the stability index M,β if*

$$\left\|\Pi_{j=1}^{k}(\mathcal{A}(t_j)+\lambda)^{-1}\right\|_X \leq M(\lambda-\beta)^{-k}, \quad \text{for some} \quad M>0, \quad \beta<\lambda, \tag{10}$$

for every finite subdivision $0\leq t_1\leq t_2\leq ...\leq t_k\leq L$ of $[0,L]$, $k\in\mathbb{N}$.

It is worth noticing that the definition is hard to apply, and to decide whether a family is of this type, we shall give a characterization for the proof (see **[26, Proposition 3.4]**).

Proposition 2.5. [26] *For each $t\in[0,L]$ let $||.||_t$ ba a new norm in X equivalent to the original one, depending on t smoothly in the sense that*

$$\frac{||x||_t}{||x||_s}\leq e^{c|t-s|}, \quad x\in X \quad s,t\in[0,L].$$

Assume for each $t\in[0,L]$, $\mathcal{A}(t)\in\mathcal{G}(X,1,\beta)$, where X_t is the space X with norm $||.||_t$. Then $(\mathcal{A}(t))_{0\leq t\leq L}$ is stable, with the stability index $M=e^{2cL}$ and β with respect to $||.||_t$ for any $t\in[0,L]$.

The study of system (9) is reduced to the construction of the evolution operator $\S(t,s)\in\mathcal{B}(X)$, defined on the triangle $\Delta: 0\leq s\leq t\leq L$. Where $\{\mathcal{X}(t,s)\}$ is a family of operators such that $u(t)=\mathcal{X}(t,s)u_0$ is the solution of the homogeneous differential equation

$$\begin{cases} \dfrac{d}{dt}u(t)+Q(t)u(t)=0, \quad \text{On} \quad [s,L], \\ u(s)=u_0. \end{cases} \tag{11}$$

From this family, we can express the solution of system (9) as an integral equation given by

$$u(t)=\mathcal{X}(t,0)u_0+\int_0^t\mathcal{X}(t,s)f(s)ds. \tag{12}$$

at least in a formal way. The construction of the evolution family is based on the following theorem (see Kato[28]).

Theorem 2.6. [28] *Assume the following conditions holds*

- *$\{\mathcal{A}(t)\}_{0\leq t\leq L}$ is a stable family of operators in $\mathcal{G}(X)$ with the stability index M and β.*
- *There is a Hilbert space Y, continuously and densely embedded in X, and an isomorphism S of Y onto X, such that $S\mathcal{A}(t)S^{-1}=\mathcal{A}(t)$, $0\leq t\leq L$.*
- *$Y\subset\mathcal{V}$, $0\leq t\leq L$, so that $\mathcal{A}(t)\in\mathcal{B}(Y,X)$. The mapping $t\to\mathcal{A}(t)\in\mathcal{B}(Y,X)$ is continuous in the operator norm.*

Then there is a unique evolution operator $\{\mathcal{X}(t,s)\}$ defined on the triangle Δ such that

1. *$\mathcal{X}$ is strongly continuous on Δ to $\mathcal{B}(X)$.*
2. *$\mathcal{X}(t,s)\mathcal{X}(s,r)=\mathcal{X}(t,r)$ and $\mathcal{X}(s,s)=\mathcal{I}$.*
3. *$\mathcal{X}(t,s)Y\subset Y$ and $\mathcal{X}$ is strongly continuous on Δ to $\mathcal{B}(Y)$.*
4. *$\dfrac{d}{dt}\mathcal{X}(t,s)=-\mathcal{A}(t)\mathcal{X}(t,s)$, $\dfrac{d}{ds}\mathcal{X}(t,s)=\mathcal{X}(t,s)\mathcal{A}(s)$, which exist in the strong sense in $\mathcal{B}(Y,X)$ and are strongly continuous on Δ to $\mathcal{B}(Y,X)$.*

Furthermore we have the estimates for $\mathcal{X}(t,s)$:

$$\|\mathcal{X}\|_{\infty,X} \leq Me^{\beta L}, \tag{13}$$

and

$$\|\mathcal{X}\|_{\infty,Y} \leq \|S\|_{\mathcal{B}}\|S^{-1}\|_{\mathcal{B}}Me^{\beta L}, \tag{14}$$

where

$$\|\mathcal{X}\|_{\infty,X} := \sup\left\{\|\mathcal{X}(t,s)\|_{\infty,X}|t,s \in \Delta\right\}. \tag{15}$$

Let u be the function defined in equation (12). In order for u to be a strong solution, we require additional conditions on both u_0 and f; thus, the following theorem is required.

Theorem 2.7. **[27, 28]** *Let u be given by equation (12). If $u_0 \in Y$ and $f \in C([0,L];X) \cap L^1([0,L];Y)$, then $u \in C([0,L];Y) \cap C^1([0,L];X)$ and u satisfies (11) and the estimates*

$$\|u\|_{\infty,Y} \leq \|\mathcal{X}\|_{\infty,Y}\left(\|u_0\|_{\infty,Y} + \|f\|_{1,Y}\right), \tag{16}$$

where

$$\|u\|_{\infty,Y} := \sup\left\{\|u(t)\|_{\infty,X}|t \in [0,L]\right\} \quad \text{and} \quad \|f\|_{1,Y} := \int_0^L \|f(t)\|_Y dt. \tag{17}$$

Finally, we present the perturbation theory that will be used later (see Kato**[27, 28]**). Let us consider another equation

$$\begin{cases} \dfrac{d}{dt}v(t) + \widetilde{Q}(t)v(t) = g(t), \quad \text{On} \quad [0,L], \\ v(0) = u_0. \end{cases} \tag{18}$$

In order for the evolution operator $\widetilde{\mathcal{X}}(t,s)$ to exist and for the system (18) to be solvable, we assume that the family $\{\widetilde{Q}\}$ satisfies the requirements of Theorem 2.6 with the same Y and S. Then we have the following theorem.

Theorem 2.8. **[27, 28]***Let $u_0 \in Y$ and $f, g \in C([0,L];X) \cap L^1([0,L];Y)$. If u and v are the strong solutions of (9) and (18), respectively, then we have*

$$\|u - v\|_{\infty,X} \leq \|\widetilde{\mathcal{X}}\|_{\infty,Y}\left(\|f - g\|_{1,X} + \|\widetilde{Q}u - Qu\|_{1,X}\right). \tag{19}$$

3. The Augmented Model

This section is concerned with the reformulation of model (1) into an augmented system. For this, we need the following result.

Theorem 3.1. **[4]** *Let μ be the function*

$$\mu(\tau) = |\tau|^{\frac{2\alpha-1}{2}}. \tag{20}$$

The relationship between the system's input "U" and output "O" is then established.

$$\begin{aligned} \theta_t + (\tau^2 + \eta)\theta - \mathcal{U}(t)\mu(\tau) &= 0 \;\; \eta \geq 0 \;\; \mathbb{R} \times (0, +\infty), && (21) \\ \theta(\tau, 0) &= 0, && (22) \\ O(t) &= \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\tau)\theta d\tau, && (23) \end{aligned}$$

is given by

$$O(t) = \mathcal{I}^{1-\alpha,\eta}\mathcal{U}(t),$$

where

$$\mathcal{I}^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)d\tau.$$

Before transforming system (1) we give the following lemma which will be used in the coming sections.

Lemma 3.2. [4] *If* $\varrho \in D_\eta = \mathbb{C}\backslash]-\infty, -\eta[$, *then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\tau)}{\varrho + \tau^2 + \eta} d\tau = \frac{\zeta}{\gamma}(\varrho+\eta)^{\alpha-1}. \tag{24}$$

Here $\zeta = \gamma\frac{\sin(\alpha\pi)}{\pi}$.

At this stage we can use Theorem 3.1 and transform equivalently system(1) into

$$\begin{cases} \Psi''(t) + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A}\Psi(t) + \zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\tau)\theta(\tau,t)d\tau = f(\Psi(t)), & \text{in } [0,L], \\ \theta_t(\tau,t) + (\tau^2+\eta)\theta(\tau,t) - \Psi'(t)\mu(\tau) = 0, & \text{in } \mathbb{R}\times[0,L], \\ \Psi(0) = \Psi_0,\ \Psi'(0) = \Psi_1,\ \theta(\tau,0) = \theta_0. \end{cases} \tag{25}$$

4. Local existence

We will apply a technique presented in Kato**[28]** to demonstrate the local existence of solutions of system (25) in this section. The following is the local existence theorem.

Theorem 4.1. *(Existence and Uniqueness) Let* $\gamma \in \mathbb{R}$. *Suppose that conditions (5)-(6) are satisfied. Then for any* $\theta_0, \Psi_0 \in \mathcal{V}$ *and* $\Psi_1 \in \mathcal{W}$, *there exists a real number* L_0 *depending only on* $\|\theta_0\|_{\mathcal{V}}$, $\|\Psi_0\|_{\mathcal{V}}$ *and* $\|\Psi_1\|_{\mathcal{W}}$ *such that problem (25) has a unique solution* $\mathcal{U}(t) = (\Psi(t), \theta(t))^T$ *on* $[0, L_0]$.

The proof of this theorem will be a consequence of a series of lemmas, but before that we need to prepare some notations. Let $\nu > 0$ be an arbitrary constant satisfying

$$\nu \geq \max\left[\left[\frac{2}{\min\{1,\zeta,m_0\}}\left(\|\Psi_1\|^2_{\mathcal{W}} + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi_0\|^2_{\mathcal{H}})\left\|\mathcal{A}^{\frac{1}{2}}\Psi_0\right\|^2_{\mathcal{W}} + \zeta\|\theta_0\|^2_{\widetilde{\mathcal{V}}} + 1\right)\right]^{\frac{1}{2}}, \sqrt{\tau^2+\eta}\right]. \tag{26}$$

Set

$$\Lambda_0 := \max\left\{\phi(s) | 0 \leq s \leq \nu^2\right\}, \tag{27}$$

$$\Lambda_1 := \max\left\{|\phi'(s)| | 0 \leq s \leq \nu^2\right\}, \tag{28}$$

and

$$\Lambda_2 := \sqrt{\frac{\zeta^3}{\gamma\eta^{1-\alpha}}} \tag{29}$$

Let $L_\nu > 0$ be a constant, the existence of which is guaranteed by (6) and set

$$\omega_1 := \max\{L_\nu, \nu L_\nu\}, \tag{30}$$

and

$$\omega_2 := 2\left(\frac{2\Lambda_1\nu^2}{m_0} + \omega_1 + 2\eta\right). \tag{31}$$

Moreover, let $L_0 > 0$ be a constant such that

$$e^{\omega_2 L_0} \leq 2, \tag{32}$$

and

$$2\omega_1 L_0 \leq 1. \tag{33}$$

In order to use the method in Kato[28] we need to write system (25) as an abstract quasilinear evolution equation, and to do so we set $\mathcal{Y} = (\Psi, \varphi, \theta)^T$ where $\varphi = \Psi'$ and we write

$$\begin{cases} \mathcal{Y}'(t) + \Theta(\mathcal{Y}(t))\mathcal{Y}(t) = F(\mathcal{Y}(t)), \\ \mathcal{Y}(0) = \mathcal{Y}_0, \end{cases} \tag{34}$$

where $\mathcal{Y}_0 = (\Psi_0, \varphi_0, \theta_0)$, $F(\mathcal{Y}(t)) = (0, f(\Psi(t)), 0)^T$ and

$$\Theta(\mathcal{Y}(t)) = \begin{bmatrix} 0 & -\mathcal{I} & 0 \\ \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A} & 0 & \mathcal{G} \\ 0 & -\mu(\tau)\mathcal{I} & \left(\tau^2 + \eta\right)\mathcal{I} \end{bmatrix}.$$

Where $\mathcal{G}\theta = \zeta \int_{-\infty}^{+\infty} \mu(\tau)\theta d\tau$. We fix $X = \mathcal{W} \times \mathcal{H} \times \widetilde{\mathcal{V}}$ and $Y = \mathcal{V} \times \mathcal{W} \times \widetilde{\mathcal{V}}$ endowed with norms

$$\left\|\mathcal{Y}\right\|_X^2 = \|\Psi\|_{\mathcal{W}}^2 + \left\|\varphi\right\|_{\mathcal{H}}^2 + \zeta\|\theta\|_{\widetilde{\mathcal{V}}}^2, \quad \text{for} \quad \mathcal{Y} \in X,$$

$$\left\|\mathcal{Y}\right\|_Y^2 = \|\Psi\|_{\mathcal{V}}^2 + \left\|\varphi\right\|_{\mathcal{W}}^2 + \zeta\|\theta\|_{\widetilde{\mathcal{V}}}^2, \quad \text{for} \quad \mathcal{Y} \in X.$$

Let ν as in equation (26). Let us define the following set

$$\mathcal{K} = \left\{ \mathcal{N}(.) = \begin{bmatrix} \xi_1(.) \\ \xi_2(.) \\ \xi_3(.) \end{bmatrix} : [0, L_0] \to Y \;\middle|\; \begin{array}{l} \mathcal{N}(0) = \mathcal{Y}_0, \xi_1 \in C^1([0, L_0]; \mathcal{H}); \\ \left\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\right\|_{\mathcal{W}} \leq \nu, \left\|\xi_1'(t)\right\|_{\mathcal{W}} \leq \nu, \|\xi_3(t)\|_{\widetilde{\mathcal{V}}} \leq \nu; \\ \|\mathcal{N}(t) - \mathcal{N}(s)\|_X \leq L|t - s|; \end{array} \right\}. \tag{35}$$

Where

$$L = \nu\left[1 + (\Lambda_0 + \Lambda_2 + L_\nu)^2 + \zeta\left(\sqrt{2}(\frac{\widetilde{\zeta}}{\nu} + \eta) + \frac{\nu\Lambda_2}{\zeta}\right)^2\right]^{\frac{1}{2}}. \tag{36}$$

Let us now fixe $\mathcal{N} \in \mathcal{K}$ and consider the following linearized problem

$$\begin{cases} \mathcal{Y}'(t) + \Theta(\mathcal{N}(t))\mathcal{Y}(t) = F(\mathcal{N}(t)), \\ \mathcal{Y}(0) = \mathcal{Y}_0 \in Y. \end{cases} \tag{37}$$

Our aim now is to prove that system (37), has a unique mild solution $\mathcal{Y}(t)$ given by

$$\mathcal{Y}(t) = \mathcal{X}^{\mathcal{N}}(t, 0)\mathcal{Y}_0 + \int_0^t \mathcal{X}^{\mathcal{N}}(t, s)F(\mathcal{N}(s))ds, \tag{38}$$

where $\left\{\mathcal{X}^{\mathcal{N}}(t,s)|0 \le t \le s \le L_0\right\}$ is an evolution operator associated with the family $\{\Theta(\mathcal{N}(t))|0 \le t \le L_0\}$, consisting of linear operators, and to do so we should construct the above mentioned family. Since system (37) is linear, we are going to apply Theorem 2.6. First we induce X with a new norm and inner-product denoted $\|.\|_t$ and $< .,. >_t$, respectively, as follows:

$$\left\|\mathcal{Y}\right\|_t^2 = \left\|\mathcal{Y}\right\|_{\mathcal{N}(t)}^2 = \phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|_{\mathcal{H}}^2)\left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|_{\mathcal{H}}^2 + \|\Psi\|_{\mathcal{H}}^2 + \left\|\varphi\right\|_{\mathcal{H}}^2 + \zeta\left\|\theta\right\|_{\widetilde{\mathcal{V}}}^2 \quad \text{for} \quad \mathcal{Y} \in X. \tag{39}$$

We have the following lemma

Lemma 4.2. *There is a constant $\varsigma > 0$, independent of $\mathcal{N} \in \mathcal{K}$ and $t \in [0, L_0]$, such that*

$$\frac{1}{\varsigma}\left\|\mathcal{Y}\right\|_X \le \left\|\mathcal{Y}\right\|_t \le \varsigma\left\|\mathcal{Y}\right\|_X \quad \textit{for} \quad \mathcal{Y} \in \mathcal{X}.$$

Proof. From the fact that $\mathcal{N} \in \mathcal{K}$ and the equation (39), we get

$$\begin{aligned} \left\|\mathcal{Y}\right\|_t^2 &\le \Lambda_0\left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|_{\mathcal{H}}^2 + \|\Psi\|_{\mathcal{H}}^2 + \left\|\varphi\right\|_{\mathcal{H}}^2 + \zeta\left\|\theta\right\|_{\widetilde{\mathcal{V}}}^2 \\ &\le \max\{\Lambda_0, 1\}\left\|\mathcal{Y}\right\|_X^2. \end{aligned} \tag{40}$$

Also from equations (39) and (4), we have

$$\begin{aligned} \left\|\mathcal{Y}\right\|_t^2 &\ge m_0\left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|_{\mathcal{H}}^2 + \|\Psi\|_{\mathcal{H}}^2 + \left\|\varphi\right\|_{\mathcal{H}}^2 + \zeta\left\|\theta\right\|_{\widetilde{\mathcal{V}}}^2 \\ &\ge \min\{m_0, 1\}\left\|\mathcal{Y}\right\|_X^2. \end{aligned} \tag{41}$$

Now, we choose $\varsigma = \max\left\{\sqrt{\max\{\Lambda_0, 1\}}, \dfrac{1}{\sqrt{\min\{m_0, 1\}}}\right\}$ and obtain the desired inequality. □

Lemma 4.3. *Let $\mathcal{N} \in \mathcal{K}$, then we have the following estimate*

$$\frac{\|\mathcal{Y}\|_t}{\|\mathcal{Y}\|_s} \le e^{c|t-s|}, \quad \textit{for} \quad 0 \ne \mathcal{Y} \in X,$$

where $c = L\varsigma^2\nu\Lambda_1$.

Proof. Let us fix $\mathcal{N} \in \mathcal{K}$, then from equations (30), (39) and Lemma 4.2, we get

$$\begin{aligned} \left\|\mathcal{Y}\right\|_t^2 - \left\|\mathcal{Y}\right\|_s^2 &\le \left[\phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|_{\mathcal{H}}^2) - \phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(s)\|_{\mathcal{H}}^2)\right]\left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|_{\mathcal{H}}^2 \\ &\le \Lambda_1\left|\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|_{\mathcal{H}}^2 - \|\mathcal{A}^{\frac{1}{2}}\xi_1(s)\|_{\mathcal{H}}^2\right|\left\|\mathcal{Y}\right\|_X^2 \\ &\le 2\varsigma^2\nu\Lambda_1\left\|\mathcal{A}^{\frac{1}{2}}\xi_1(t) - \mathcal{A}^{\frac{1}{2}}\xi_1(s)\right\|_{\mathcal{H}}\left\|\mathcal{Y}\right\|_s^2 \\ &\le 2\varsigma^2\nu\Lambda_1\|\mathcal{N}(t) - \mathcal{N}(s)\|_X\left\|\mathcal{Y}\right\|_s^2 \\ &\le 2L\varsigma^2\nu\Lambda_1|t-s|\left\|\mathcal{Y}\right\|_s^2. \end{aligned} \tag{42}$$

and from this last inequality we can conclude. □

Lemma 4.4. *Let $\mathcal{N} \in \mathcal{K}$, then we have*

$$\Theta(\mathcal{N}(t)) \in \mathcal{G}(X, 1, \tfrac{1}{2} + \eta) \quad \textit{for all} \quad t \in [0, L_0].$$

Proof. Let us fix $\mathcal{N} \in \mathcal{K}$, $\lambda > \frac{1}{2} + \eta$, $\mathcal{Y} \in X$ and $\Theta(\mathcal{N}(t))$ is given by

$$\begin{bmatrix} 0 & -\mathcal{I} & 0 \\ \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A} & 0 & \mathcal{G} \\ 0 & -\mu(\tau)\mathcal{I} & \left(\tau^2 + \eta\right)\mathcal{I} \end{bmatrix}.$$

Where $\mathcal{G}\theta = \zeta \int_{-\infty}^{+\infty} \mu(\tau)\theta d\tau$.
Since $\Theta(\mathcal{N}(t))$ is densely defined and closed in X_t, then we should only prove that

$$\left\|(\Theta(\mathcal{N}(t)) + \lambda)^{-1}\mathcal{Y}\right\|_t \leq \frac{1}{\lambda - \left(\frac{1}{2} + \eta\right)} \left\|\mathcal{Y}\right\|_t, \tag{43}$$

we have the following

$$\begin{aligned} < \Theta(\mathcal{N}(t))\mathcal{Y} + \lambda\mathcal{Y}, \mathcal{Y} >_t \quad &= \phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}}) < \lambda\mathcal{A}^{\frac{1}{2}}\Psi - \mathcal{A}^{\frac{1}{2}}\varphi, \mathcal{A}^{\frac{1}{2}}\Psi >_{\mathcal{H}} \\ &+ < \lambda\Psi - \varphi, \Psi >_{\mathcal{H}} + \zeta < \left(\tau^2 + \eta\right)\theta - \mu(\tau)\varphi, \theta >_{\widetilde{\mathcal{V}}} \\ &+ < \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A}\Psi + \lambda\varphi + \zeta \int_{-\infty}^{+\infty} \mu(\tau)\theta d\tau, \varphi >_{\mathcal{H}} \\ &= \lambda\|\mathcal{Y}\|^2_t + \zeta \int_{-\infty}^{+\infty} \left(\tau^2 + \eta\right)\|\theta\|^2_{\mathcal{H}}\, d\tau - < \Psi, \varphi >_{\mathcal{H}} \\ &\geq \lambda\|\mathcal{Y}\|^2_t + \zeta \int_{-\infty}^{+\infty} \left(\tau^2 + \eta\right)\|\theta\|^2_{\mathcal{H}}\, d\tau - \|\Psi\|_{\mathcal{H}} \left\|\varphi\right\|_{\mathcal{H}}. \end{aligned} \tag{44}$$

We have at first that

$$\eta\,\|\theta\|^2_{\widetilde{\mathcal{V}}} \leq \int_{-\infty}^{+\infty} \left(\tau^2 + \eta\right)\|\theta\|^2_{\mathcal{H}}\, d\tau. \tag{45}$$

Using equation (45) and Yong's inequality, we obtain

$$\begin{aligned} -\zeta \int_{-\infty}^{+\infty} \left(\tau^2 + \eta\right)\|\theta\|^2_{\mathcal{H}}\, d\tau + \|\Psi\|_{\mathcal{H}} \left\|\varphi\right\|_{\mathcal{H}} \quad &\leq \eta\zeta\,\|\theta\|^2_{\widetilde{\mathcal{V}}} + \tfrac{1}{2}(\|\Psi\|^2_{\mathcal{H}} + \left\|\varphi\right\|^2_{\mathcal{H}}) \\ &\leq \left(\tfrac{1}{2} + \eta\right)\|\mathcal{Y}\|^2_t. \end{aligned} \tag{46}$$

From equations (44) and (46), we obtain

$$< \Theta(\mathcal{N}(t))\mathcal{Y} + \lambda\mathcal{Y}, \mathcal{Y} >_t \geq \left(\lambda - \left(\frac{1}{2} + \eta\right)\right)\|\mathcal{Y}\|^2_t. \tag{47}$$

From equation (47) we deduce equation (43). Lemma 4.4 follows as an application of the Hille-Yosida theorem (see [36]). □

Lemma 4.5. *The family $\left\{\mathcal{X}^{\mathcal{N}}(t,s) | 0 \leq t \leq s \leq L_0\right\}$ is stable in the sense of Definition 2.4 with stability constants $M = \varsigma^2 e^{2cL_0}$ and $\beta = \frac{1}{2} + \eta$, where c is the constant defined in Lemma 4.3*

Proof. This lemma follows as a consequence of Lemma 4.3, Lemma 4.4 and Proposition 2.5. □

We give next a lemma, the proof of which is a straightforward

Lemma 4.6. *Let $\mathcal{S} : Y \to X$ be an operator defined by*

$$\mathcal{S} = \begin{bmatrix} (\mathcal{I} + \mathcal{A})^{\frac{1}{2}} & 0 & 0 \\ 0 & (\mathcal{I} + \mathcal{A})^{\frac{1}{2}} & 0 \\ 0 & 0 & (\mathcal{I} + \mathcal{A})^{\frac{1}{2}} \end{bmatrix}. \tag{48}$$

Then $\mathcal{S}$ is an isomorphism between Y and X and satisfies

$$\mathcal{S}\Theta(\mathcal{N}(t))\mathcal{S}^{-1} = \Theta(\mathcal{N}(t)), \quad \textit{for any} \quad \mathcal{N} \in \mathcal{K} \quad \textit{and} \quad t \in [0, L_0].$$

Lemma 4.7. *Set $\varrho = 2\nu\Lambda_1$, let $\mathcal{N} \in \mathcal{K}$ and $t, s \in [0, L_0]$. Then we have the following estimate*

$$\|\Theta(\mathcal{N}(t)) - \Theta(\mathcal{N}(s))\|_{\mathcal{B}} \leq \varrho \|\mathcal{N}(t) - \mathcal{N}(s)\|_X .$$

Proof. Let $\mathcal{N} \in \mathcal{K}$ and $\mathcal{Y} \in Y$. From the calculation in equation (42), then we have

$$\begin{aligned} \left\|\Theta(\mathcal{N}(t))\mathcal{Y} - \Theta(\mathcal{N}(s))\mathcal{Y}\right\|_X &= \left|\left(\phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}}) - \phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(s)\|^2_{\mathcal{H}})\right)\mathcal{A}\Psi\right| \\ &\leq \left|\left(\phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}}) - \phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(s)\|^2_{\mathcal{H}})\right)\right| \|\mathcal{Y}\|_Y \\ &\leq 2\nu\Lambda_1 \|\mathcal{N}(t) - \mathcal{N}(s)\|_X . \end{aligned} \tag{49}$$

Thus, we obtain the desired estimate. □

The construction of the family $\left\{\mathcal{X}^{\mathcal{N}}(t,s) | 0 \leq t \leq s \leq L_0\right\}$ associated with the family $\{\Theta(\mathcal{N}(t)) | 0 \leq t \leq L_0\}$ is now assured by Theorem 2.6 Lemmas 4.2–4.7 and the last equation in (35), and then $\mathcal{Y}(t)$ given by equation (38) is the unique mild solution to system (37). Our task now is to prove that $\mathcal{Y}(t)$ is in fact a strong solution to problem (37), and to this end we have the following lemma

Lemma 4.8. *Let $\mathcal{N} \in \mathcal{K}$, the we have the following estimates*

$$\|F(\mathcal{N}(t)) - F(\mathcal{N}(s))\|_X \leq L_\nu \|\mathcal{N}(t) - \mathcal{N}(s)\|_X , \tag{50}$$

and

$$\|F(\mathcal{N}(t))\|_X \leq \widetilde{\varrho}_\nu \quad \textit{with} \quad \widetilde{\varrho}_\nu = L_\nu \sqrt{1+\nu^2}. \tag{51}$$

Proof. Let $\mathcal{N} \in \mathcal{K}$. From equation (6), we obtain

$$\begin{aligned} \|F(\mathcal{N}(t)) - F(\mathcal{N}(s))\|_X &= \left\|f(\xi_1(t)) - f(\xi_1(s))\right\|_{\mathcal{H}} \\ &\leq L_\nu \left\|\mathcal{A}^{\frac{1}{2}}\xi_1(t) - \mathcal{A}^{\frac{1}{2}}\xi_1(s)\right\|_{\mathcal{H}} \\ &\leq L_\nu \|\mathcal{N}(t) - \mathcal{N}(s)\|_X . \end{aligned} \tag{52}$$

Likewise, we obtain

$$\begin{aligned} \|F(\mathcal{N}(t))\|_X &= \left\|f(\xi_1(t))\right\|_{\mathcal{H}} \\ &= \left(\left\|\mathcal{A}^{\frac{1}{2}} f(\xi_1(t))\right\|^2_{\mathcal{H}} + \left\|f(\xi_1(t))\right\|^2_{\mathcal{H}}\right)^{\frac{1}{2}} \\ &\leq \left(L_\nu^2 + (\nu L_\nu)^2\right)^{\frac{1}{2}} . \end{aligned} \tag{53}$$

□

We can observe that Lemma 4.8 asserts that $F(\mathcal{N}(.)) \in C([0, L_0]; X) \cap L^\infty([0, L_0]; Y)$ for any $\mathcal{N} \in \mathcal{K}$. Then Theorem 2.7 claims that $\mathcal{Y}(t)$ given by equation (38) is in fact a strong solution and it satisfies

$$\mathcal{Y}(.) \in C([0, L_0]; Y) \cap C^1([0, L_0]; X). \tag{54}$$

Let us now, define

$$\Psi : \mathcal{K} \to X, \tag{55}$$

by $\mathcal{Y} = \Psi\mathcal{N}$. We are going to show that $\Psi(\mathcal{K}) \subset \mathcal{K}$, and to this end we need the following lemma

Lemma 4.9. *Let $\mathcal{Y}$ be a solution to problem (37) for any $\mathcal{N} \in \mathcal{K}$ and $t \in [0, L_0]$, then the following inequality holds*

$$\widetilde{\mathcal{I}}(\mathcal{Y}) = \|\Psi'\|^2_{\mathcal{W}} + m_0 \left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|^2_{\mathcal{W}} + \zeta \|\theta\|^2_{\widetilde{\mathcal{V}}} \leq \min\{1, \zeta, m_0\}\nu^2. \tag{56}$$

Proof. Let $\mathcal{Y}$ be the solution to system (37). Then it satisfies

$$\begin{cases} \Psi''(t) + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A}\Psi(t) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\tau)\theta(\tau,t)d\tau = f(\xi_1(t)), & \text{in } [0,L_0], \\ \theta_t(\tau,t) + (\tau^2+\eta)\theta(\tau,t) - \varphi(t)\mu(\tau) = 0, & \text{in } \mathbb{R}\times[0,L_0], \\ \Psi(0) = \Psi_0,\ \Psi'(0) = \Psi_1,\ \theta(\tau,0) = \theta_0. \end{cases} \tag{57}$$

Multipying equation $(57)_1$ by $2\Psi'$, we get

$$\begin{cases} \dfrac{d}{dt}\left[\|\Psi'\|^2_{\mathcal{H}} + \phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}})\left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|^2_{\mathcal{H}}\right] = -2\zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\tau) < \theta(\tau,t),\Psi' >_{\mathcal{H}} d\tau \\ +2 < f(\xi_1(t)),\Psi' >_{\mathcal{H}} +2\phi'(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}}) < \mathcal{A}^{\frac{1}{2}}\xi_1(t),\mathcal{A}^{\frac{1}{2}}\xi_1'(t) >_{\mathcal{H}} \left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|^2_{\mathcal{H}}. \end{cases} \tag{58}$$

Multipying equation $(57)_2$ by $2\zeta\theta$, we get

$$\zeta\frac{d}{dt}\|\theta\|^2_{\widetilde{\mathcal{V}}} = -2\zeta\int_{-\infty}^{+\infty}(\tau^2+\eta)\|\theta(t)\|^2_{\mathcal{H}}d\tau + 2\zeta\int_{-\infty}^{+\infty}\mu(\tau) < \theta(\tau,t),\Psi' >_{\mathcal{H}} d\tau. \tag{59}$$

Set

$$\begin{aligned} &\widetilde{\mathcal{D}}(\mathcal{Y}(t)) = \|\Psi'\|^2_{\mathcal{H}} + \phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}})\left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|^2_{\mathcal{H}} + \zeta\|\theta\|^2_{\widetilde{\mathcal{V}}}. \\ &\widetilde{\mathcal{D}}(\mathcal{Y}(0)) = \|\Psi_1\|^2_{\mathcal{H}} + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi_0\|^2_{\mathcal{H}})\left\|\mathcal{A}^{\frac{1}{2}}\Psi_0\right\|^2_{\mathcal{H}} + \zeta\|\theta_0\|^2_{\mathcal{H}}. \end{aligned} \tag{60}$$

Take the sum of equations (58), (59) and the consideration of equation (60), we get

$$\begin{cases} \dfrac{d}{dt}\widetilde{\mathcal{D}}(\mathcal{Y}(t)) = 2 < f(\xi_1(t)),\Psi' >_{\mathcal{H}} +2\phi'(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}}) < \mathcal{A}^{\frac{1}{2}}\xi_1(t),\mathcal{A}^{\frac{1}{2}}\xi_1'(t) >_{\mathcal{H}} \left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|^2_{\mathcal{H}} \\ -2\zeta\displaystyle\int_{-\infty}^{+\infty}(\tau^2+\eta)\|\theta(t)\|^2_{\mathcal{H}}d\tau. \end{cases} \tag{61}$$

Integrating equation (61) on $[0,t]$, taking into account the hypotheses imposed on ϕ, equation(6), equation (45) and the fact that $\mathcal{N} \in \mathcal{K}$, and by the help of Cauchy-Schwartz inequality, we get

$$\begin{aligned} \widetilde{\mathcal{D}}(\mathcal{Y}(t)) \ &\le \|\Psi_1\|^2_{\mathcal{H}} + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi_0\|^2_{\mathcal{H}})\left\|\mathcal{A}^{\frac{1}{2}}\Psi_0\right\|^2_{\mathcal{H}} + \zeta\|\theta_0\|^2_{\widetilde{\mathcal{V}}} + 2\int_0^t \left\|f(\xi_1(s)\right\|_{\mathcal{H}}\|\Psi'(s)\|_{\mathcal{H}}ds \\ &+2\int_0^t \phi'(\|\mathcal{A}^{\frac{1}{2}}\xi_1(s)\|^2_{\mathcal{H}}) < \mathcal{A}^{\frac{1}{2}}\xi_1(s),\mathcal{A}^{\frac{1}{2}}\xi_1'(s) >_{\mathcal{H}} \left\|\mathcal{A}^{\frac{1}{2}}\Psi(s)\right\|^2_{\mathcal{H}} ds \\ &+2\eta\zeta\int_0^t \|\theta\|^2_{\widetilde{\mathcal{V}}} ds \\ &\le \widetilde{\mathcal{D}}(\mathcal{Y}(0)) + 2\Lambda_1\nu^2\int_0^t \left\|\mathcal{A}^{\frac{1}{2}}\Psi(s)\right\|^2_{\mathcal{H}} ds + \nu L_\nu\int_0^t\left(1+\|\Psi'(s)\|^2_{\mathcal{H}}\right)ds \\ &+2\eta\zeta\int_0^t \|\theta\|^2_{\widetilde{\mathcal{V}}} ds \\ &\le \widetilde{\mathcal{D}}(\mathcal{Y}(0)) + \frac{2\Lambda_1\nu^2}{m_0}\int_0^t \widetilde{\mathcal{D}}(\mathcal{Y}(s))ds + \nu L_\nu\left(t + \int_0^t \widetilde{\mathcal{D}}(\mathcal{Y}(s))ds\right) + 2\eta\int_0^1 \widetilde{\mathcal{D}}(\mathcal{Y}(s))ds \\ &\le \widetilde{\mathcal{D}}(\mathcal{Y}(0)) + 2\nu L_\nu L_0 + \left(\frac{2\Lambda_1\nu^2}{m_0} + \nu L_\nu + 2\eta\right)\int_0^t \widetilde{\mathcal{D}}(\mathcal{Y}(s))ds. \end{aligned} \tag{62}$$

Now, we operate $\mathcal{J}_\lambda = (\mathcal{I} + \lambda\mathcal{A})^{-1}$, introduced in Lemma 2.3, on both sides of equations $(57)_1$ and $(57)_2$ to obtain

$$\begin{cases} \mathcal{J}_\lambda\Psi''(t) + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{J}_\lambda\mathcal{A}\Psi(t) + \zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\tau)\mathcal{J}_\lambda\theta(\tau,t)d\tau = \mathcal{J}_\lambda f(\xi_1(t)), \\ \mathcal{J}_\lambda\theta_t(\tau,t) + (\tau^2+\eta)\mathcal{J}_\lambda\theta(\tau,t) - \mathcal{J}_\lambda\varphi(t)\mu(\tau) = 0, \end{cases} \tag{63}$$

Multipying equation (63)$_1$ by $2\mathcal{A}_\lambda\Psi'$, we get

$$\left\{\begin{array}{l}\frac{d}{dt}\left[\left\|\mathcal{A}^{\frac{1}{2}}\mathcal{J}_\lambda\Psi'(t)\right\|^2_{\mathcal{H}}+\phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}})\,\|\mathcal{A}_\lambda\Psi(t)\|^2_{\mathcal{H}}\right]=-2\zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\tau)<\mathcal{J}_\lambda\theta(\tau,t),\Psi'>_{\mathcal{H}}d\tau\\ +2<\mathcal{J}_\lambda f(\xi_1(t)),\mathcal{A}_\lambda\Psi'(t)>_{\mathcal{H}}+2\phi'(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}})<\mathcal{A}^{\frac{1}{2}}\xi_1(t),\mathcal{A}^{\frac{1}{2}}\xi_1'(t)>_{\mathcal{H}}\|\mathcal{A}_\lambda\Psi(t)\|^2_{\mathcal{H}}.\end{array}\right. \tag{64}$$

Multiplying equation (63)$_2$ by $2\zeta\mathcal{J}_\lambda\theta$, we get

$$\zeta\frac{d}{dt}\|\mathcal{J}_\lambda\theta\|^2_{\mathcal{V}}=-2\zeta\int_{-\infty}^{+\infty}(\tau^2+\eta)\|\mathcal{J}_\lambda\theta(t)\|^2_{\mathcal{H}}d\tau+2\zeta\int_{-\infty}^{+\infty}\mu(\tau)<\mathcal{J}_\lambda\theta(\tau,t),\Psi'>_{\mathcal{H}}d\tau. \tag{65}$$

Set

$$\begin{array}{l}\widetilde{C}_\lambda(\mathcal{Y}(t))=\left\|\mathcal{A}^{\frac{1}{2}}\mathcal{J}_\lambda\Psi'(t)\right\|^2_{\mathcal{H}}+\phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}})\,\|\mathcal{A}_\lambda\Psi(t)\|^2_{\mathcal{H}}+\zeta\,\|\mathcal{J}_\lambda\theta(t)\|^2_{\mathcal{V}}.\\ \widetilde{C}(\mathcal{Y}(t))=\left\|\mathcal{A}^{\frac{1}{2}}\Psi'(t)\right\|^2_{\mathcal{H}}+\phi(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}})\,\|\mathcal{A}\Psi(t)\|^2_{\mathcal{H}}+\zeta\,\|\theta(t)\|^2_{\mathcal{V}}.\\ \widetilde{C}_\lambda(\mathcal{Y}(0))=\left\|\mathcal{A}^{\frac{1}{2}}\mathcal{J}_\lambda\Psi_1\right\|^2_{\mathcal{H}}+\phi(\|\mathcal{A}^{\frac{1}{2}}\Psi_0\|^2_{\mathcal{H}})\,\|\mathcal{A}_\lambda\Psi_0\|^2_{\mathcal{H}}+\zeta\,\|\mathcal{J}_\lambda\theta_0\|^2_{\mathcal{V}}.\\ \widetilde{C}(\mathcal{Y}(0))=\left\|\mathcal{A}^{\frac{1}{2}}\Psi_1\right\|^2_{\mathcal{H}}+\phi(\|\mathcal{A}^{\frac{1}{2}}\Psi_0\|^2_{\mathcal{H}})\,\|\mathcal{A}\Psi_0\|^2_{\mathcal{H}}+\zeta\,\|\theta_0\|^2_{\mathcal{V}}.\end{array} \tag{66}$$

Remark 4.10. *As a consequence of Lemma 2.3, we have* $\widetilde{C}_\lambda(\mathcal{Y}(t))\underset{\lambda\to 0}{\longrightarrow}\widetilde{C}(\mathcal{Y}(t))$, *and* $\widetilde{C}_\lambda(\mathcal{Y}(0))\underset{\lambda\to 0}{\longrightarrow}\widetilde{C}(\mathcal{Y}(0))$.

Take the sum of equations (64), (65) and the consideration of equation (60), we get

$$\left\{\begin{array}{l}\frac{d}{dt}\widetilde{C}_\lambda(\mathcal{Y}(t))=2<\mathcal{J}_\lambda f(\xi_1(t)),\mathcal{A}_\lambda\Psi'(t)>_{\mathcal{H}}-2\zeta\displaystyle\int_{-\infty}^{+\infty}(\tau^2+\eta)\|\mathcal{J}_\lambda\theta(t)\|^2_{\mathcal{H}}d\tau\\ +2\phi'(\|\mathcal{A}^{\frac{1}{2}}\xi_1(t)\|^2_{\mathcal{H}})<\mathcal{A}^{\frac{1}{2}}\xi_1(t),\mathcal{A}^{\frac{1}{2}}\xi_1'(t)>_{\mathcal{H}}\|\mathcal{A}_\lambda\Psi(t)\|^2_{\mathcal{H}}.\end{array}\right. \tag{67}$$

Integrating equation (67) on $[0,t]$, taking into account the hypotheses imposed on ϕ, equations (6) and (45), Lemma 2.3 and the fact that $\mathcal{N}\in\mathcal{K}$ and by the help of Cauchy-Schwartz inequality, we get

$$\begin{array}{rl}\widetilde{C}_\lambda(\mathcal{Y}(t)) & \leq\left\|\mathcal{A}^{\frac{1}{2}}\mathcal{J}_\lambda\Psi_1\right\|^2_{\mathcal{H}}+\phi(\|\mathcal{A}^{\frac{1}{2}}\Psi_0\|^2_{\mathcal{H}})\,\|\mathcal{A}_\lambda\Psi_0\|^2_{\mathcal{H}}+\zeta\,\|\mathcal{J}_\lambda\theta_0\|^2_{\mathcal{V}}\\ & +2\displaystyle\int_0^t\left\|\mathcal{A}^{\frac{1}{2}}f(\xi_1(s)\right\|_{\mathcal{H}}\left\|\mathcal{A}^{\frac{1}{2}}\Psi'(s)\right\|_{\mathcal{H}}ds\\ & +2\displaystyle\int_0^t\phi'(\|\mathcal{A}^{\frac{1}{2}}\xi_1(s)\|^2_{\mathcal{H}})<\mathcal{A}^{\frac{1}{2}}\xi_1(s),\mathcal{A}^{\frac{1}{2}}\xi_1'(s)>_{\mathcal{H}}\|\mathcal{A}\Psi(s)\|^2_{\mathcal{H}}ds\\ & +2\eta\zeta\displaystyle\int_0^t\|\mathcal{J}_\lambda\theta\|^2_{\mathcal{V}}ds\\ & \leq\widetilde{C}_\lambda(\mathcal{Y}(0))+2\Lambda_1\nu^2\displaystyle\int_0^t\|\mathcal{A}\Psi(s)\|^2_{\mathcal{H}}ds+L_\nu\int_0^t\left(1+\left\|\mathcal{A}^{\frac{1}{2}}\Psi'(s)\right\|^2_{\mathcal{H}}\right)ds\\ & +2\eta\zeta\displaystyle\int_0^t\|\mathcal{J}_\lambda\theta\|^2_{\mathcal{V}}ds\\ & \leq\widetilde{C}_\lambda(\mathcal{Y}(0))+L_\nu\left(t+\displaystyle\int_0^t\widetilde{C}_\lambda(\mathcal{Y}(s))ds\right)+2\eta\int_0^t\widetilde{C}_\lambda(\mathcal{Y}(s))ds\\ & +\dfrac{2\Lambda_1\nu^2}{m_0}\displaystyle\int_0^1\widetilde{C}_\lambda(\mathcal{Y}(s))ds\\ & \leq\widetilde{C}_\lambda(\mathcal{Y}(0))+2\nu L_\nu L_0+\left(\dfrac{2\Lambda_1\nu^2}{m_0}+L_\nu+2\eta\right)\displaystyle\int_0^t\widetilde{C}_\lambda(\mathcal{Y}(s))ds.\end{array} \tag{68}$$

Letting $\lambda\to 0$ in equation (68) and Remark 4.10, we get

$$\widetilde{C}(\mathcal{Y}(t))\leq\widetilde{C}(\mathcal{Y}(0))+2\nu L_\nu L_0+\left(\frac{2\Lambda_1\nu^2}{m_0}+L_\nu+2\eta\right)\int_0^t\widetilde{C}(\mathcal{Y}(s))ds. \tag{69}$$

From equations (30), (5), (62) and (69), we get

$$\begin{aligned}\widetilde{\mathcal{I}}(\mathcal{Y}) &\leq \widetilde{\mathcal{D}}(\mathcal{Y}(t)) + \widetilde{C}(\mathcal{Y}(t)) \\ &\leq \widetilde{\mathcal{D}}(\mathcal{Y}(0)) + \widetilde{C}(\mathcal{Y}(0)) + 2\omega_1 L_0 + 2\left(\frac{2\Lambda_1 \nu^2}{m_0} + \omega_1 + 2\eta\right)\int_0^t \left(\widetilde{\mathcal{D}}(\mathcal{Y}(s)) + \widetilde{C}(\mathcal{Y}(s))\right) ds.\end{aligned} \tag{70}$$

By Gronwall's lemma, equations (32) and (33), we get

$$\begin{aligned}\widetilde{\mathcal{I}}(\mathcal{Y}) &\leq \left(\widetilde{\mathcal{D}}(\mathcal{Y}(0)) + \widetilde{C}(\mathcal{Y}(0)) + 2\omega_1 L_0\right) e^{\omega_2 t} \\ &\leq \frac{\min\{1,\zeta,m_0\}}{2}\nu^2 e^{\omega_2 L_0}.\end{aligned} \tag{71}$$

Thus, we attained the desired inequality. □

Lemma 4.11. *Let Ψ be the mapping defined in equation (55). Then Ψ maps $\mathcal{K}$ to itself.*

Proof. Let $\mathcal{Y}$ be the solution to system (37) and $\mathcal{N} \in \mathcal{K}$, our aim is to show that $\Psi\mathcal{N} = \mathcal{Y} \in \mathcal{K}$. From Lemma 4.11, we get

$$\|\Psi'\|_{\mathcal{W}}^2 + \left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|_{\mathcal{W}}^2 + \|\theta\|_{\widetilde{\mathcal{V}}}^2 \leq \nu^2. \tag{72}$$

Hence,

$$\|\Psi'\|_{\mathcal{W}} \leq \nu, \quad \left\|\mathcal{A}^{\frac{1}{2}}\Psi\right\|_{\mathcal{W}} \leq \nu, \quad \|\theta\|_{\widetilde{\mathcal{V}}} \leq \nu. \tag{73}$$

From equations (24), (73) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\zeta\int_{-\infty}^{+\infty}\mu(\tau)\,\|\theta\|_H\,d\tau &\leq \zeta\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\tau)}{\tau^2+\eta}\right)^{\frac{1}{2}}\|\theta\|_{\widetilde{\mathcal{V}}} \\ &\leq \Lambda_2\nu.\end{aligned} \tag{74}$$

From equations (7), (73) and the inequality $2(a^2+b^2) \geq (a+b)^2$, we obtain

$$\begin{aligned}\left(\int_{-\infty}^{+\infty}\left(\tau^2+\eta\right)^2\|\theta(\tau,t)\|_{\mathcal{H}}^2\,d\tau\right)^{\frac{1}{2}} &\leq \left(2\int_{-\infty}^{+\infty}\tau^2\,\|\theta(\tau,t)\|_{\mathcal{H}}^2\,d\tau + 2\eta\,\|\theta(t)\|_{\widetilde{\mathcal{V}}}^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{2}(\widetilde{\zeta}+\eta\nu).\end{aligned} \tag{75}$$

Now, we should show that $\left\|\mathcal{Y}(t)-\mathcal{Y}(s)\right\|_X \leq L|t-s|$, where L is defined in equation (36). From equations (28), (6), (45), (75), (26) and (73), we obtain

$$\begin{aligned}\left\|\mathcal{Y}'(t)\right\|_X &= \left[\|\Psi'(t)\|_{\mathcal{W}}^2 + \|\Psi''(t)\|_{\mathcal{H}}^2 + \zeta\,\|\theta'(t)\|_{\widetilde{\mathcal{V}}}^2\right]^{\frac{1}{2}} \\ &\leq \left[\nu^2 + \left\|-\phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A}\Psi(t) - \zeta\int_{-\infty}^{+\infty}\mu(\tau)\theta(\tau,t)d\tau + f(\xi_1(t))\right\|_{\mathcal{H}}^2\right. \\ &\quad \left. +\zeta\left\|-(\tau^2+\eta)\theta(\tau,t) + \Psi'(t)\mu(\tau)\right\|_{\widetilde{\mathcal{V}}}^2\right]^{\frac{1}{2}} \\ &\leq \left[\nu^2 + (\Lambda_0\nu + \Lambda_2\nu + \nu L_\nu)^2 + \zeta\left(\sqrt{2}(\widetilde{\zeta}+\eta\nu) + \tfrac{\nu^2\Lambda_2}{\zeta}\right)^2\right]^{\frac{1}{2}} = L.\end{aligned} \tag{76}$$

Thus we have from equation (76)

$$\left\|\mathcal{Y}(t)-\mathcal{Y}(s)\right\|_X \leq \left\|\int_s^t \mathcal{Y}'(t)\right\|_X \leq L|t-s|. \tag{77}$$

From equations (54), (73) and (76) we conclude that $\mathcal{Y}(.) \in \mathcal{K}$ and this achieve the proof. □

Let $\mathcal{S}$ be the operator introduced in equation (48). Set $\eta_1 = \|\mathcal{S}\|_{\mathcal{B}}$, $\eta_2 = \|\mathcal{S}^{-1}\|_{\mathcal{B}}$ and

$$C(L_0) = \varsigma^2 \mathcal{E}(c, L_0)\left(L_\nu + \varrho\eta_1\eta_2\varsigma^2\mathcal{E}(c, L_0)\left(\|\mathcal{Y}_0\|_Y + \widetilde{\varrho}_\nu L_0\right)\right)L_0.$$

Where $\widetilde{\varrho}_\nu = L_\nu\sqrt{1+\nu^2}$, $\mathcal{E}(c, L_0) = e^{(2c+\frac{1}{2}+\eta)L_0}$, $\varrho = 2\nu\Lambda_1$ and
$c = L\varsigma^2\nu\Lambda_1$, in the sequel we will assume further that L_0 satisfy

$$C(L_0) < 1. \tag{78}$$

Lemma 4.12. *Let $\Psi : \mathcal{K} \to \mathcal{K}$ be the mapping defined in equation (55). Assume also that condition (78) holds. Then Ψ is a strict contraction with respect to the metric of $\mathcal{K}$ defined by*

$$d(\mathcal{N}_1, \mathcal{N}_2) := \sup\{\|\mathcal{N}_1(t) - \mathcal{N}_2(t)\| \,|0 \le t \le L_0\}, \quad \textit{for} \quad \mathcal{N}_1, \mathcal{N}_2 \in \mathcal{K}. \tag{79}$$

Proof. Let $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{K}$. From equation(19) of Theorem 2.8, we get

$$\begin{aligned} d(\Psi\mathcal{N}_1, \Psi\mathcal{N}_2) \;&\le \|\mathcal{X}^{\mathcal{N}_2}\|_{\infty,Y}\|F(\mathcal{N}_1(.)) - F(\mathcal{N}_2(.))\|_{1,X} \\ &+\|\mathcal{X}^{\mathcal{N}_2}\|_{\infty,Y}\|(\Theta(\mathcal{N}_1(.)) - \Theta(\mathcal{N}_2(.)))\,\Psi\mathcal{N}_1(.)\|_{1,X}\,. \end{aligned} \tag{80}$$

Here $\{\mathcal{X}^{\mathcal{N}_2}(t,s)\}$ is an evolution operator associated with the family $\{\Theta(\mathcal{N}_2(.))\}$ of generators. From Lemma 4.5 and equation (13), we obtain

$$\|\mathcal{X}^{\mathcal{N}_2}\|_{\infty,Y} \le e^{(2c+\frac{1}{2}+\eta)L_0} = \mathcal{E}(c, L_0). \tag{81}$$

From Lemma 4.7 and equation (50), we obtain

$$\|F(\mathcal{N}_1(t)) - F(\mathcal{N}_2(t))\|_X \le L_\nu d(\mathcal{N}_1, \mathcal{N}_2)\,. \tag{82}$$

and

$$\|(\Theta(\mathcal{N}_1(s)) - \Theta(\mathcal{N}_2(s)))\,\Psi\mathcal{N}_1(s)\|_X \le \varrho d(\mathcal{N}_1, \mathcal{N}_2)\,\|\Psi\mathcal{N}_1(s)\|_Y\,. \tag{83}$$

From equations (14), (16) and (51), we obtain

$$\begin{aligned} \|\Psi\mathcal{N}_1(s)\|_Y \;&\le \|\mathcal{X}^{\mathcal{N}_1}\|_{\infty,Y}\left(\|\mathcal{Y}_0\|_{\infty,Y} + \|F(\mathcal{N}_1(t))\|_{1,Y}\right) \\ &\le \eta_1\eta_2\varsigma^2\mathcal{E}(c, L_0)\left(\|\mathcal{Y}_0\|_Y + \widetilde{\varrho}_\nu L_0\right). \end{aligned} \tag{84}$$

We combine Eq(72)–equation(84), we obtain

$$d(\Psi\mathcal{N}_1, \Psi\mathcal{N}_2) \le C(L_0)d(\Psi\mathcal{N}_1, \Psi\mathcal{N}_2)\,. \tag{85}$$

Thus the proof is conclude. □

Since $\mathcal{K}$ is not necessarily closed with respect to the metric defined in equation (79), we should use another approach to prove that Ψ has a fixed point $\mathcal{Y}$, and to do so we introduce $(\mathcal{Y}_n(.))_n$ a sequence in $\mathcal{K}$ such that

$$\begin{cases} \mathcal{Y}_0(t) = \mathcal{Y}_0, \quad \text{for} \quad 0 \le t \le L_0, \\ \mathcal{Y}_n = \Psi\mathcal{Y}_n. \end{cases} \tag{86}$$

Where $\mathcal{Y}_0$ is the initial data of system (37). From Lemma 4.12 we deduce that there is $\mathcal{Y} \in C([0, L_0]; X)$ such that

$$\mathcal{Y}_n \xrightarrow[n\to\infty]{} \mathcal{Y} \quad \text{in} \quad C([0, L_0]; X). \tag{87}$$

We have also from equation (38) that $\Psi \in C^1([0, L_0]; \mathcal{H})$ and that $\Psi' = \varphi$, and this means that $\mathcal{Y} = (\Psi, \Psi', \theta)^T$. Let $\mathcal{Y}_n = (\xi_{1,n}, \xi_{2,n}, \xi_{3,n})^T \in \mathcal{K}$, then we have,

$$\begin{aligned} &\mathcal{Y}_n(0) = \mathcal{Y}_0, \\ &\left\|\mathcal{A}^{\frac{1}{2}} \xi_{1,n}(t)\right\|_{\mathcal{W}} \leq \nu, \qquad \left\|\xi'_{1,n}(t)\right\|_{\mathcal{W}} \leq \nu, \quad \left\|\xi_{3,n}(t)\right\|_{\widetilde{\mathcal{V}}} \leq \nu, \\ &\left\|\mathcal{Y}_n(t) - \mathcal{Y}_n(s)\right\|_X \leq L|t-s|. \end{aligned} \tag{88}$$

In one hand we have from equations (87) and (88) as $n \to \infty$, gives us

$$\begin{aligned} &\mathcal{Y}(0) = \mathcal{Y}_0, \\ &\left\|\mathcal{Y}(t) - \mathcal{Y}(s)\right\|_X \leq L|t-s|. \end{aligned} \tag{89}$$

On the other hand

$$\left\|\mathcal{A}^{\frac{1}{2}} \xi_{1,n}(t)\right\|_{\mathcal{H}} \leq \nu, \quad \left\|\xi'_{1,n}(t)\right\|_{\mathcal{H}} \leq \nu, \quad \left\|\xi_{3,n}(t)\right\|_{\widetilde{\mathcal{V}}} \leq \nu. \tag{90}$$

Another time from the consideration in (87), there are two subsequences (ξ_{1,n_k}) and (ξ_{3,n_k}) and $\mathcal{Y} \in Y$ such that

$$\begin{array}{llll} \mathcal{A}\xi_{1,n}(t) \to \mathcal{A}\Psi & \text{weakly in,} & \mathcal{H} & \text{uniformly on} \quad [0, L_0]. \\ \mathcal{A}^{\frac{1}{2}} \xi'_{1,n}(t) \to \mathcal{A}^{\frac{1}{2}} \Psi' & \text{weakly in,} & \mathcal{H} & \text{uniformly on} \quad [0, L_0]. \\ \xi_{3,n}(t) \to \theta & \text{weakly in,} & \widetilde{\mathcal{V}} & \text{uniformly on} \quad [0, L_0]. \end{array} \tag{91}$$

From equations (88) and (91), we get

$$\left\|\mathcal{A}^{\frac{1}{2}} \Psi(t)\right\|_{\mathcal{W}} \leq \nu, \quad \|\Psi(t)\|_{\mathcal{W}} \leq \nu, \quad \|\theta'(t)\|_{\widetilde{\mathcal{V}}} \leq \nu. \tag{92}$$

From equations (89) and (92) we conclude that $\mathcal{Y} \in \mathcal{K}$. We can prove further that this $\mathcal{Y}$ is unique as follows

$$\begin{aligned} d(\mathcal{Y}, \Psi\mathcal{Y}) &\leq d(\mathcal{Y}, \mathcal{Y}_n) + d(\mathcal{Y}_n, \Psi\mathcal{Y}) \\ &= d(\mathcal{Y}, \mathcal{Y}_n) + d(\Psi\mathcal{Y}_{n-1}, \Psi\mathcal{Y}) \\ &\leq d(\mathcal{Y}, \mathcal{Y}_n) + C(l_0) d(\mathcal{Y}_{n-1}, \mathcal{Y}). \end{aligned} \tag{93}$$

Letting $n \to \infty$ in equation (93), we get $d(\mathcal{Y}, \Psi\mathcal{Y}) = 0$ from which we conclude that $\mathcal{Y}$ is a unique fixed point; it is in fact a unique solution to system (34) or equivalently system (25), and this proves Theorem 4.1.

5. Application

Let $\delta, \gamma > 0$, $\Omega \subset \mathbb{R}^N$ $(1 \leq N \leq 3)$ a bounded open subset with a smooth boundary, ϕ be a function satisfying the hypotheses in (4). We consider the following system

$$\begin{cases} \Psi_{tt}(x,t) - \phi(\|\nabla\Psi(x,t)\|_2^2)\Delta\Psi(x,t) + \gamma\partial_t^{\alpha,\eta}\Psi(t) = |\Psi(x,t)|^2\Psi(x,t) - \delta\Psi(x,t), & \text{in} \quad \Omega, t > 0, \\ \Psi = 0, & \text{on} \quad \partial\Omega, t > 0, \\ \Psi(x,0) = \Psi_0, \ \Psi_t(x,0) = \Psi_1, \ \theta(x,0) = \theta_0, & \text{in} \quad \Omega. \end{cases} \tag{94}$$

The local solvability of problem (94) is a consequence of Theorem 4.1, to see this we consider the classical Lebesgue spaces $L^p(\Omega)$ with their well-known norm for $1 \leq p \leq \infty$. In fact we have $\mathcal{A} = -\Delta$ is a positive definite self-adjoint operator on $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$ (see **[13]**), and $\mathcal{A}^{\frac{1}{2}} = \nabla$ with $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega)$. In this case the non-linear operator is taken to be $f(u) = u^3 - \delta u$, the domain of this operator is $\mathcal{D}(f) = L^6(\Omega)$ which is an example that fulfills the conditions (4) (See **[25]**). Under this circumstances we can apply Theorem 4.1 and get the local existence of system (94).

6. Conclusion

We studied the local existence of solutions to a quasilinear Kirchhoff equation with a nonlinear inhomogeneous term submitted to an internal viscous damping of fractional type; to tackle the equation directly is much harder, and to overcome the difficulty, we used an auxiliary and equivalent system. We combined the approach introduced in [28], the multiplier method, with an iterative scheme to achieve this result. In fact, we lived the degenerate case when $\eta = 0$ and the nonexistence case as open questions. We can look for an interesting and more complicated system that goes as follows:

$$\begin{cases} \Psi''(t) + \phi(\|\mathcal{A}^{\frac{1}{2}}\Psi(t)\|^2)\mathcal{A}\Psi(t) + \gamma\partial_t^{\alpha,\eta}\Psi(t-\tau) = f(\Psi(t)), & \text{in } [0, L], \\ \Psi(0) = \Psi_0, \ \Psi'(0) = \Psi_1, \end{cases} \tag{95}$$

$(t - \tau)$ this notation means a delay.

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