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New characterizations of SD operator and its generalized Alutghe transform

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Abstract. We investigate the class of star-dagger operators for which A^* and $A^\dagger$ commute. Let $A = U|A|$ be the polar decomposition, $\widetilde{A}(s,t) = |A|^s U|A|^t$ be the generalized Aluthge transformation, and $\widetilde{A}^{(*)}(s,t) = |A^*|^s U|A^*|^t$ be the generalized $*$-Aluthge transformation of A, respectively. We have discovered new characterizations for star-dagger operators, specifically that A is a star-dagger operator if and only if U and A commute. In this particular case, we have proven that $\widetilde{A}(s,t) = P_{\mathscr{R}(A^*)}A$ and $\widetilde{A}^{(*)}(s,t) = AP_{\mathscr{R}(A)}$ when $s,t > 0$ and $s + t = 1$.

1. Introduction and Preliminaries

In the context of a complex Hilbert space $\mathscr{H}$, let $\mathscr{B}(\mathscr{H})$ represent the C^*-algebra consisting of all bounded linear operators on $\mathscr{H}$. If we have an operator $A \in \mathscr{B}(\mathscr{H})$, we use $\mathscr{R}(A)$ and $\mathscr{N}(A)$ to denote the range and null space of A, respectively. A class of operators that receives less attention but is encountered intermittently in operator and matrix theory is known as star-dagger operators, abbreviated as SD. In this class, the operators satisfy the condition that A^* and $A^\dagger$ commute when the range of A, denoted as $\mathscr{R}(A)$, is closed. Here, $A^\dagger$ represents the Moore-Penrose inverse of A (refer to [13]), which exists if and only if $\mathscr{R}(A)$ is closed. These operators also exhibit close connections with other types of operators, including idempotent operators (i.e., $A^2 = A$), partial isometries (i.e., $A^* = A^\dagger$), and bi-dagger operators (i.e., $(A^2)^\dagger = (A^\dagger)^2$); references for these connections can be found in [7, 13, 18]. Moreover, star-dagger operators appear in diverse fields, such as the study of normal, positive-semidefinite, and partial orderings of complex matrices and operators. There are independent classes of operators that are related to the SD concept, but that relations exist between the intersection of some of these classes. It is well-known, for a closed range operator A, that A is both SD and EP if and only if A is normal; see [14, 17]. Also, Orthogonal projections $\subseteq$ Partial Isometries $\subseteq$ SD and Orthogonal projections $\subseteq$ Idempotents $\subseteq$ SD; see [7]. Each of the conditions $A^* = A^*A^\dagger$, $A^* = A^\dagger A^*$, $A^\dagger = A^\dagger A^\dagger$, $A^* = A^\dagger A^\dagger$, $A^\dagger = A^*A^*$ is sufficient for matrix $A \in \mathbb{C}_{n\times n}$ to be star-dagger; see [5]. Given an operator $A \in \mathscr{B}(\mathscr{H})$, we can consider its polar decomposition as $A = U|A|$. Aluthge, in his work [2], introduced a transformation called the Aluthge transformation

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denoted as $\widetilde{A}(\frac{1}{2},\frac{1}{2}) : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$, defined as $\widetilde{A}(\frac{1}{2},\frac{1}{2}) = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. This transformation, known as the Aluthge transformation, has proven to be very useful, and many authors have obtained significant results by utilizing it. The Aluthge transformation and its generalizations have received considerable attention in operator theory, with many authors focusing on studying and extending its properties for various classes of operators. This emphasizes the importance and applicability of these transformations in the field. For further details and related research, you can refer to the works of [8, 11, 12, 16, 20]. In addition to the original Aluthge transformation, Yamazaki introduced the notion of the $*$-Aluthge transform denoted as $\widetilde{A}^{(*)}(\frac{1}{2},\frac{1}{2})$. This transformation is defined as $\widetilde{A}^{(*)}(\frac{1}{2},\frac{1}{2}) = |A^*|^{\frac{1}{2}}U|A^*|^{\frac{1}{2}}$; refer to [21] for more details. Similarly, Jabbarzadeh, in the work of [12], introduced the †-Aluthge transformation denoted as $\widetilde{A}^{(\dagger)}$. This transformation is defined as $\widetilde{A}^{(\dagger)}(\frac{1}{2},\frac{1}{2}) = \left(\widetilde{A^\dagger}(\frac{1}{2},\frac{1}{2})\right)^\dagger$, where $\widetilde{A^\dagger}(\frac{1}{2},\frac{1}{2}) = \left|A^\dagger\right|^{\frac{1}{2}} U^* \left|A^\dagger\right|^{\frac{1}{2}}$. These concepts were further generalized by considering parameters r and s as follows: For every $r,s > 0$, the generalized Aluthge transformation denoted as $\widetilde{A}(r,s)$ is defined as $\widetilde{A}(r,s) = |A|^rU|A|^s$. Additionally, the generalized $*$-Aluthge transformation denoted as $\widetilde{A}^{(*)}(r,s)$ is defined as $\widetilde{A}^{(*)}(r,s) = |A^*|^rU|A^*|^s$; see [6, 19] for more information on these generalizations. Furthermore, we can introduce a generalized †-Aluthge transformation, which is analogous to the generalized Aluthge transformation, but with the additional property associated with the Moore-Penrose of operator. If $A = U|A|$ is normal, then U commutes with $|A|$ and so with $|A|^{\frac{1}{2}}$, hence $\widetilde{A}(\frac{1}{2},\frac{1}{2}) = A$. It is also easy to show that if $\widetilde{A}(\frac{1}{2},\frac{1}{2}) = A = U|A|$ then U commutes with $|A|$, so that A is normal [4].

If A is idempotent, that is $A^2 = A$, then $\widetilde{A}(s, 1-s)$ is the orthogonal projection onto the range of $|A|^{\frac{1}{2}}$, for any $s \in (0,1)$; see [6].

The aim of this paper is to investigate star-dagger class of operators and represented new characterizations by applying matrix representations of operators. More characterization of star-dagger operators show where it is situated relative to the other well-known classes of special operator. Also, we prove that generalized Aluthge transformation and generalized $*$-Aluthge transformation of a SD operator A are $\widetilde{A}(s,t) = P_{\mathscr{R}(A^*)}A$ and $\widetilde{A}^{(*)}(s,t) = AP_{\mathscr{R}(A)}$ when $s,t > 0$ and $s+t = 1$.

Lemma 1.1. *[15, Theorem 2.1] Let $A \in \mathscr{B}(\mathscr{H})$ have a closed range, and let $A = U|A|$ be a polar decomposition of A. Then $U = A|A|^\dagger$.*

Theorem 1.2. *[16, Theorem 2.4.] Suppose that $A \in \mathscr{B}(\mathscr{H})$ has a closed range, and that $A = U|A|$ is the polar decomposition of A. If $A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix}$, then*

$$A = U|A| = \begin{bmatrix} A_1D^{-\frac{1}{2}} & 0 \\ A_2D^{-\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix},$$

where $D = A_1^*A_1 + A_2^*A_2$ is positive and invertible. The first matrix of the last equation coincides with U and the second one coincides with $|A|$.

In addition, we use the following simple property of reverse order law for a closed range $A \in \mathscr{B}(\mathscr{H})$, by [17],

$$(AA^*A)^\dagger = A^\dagger(A^*)^\dagger A^\dagger. \tag{1}$$

2. New characterization of SD operators

In this section, we represent new characterizations of a less attention class of operators that appears intermittently throughout operator and matrix theory, the class of star-dagger operators. As always, we write $[A,B] = AB - BA$.

Theorem 2.1. *Let $A \in \mathscr{B}(\mathscr{H})$ have a closed range and the polar decomposition $A = U|A|$. Then the following statements are equivalent:*

(1) *A is SD*
(2) $[A(A^*A)^\alpha, A] = 0$ *for all* $\alpha > 0$
(3) *There exists* α *such that* $[A(A^*A)^\alpha, A] = 0$
(4) $(A^*A)^\alpha A = (A^*A)^{\alpha-\beta}A(A^*A)^\beta$ *for all* $\alpha > \beta > 0$
(5) *There exists* α, β *such that* $\alpha > \beta > 0$ *and* $(A^*A)^\alpha A = (A^*A)^{\alpha-\beta}A(A^*A)^\beta$
(6) $\left[A((A^*A)^\dagger)^\alpha, A\right] = 0$ *for all* $\alpha > 0$
(7) *There exists* $\alpha > 0$ *such that* $\left[A((A^*A)^\dagger)^\alpha, A\right] = 0$
(8) $((A^*A)^\dagger)^\alpha A = ((A^*A)^\dagger)^{\alpha-\beta}A((A^*A)^\dagger)^\beta$ *for all* $\alpha > \beta > 0$
(9) *There exists* α, β *such that* $\alpha > \beta > 0$ *and* $((A^*A)^\dagger)^\alpha A = ((A^*A)^\dagger)^{\alpha-\beta}A((A^*A)^\dagger)^\beta$
(10) $AA^\dagger(A^*A)^\dagger = (A^\dagger)^*(A^\dagger)^2A$
(11) $A^2 = (A^\dagger)^*AA^*A$
(12) $A^2 = AA^*A(A^\dagger)^*$
(13) $A^\dagger A^2 = A^*A\left(A^\dagger\right)^*$
(14) $(A^*A)^\dagger A = A^\dagger A(A^\dagger)^*$
(15) $UA = AU$
(16) $U^2 = \left(A^\dagger\right)^* A$
(17) $UAA^*A = AA^*UA$
(18) $A^\dagger$ *is SD*
(19) $\left[A^\dagger((AA^*)^\dagger)^\alpha, A^\dagger\right] = 0$ *for all* $\alpha > 0$
(20) *There exists* $\alpha > 0$ *such that* $\left[A^\dagger((AA^*)^\dagger)^\alpha, A^\dagger\right] = 0$
(21) $((AA^*)^\dagger)^\alpha A^\dagger = ((AA^*)^\dagger)^{\alpha-\beta}A^\dagger((AA^*)^\dagger)^\beta$ *for all* $\alpha > \beta > 0$
(22) *There exists* α, β *such that* $\alpha > \beta > 0$ *and* $((AA^*)^\dagger)^\alpha A^\dagger = ((AA^*)^\dagger)^{\alpha-\beta}A^\dagger((AA^*)^\dagger)^\beta$
(23) $\left[A^\dagger(AA^*)^\alpha, A^\dagger\right] = 0$ *for all* $\alpha > 0$
(24) *There exists* $\alpha > 0$ *such that* $\left[A^\dagger(AA^*)^\alpha, A^\dagger\right] = 0$
(25) $(AA^*)^\alpha A^\dagger = (AA^*)^{\alpha-\beta}A^\dagger(AA^*)^\beta$ *for all* $\alpha > \beta > 0$
(26) *There exists* α, β *such that* $\alpha > \beta > 0$ *and* $(AA^*)^\alpha A^\dagger = (AA^*)^{\alpha-\beta}A^\dagger(AA^*)^\beta$
(27) $A^\dagger AAA^* = A^*AAA^\dagger$
(28) $\left(A^\dagger\right)^2 = A^*\left(AA^*A\right)^\dagger$
(29) $\left(A^\dagger\right)^2 = \left(AA^*A\right)^\dagger A^*$
(30) $A\left(A^\dagger\right)^2 = \left(A^\dagger\right)^* A^\dagger A^*$
(31) $AA^*A^\dagger = AA^\dagger A^*$
(32) $U^*A^\dagger = A^\dagger U^*$
(33) $(U^*)^2 = A^*A^\dagger$
(34) $U^*(AA^*A)^\dagger = (A^*A)^\dagger U^*A^\dagger$
(35) $A^2A^\dagger = (A^\dagger)^*AA^*$
(36) $A^*A^\dagger A = A^\dagger A^*A$
(37) $(A^*A)^\dagger AA^* = A^\dagger A^2A^\dagger$
(38) $AA^\dagger A^*A = AA^*A^\dagger A$
(39) $A(AA^*)^\dagger = (A^\dagger)^*AA^\dagger$
(40) $(A^\dagger)^2A = A^*A^\dagger(A^\dagger)^*$
(41) $AA^*(A^*A)^\dagger = A(A^\dagger)^2A$
(42) $A^\dagger A(AA^*)^\dagger = (A^*A)^\dagger AA^\dagger$
(43) $[A(A^*U)^\alpha, A] = 0$ *for all* $\alpha > 0$
(44) *There exists* $\alpha > 0$ *such that* $[A(A^*U)^\alpha, A] = 0$

(45) $(A^*U)^\alpha A = (A^*U)^{\alpha-\beta} A(A^*U)^\beta$ *for all* $\alpha > \beta > 0$
(46) *There exists* α, β *such that* $\alpha > \beta > 0$ *and* $(A^*U)^\alpha A = (A^*U)^{\alpha-\beta} A(A^*U)^\beta$
(47) $\left[A((A^*U)^\dagger)^\alpha, A\right] = 0$ *for all* $\alpha > 0$
(48) *There exists* $\alpha > 0$ *such that* $\left[A((A^*U)^\dagger)^\alpha, A\right] = 0$
(49) $((A^*U)^\dagger)^\alpha A = ((A^*U)^\dagger)^{\alpha-\beta} A((A^*U)^\dagger)^\beta$ *for all* $\alpha > \beta > 0$
(50) *There exists* α, β *such that* $\alpha > \beta > 0$ *and* $((A^*U)^\dagger)^\alpha A = ((A^*U)^\dagger)^{\alpha-\beta} A((A^*U)^\dagger)^\beta$
(51) $AU^*(A^*A)^\dagger = U(A^\dagger)^2 A$
(52) $A^2U^*A = UAA^*A$
(53) $A^2U^*A = AA^*AU$
(54) $AU^*A^2 = AA^*AU$
(55) $U^*A^2 = A^*AU$
(56) $U^*(A^*)^\dagger A = U^*A(A^\dagger)^*$
(57) $A^\dagger UA = A^\dagger AU$
(58) $U^*UA = U^*AU$
(59) $U^*A^2A^* = A^*A^2U^*$
(60) $(U^*)^2 = A^*U^*UA^\dagger$
(61) $(U^*)^2 = A^*U^*(A^*)^\dagger U^*$
(62) $(U^*)^2 = A^*A^\dagger UU^*$
(63) $U^*A^\dagger = A^*A^\dagger UA^\dagger$
(64) $(U^*)^2 = U^*UA^\dagger A^*$
(65) $(U^*)^2 = U^*(A^*)^\dagger U^*A^*$
(66) $(U^*)^2 = A^\dagger UU^*A^*$
(67) $A\,(U^*)^2 = UU^*A^*$
(68) $AA^\dagger U^* = UA^\dagger A^*$
(69) $AU^*A^\dagger = \left(A^\dagger\right)^* U^*A^*$
(70) $AA^*U^* = AU^*A^*$
(71) $(U^*)^2(AA^*)^\dagger = (A^*A)^\dagger(U^*)^2$
(72) $A^2U^* = UAA^*$
(73) $A^*U^*A = U^*A^*A$
(74) $U^*UAA^* = U^*A^2U^*$
(75) $AU^*A^*A = AA^*U^*A$
(76) $AUU^* = UAU^*$
(77) $(U^*)^2A = A^*U^*U$
(78) $AA^*U^*U = A(U^*)^2A$
(79) $A^\dagger AUU^* = U^*UAA^\dagger$
(80) $U^*AU^*U = U^*UAU^*$

Proof. (1) ⇒ (2) Matrix form of closed range operator A is the following form

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix}. \tag{2}$$

Since A is SD, it implies that

$$\begin{aligned} \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_1^* & A_2^* \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} A_1^* & A_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} D^{-1}A_1^*A_1^* & D^{-1}A_1^*A_2^* \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} A_1^*D^{-1}A_1^* & A_1^*D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where $D = A_1^*A_1 + A_2^*A_2$. Therefore, $D^{-1}A_1^*A_1^* = A_1^*D^{-1}A_1^*$ and $D^{-1}A_1^*A_2^* = A_1^*D^{-1}A_2^*$. Now, post-multiplying by A_1 the first equation and by A_2 the second equation, we have $D^{-1}A_1^*A_1^*A_1 = A_1^*D^{-1}A_1^*A_1$ and $D^{-1}A_1^*A_2^*A_2 = A_1^*D^{-1}A_2^*A_2$. By additive the obtained equalities, we conclude that $D^{-1}A_1^*D = A_1^*$. Therefore

$$DA_1 = A_1D, \tag{3}$$

Or equivalently

$$A_1D^{-1} = D^{-1}A_1. \tag{4}$$

Equations (3), (4) and [10, Proposition 2.4] imply that

$$D^\alpha A_1 = A_1D^\alpha \quad \text{for all} \quad \alpha \in \mathbb{R} \setminus \{0\}. \tag{5}$$

If $\alpha \in \mathbb{R}^+$. Then

$$\begin{aligned} A(A^*A)^\alpha A &= \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} \begin{bmatrix} D^\alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_1D^\alpha A_1 & 0 \\ A_2D^\alpha A_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_1A_1D^\alpha & 0 \\ A_2A_1D^\alpha & 0 \end{bmatrix} \\ &= A^2(A^*A)^\alpha, \end{aligned}$$

that is, $[A(A^*A)^\alpha, A] = 0$.

(2) ⇒ (1) Since $[A(A^*A)^\alpha, A] = 0$, then matrix form (2) allows us that

$$\begin{bmatrix} A_1D^\alpha A_1 & 0 \\ A_2D^\alpha A_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1A_1D^\alpha & 0 \\ A_2A_1D^\alpha & 0 \end{bmatrix}. \tag{6}$$

Therefore, $A_1^*A_1D^\alpha A_1 = A_1^*A_1A_1D^\alpha$ and $A_2^*A_2D^\alpha A_1 = A_2^*A_2A_1D^\alpha$. They imply that $D^\alpha A_1 = A_1D^\alpha$ holds. Letting $\alpha = 1$, ensures that A is SD.

(1) ⇒ (3) By ((1) ⇒ (2)) is clear.

(3) ⇒ (1) Matrix form (2) concludes that (6) establishes. Then $D^\alpha A_1 = A_1D^\alpha$. [10, Proposition 2.4] implies that $[A_1, D^{\frac{\alpha}{\alpha}}] = 0$, that is, A is SD.

(1) ⇒ (4) Matrix form (2) gives that (5) holds. By (5) and the matrix forms of $(A^*A)^\alpha$ and A, we have

$$\begin{aligned} (A^*A)^\alpha A &= \begin{bmatrix} D^\alpha A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D^{\alpha-\beta} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} \begin{bmatrix} D^\beta & 0 \\ 0 & 0 \end{bmatrix} & (7) \\ &= (A^*A)^{\alpha-\beta}A(A^*A)^\beta. & (8) \end{aligned}$$

(4) ⇒ (1) Matrix form (2) and equation $(A^*A)^\alpha A = (A^*A)^{\alpha-\beta}A(A^*A)^\beta$ cause for all $\alpha > \beta$, $D^\alpha A_1 = D^{\alpha-\beta}A_1D^\beta$ to establish. It is sufficient to let $\alpha = 2$ and $\beta = 1$, then equation (3) is satisfied, that is, A is SD.

(1) ⇒ (5) Similar to ((1) ⇒ (4)).

(5) ⇒ (1) Matrix form (2) and equation $(A^*A)^\alpha A = (A^*A)^{\alpha-\beta}A(A^*A)^\beta$ yield for all $\alpha > \beta$, $D^\alpha A_1 = D^{\alpha-\beta}A_1D^\beta$ to establish. Multiplying this equality on the left and right by $D^{-\alpha}$ and $D^{-\beta}$, respectively, we get $[A_1, D^{-\beta}] = 0$. By [10, Proposition 2.4], we get $[A_1, D^{\frac{-\beta}{\beta}}] = 0$. The proof of the equivalence of (5) and (1) is completed.

(1) ⇔ (6) ⇔ ... ⇔ (9) Similarly to (1) ⇔ ... ⇔ (5), applying matrix form (2) and equations (3), (4), these equivalences can be obtained.

(1) ⇔ (10) Firstly, $A^*A^\dagger = A^\dagger A^*$ gives $(A^\dagger)^*A^*A^\dagger(A^\dagger)^* = (A^\dagger)^*A^\dagger A^*(A^\dagger)^*$. Now pre- and post-multiply (10) by A^* to obtain (1).

(1) ⇔ (11) We have

$$\begin{aligned} A^*A^\dagger = A^\dagger A^* \quad &\Leftrightarrow \quad \left(A^\dagger\right)^* A = A\left(A^\dagger\right)^* \\ &\Leftrightarrow \quad \left(A^\dagger\right)^* AA^* \left(A^\dagger\right)^* = A\left(A^\dagger\right)^* \\ &\Leftrightarrow \quad \left((A^\dagger)^*AA^* - A\right)\left(A^\dagger\right)^* = 0 \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \mathscr{R}((A^\dagger)^*) \subset \mathscr{N}((A^\dagger)^*AA^* - A)\\
&\Leftrightarrow \mathscr{R}(A) \subset \mathscr{N}((A^\dagger)^*AA^* - A)\\
&\Leftrightarrow \left(\left(A^\dagger\right)^* AA^* - A\right)A = 0\\
&\Leftrightarrow \left(A^\dagger\right)^* AA^*A = A^2.
\end{aligned}$$

(1) ⇔ (12) We obtain from direct computations that

$$\begin{aligned}
A(A^\dagger)^* = (A^\dagger)^*A \quad &\Leftrightarrow \quad (A^\dagger)^*A^*A(A^\dagger)^* = (A^\dagger)^*A\\
&\Leftrightarrow \quad \mathscr{R}(A^*A(A^\dagger)^* - A) \subset \mathscr{N}((A^\dagger)^*)\\
&\Leftrightarrow \quad \mathscr{R}(A^*A(A^\dagger)^* - A) \subset \mathscr{N}(A)\\
&\Leftrightarrow \quad AA^*A(A^\dagger)^* = A^2.
\end{aligned}$$

(12) ⇔ (13) It follows from the following computation:

$$A^2 = AA^*A(A^\dagger)^* \Leftrightarrow A^\dagger A^2 = A^*A\left(A^\dagger\right)^*.$$

(12) ⇔ (14) Applying properties of the Moore-Penrose inverse, we see that

$$\begin{aligned}
A^2 = AA^*A(A^\dagger)^* \quad &\Leftrightarrow \quad A^\dagger A^2 = A^\dagger AA^*A(A^\dagger)^*\\
&\Leftrightarrow \quad A^*(A^\dagger)^*A = A^*A(A^\dagger)^*\\
&\Leftrightarrow \quad \mathscr{R}((A^\dagger)^*A - A(A^\dagger)^*) \subset \mathscr{N}(A^*)\\
&\Leftrightarrow \quad \mathscr{R}((A^\dagger)^*A - A(A^\dagger)^*) \subset \mathscr{N}(A^\dagger)\\
&\Leftrightarrow \quad A^\dagger(A^\dagger)^*A = A^\dagger A(A^\dagger)^*\\
&\Leftrightarrow \quad (A^*A)^\dagger A = A^\dagger A(A^\dagger)^*.
\end{aligned}$$

(1) ⇒ (15) Since A is SD, then matrix form (2) concludes that equality (5) holds. Therefore, by (5) and matrix representations given in Theorem 1.2, we deduce that

$$\begin{aligned}
\begin{bmatrix} A_1D^{-\frac{1}{2}}A_1 & 0\\ A_2D^{-\frac{1}{2}}A_1 & 0 \end{bmatrix} &= \begin{bmatrix} A_1A_1D^{-\frac{1}{2}} & 0\\ A_2A_1D^{-\frac{1}{2}} & 0 \end{bmatrix},\\
\begin{bmatrix} A_1D^{-\frac{1}{2}} & 0\\ A_2D^{-\frac{1}{2}} & 0 \end{bmatrix}\begin{bmatrix} A_1 & 0\\ A_2 & 0 \end{bmatrix} &= \begin{bmatrix} A_1 & 0\\ A_2 & 0 \end{bmatrix}\begin{bmatrix} A_1D^{-\frac{1}{2}} & 0\\ A_2D^{-\frac{1}{2}} & 0 \end{bmatrix},\\
UA &= AU.
\end{aligned}$$

(15) ⇒ (1) The matrix representations given in Theorem 1.2 allow us to have

$$UA = AU \Leftrightarrow \begin{bmatrix} A_1D^{-\frac{1}{2}}A_1 & 0\\ A_2D^{-\frac{1}{2}}A_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1A_1D^{-\frac{1}{2}} & 0\\ A_2A_1D^{-\frac{1}{2}} & 0 \end{bmatrix}.$$

Hence, $A_1D^{-\frac{1}{2}}A_1 = A_1A_1D^{-\frac{1}{2}}$ and $A_2D^{-\frac{1}{2}}A_1 = A_2A_1D^{-\frac{1}{2}}$. Therefore,

$$A_1^*A_1D^{-\frac{1}{2}}A_1 = A_1^*A_1A_1D^{-\frac{1}{2}}, \quad A_2^*A_2D^{-\frac{1}{2}}A_1 = A_2^*A_2A_1D^{-\frac{1}{2}}.$$

Then

$$\begin{aligned}
(A_1^*A_1 + A_2^*A_2)D^{-\frac{1}{2}}A_1 \quad &= \quad (A_1^*A_1 + A_2^*A_2)A_1D^{-\frac{1}{2}}\\
&\Leftrightarrow \quad D^{-\frac{1}{2}}A_1 = A_1D^{-\frac{1}{2}}\\
&\Leftrightarrow \quad D^{\frac{1}{2}}D^{-\frac{1}{2}}A_1D^{\frac{1}{2}} = D^{\frac{1}{2}}A_1D^{-\frac{1}{2}}D^{\frac{1}{2}}\\
&\Leftrightarrow \quad D^{\frac{1}{2}}A_1 = A_1D^{\frac{1}{2}}.
\end{aligned}$$

So, [10, Proposition 2.4] imply that $DA_1 = A_1D$. Therefore, $A_1^*D = DA_1^*$. It leads to

$$\begin{aligned} \begin{bmatrix} D^{-1}A_1^*A_1^* & D^{-1}A_1^*A_2^* \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} A_1^*D^{-1}A_1^* & A_1^*D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_1^* & A_2^* \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} A_1^* & A_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}, \\ A^\dagger A^* &= A^*A^\dagger. \end{aligned}$$

(1) ⇔ (16) and (1) ⇔ (17) By considering the matrix forms of operators, similarly to ((1) ⇔ (15)), the desired results follows.

(1) ⇔ (18) ⇔ ... ⇔ (34) An operator A is SD if and only if $A^\dagger$ is SD. Hence, we replace A and U with $A^\dagger$ and U^*, respectively, in the ((1) ⇔ .... ⇔ (17)) to obtain the desired results.

(11) ⇔ (35) Multiplying $A^2 = (A^\dagger)^*AA^*A$ from the right hand side by $A^\dagger$, it follows $A^2A^\dagger = (A^\dagger)^*AA^*$. If we multiply $A^2A^\dagger = (A^\dagger)^*AA^*$ from the right hand side by A, we get $A^2 = (A^\dagger)^*AA^*A$.

(13) ⇔ (36) This equivalence is clear by properties of conjugate operator.

(14) ⇔ (37) If we multiply $(A^*A)^\dagger A = A^\dagger A(A^\dagger)^*$ from the right hand side by A^*, we obtain (37). Multiplying $(A^*A)^\dagger AA^* = A^\dagger A^2A^\dagger$ from the right hand side by $(A^\dagger)^*$, notice that (14) holds.

(31) ⇔ (37) Obviously $AA^*A^\dagger = AA^\dagger A^*$ gives $AA^*A^\dagger A = AA^\dagger A^*A$. Multiplying $AA^\dagger A^*A = AA^*A^\dagger A$ from the right hand side by $A^\dagger$, we see that (31) is satisfied.

(35) ⇔ (39) We observe that

$$\begin{aligned} A^2A^\dagger = (A^\dagger)^*AA^* \quad &\Leftrightarrow \quad A(A^\dagger)^*A^* = (A^\dagger)^*AA^* \\ &\Leftrightarrow \quad A(A^\dagger)^*A^\dagger = (A^\dagger)^*AA^\dagger \\ &\Leftrightarrow \quad A(AA^*)^\dagger = (A^\dagger)^*AA^\dagger. \end{aligned}$$

(35) ⇔ (40) and (37) ⇔ (41) and (38) ⇔ (42) Since A is SD if and only if $A^\dagger$ is SD, these equivalences follow.

(1) ⇔ (43) and (1) ⇔ (44) and ... and (1) ⇔ (80) By considering the matrix forms of operators and matrix form of operator U of Theorem 1.2, similarly to ((1) ⇔ (2)), the desired results follows.

□

In general the SD property is not additive, hence the following theorem is important.

Theorem 2.2. *Let $A = U|A|$ be the polar decomposition of a SD operator $A \in \mathscr{B}(\mathscr{H})$. Then $A + (A^*)^\dagger$ is SD.*

Proof. Matrix form (2) of A implies that

$$A + (A^*)^\dagger = \begin{bmatrix} A_1(1 + D^{-1}) & 0 \\ A_2\left(1 + D^{-1}\right) & 0 \end{bmatrix}. \tag{9}$$

We let $H = A + (A^*)^\dagger$. If $H = V|H|$ is a polar decomposition of H, notice that

$$|H| = \begin{bmatrix} (D^{\frac{1}{2}} + D^{\frac{-1}{2}})^2 & 0 \\ 0 & 0 \end{bmatrix}^{\frac{1}{2}}.$$

Clearly, $|H|$ has a closed range and then $V = H|H|^\dagger$. Therefore, we compute V:

$$\begin{aligned} V &= \begin{bmatrix} A_1(1 + D^{-1}) & 0 \\ A_2\left(1 + D^{-1}\right) & 0 \end{bmatrix}\left(\begin{bmatrix} (1 + D^{-1})D(1 + D^{-1}) & 0 \\ 0 & 0 \end{bmatrix}^{\frac{1}{2}}\right)^\dagger \\ &= \begin{bmatrix} A_1(1 + D^{-1}) & 0 \\ A_2\left(1 + D^{-1}\right) & 0 \end{bmatrix}\left(\begin{bmatrix} (D^{\frac{1}{2}} + D^{\frac{-1}{2}})^2 & 0 \\ 0 & 0 \end{bmatrix}^{\frac{1}{2}}\right)^\dagger \\ &= \begin{bmatrix} A_1D^{\frac{-1}{2}}(D^{\frac{1}{2}} + D^{\frac{-1}{2}}) & 0 \\ A_2D^{\frac{-1}{2}}(D^{\frac{1}{2}} + D^{\frac{-1}{2}}) & 0 \end{bmatrix}\begin{bmatrix} (D^{\frac{1}{2}} + D^{\frac{-1}{2}})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} A_1D^{-\frac{1}{2}} & 0 \\ A_2D^{-\frac{1}{2}} & 0 \end{bmatrix} \\ &= U. \end{aligned}$$

Since A is SD, then (5) holds. Now, Theorem 2.1 ((1) $\Rightarrow$ (15)) and the following equalities

$$\begin{aligned} U(A+(A^*)^\dagger) &= \begin{bmatrix} A_1D^{-\frac{1}{2}} & 0 \\ A_2D^{-\frac{1}{2}} & 0 \end{bmatrix}\begin{bmatrix} A_1(1+D^{-1}) & 0 \\ A_2\left(1+D^{-1}\right) & 0 \end{bmatrix} = \begin{bmatrix} A_1D^{-\frac{1}{2}}A_1(1+D^{-1}) & 0 \\ A_2D^{-\frac{1}{2}}A_1(1+D^{-1}) & 0 \end{bmatrix}, \\ (A+(A^*)^\dagger)U &= \begin{bmatrix} A_1(1+D^{-1}) & 0 \\ A_2\left(1+D^{-1}\right) & 0 \end{bmatrix}\begin{bmatrix} A_1D^{-\frac{1}{2}} & 0 \\ A_2D^{-\frac{1}{2}} & 0 \end{bmatrix} = \begin{bmatrix} A_1(1+D^{-1})A_1D^{-\frac{1}{2}} & 0 \\ A_2(1+D^{-1})A_1D^{-\frac{1}{2}} & 0 \end{bmatrix} \end{aligned}$$

imply that $A+(A^*)^\dagger$ is SD. □

Theorem 2.3. *If $A \in \mathscr{B}(\mathscr{H})$ is a SD operator, then the following equalities hold:*

(1) *$A+(A^*)^\dagger$ is a solution of the equation $|A+X| = |A|+|X|$.*
(2) *$\langle A, A+(A^*)^\dagger\rangle = A^*A + A^\dagger A$.*
(3) *$\langle A, A+(A^*)^\dagger\rangle = |A|\,|A+(A^*)^\dagger\,| = |A+(A^*)^\dagger\,|\,|A|$.*
(4) *$\langle A, A+(A^*)^\dagger\rangle$ is positive.*
(5) *$A|A+(A^*)^\dagger\,| = (A+(A^*)^\dagger)|A|$.*

Proof. (1) Since $A = U|A|$, Theorem 2.2 deduces $A+(A^*)^\dagger = U\left|A+(A^*)^\dagger\right|$. Then [3, Theorem 2.3] implies that $A+(A^*)^\dagger$ is a solution of the equation $|A+X| = |A|+|X|$.

(2) Since (1) holds, then [1, Theorem 2.3] implies that

$$\begin{aligned} \langle A, A+(A^*)^\dagger\rangle &= |A||A+(A^*)^\dagger\,| \\ &= \begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^{\frac{1}{2}}+D^{\frac{-1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} D+1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= A^*A + A^\dagger A. \end{aligned}$$

(3) and (4) are obtained by applying (2) and its matrix forms.

(5) Corollary 2.5 of [1] implies it. □

We now study when the product of two SD operators is SD too.

Theorem 2.4. *Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of SD operators $A, B \in \mathscr{B}(\mathscr{H})$, respectively. If $|A|\,|B^*| = |B^*|\,|A|$, $VA = AV$ and $UB = BU$, then AB is SD.*

Proof. Since $|A|\,|B^*| = |B^*|\,|A|$, then [8, Theorem 2.3] implies that $AB = UV|AB|$ is the polar decomposition. Now, since $VA = AV$ and $UB = BU$, Theorem 2.1 ((1) $\Leftrightarrow$ (15)) implies that $(UV)AB = AUVB = AB(UV)$, that is, AB is SD. □

Theorem 2.5. *Let $A \in \mathscr{B}(\mathscr{H})$ be* SD *and bi-dagger operator such that $(A^\dagger A^*)^\dagger = AA^*\left(A^\dagger A^*AA^*\right)^\dagger$, then $A(A^\dagger)^*$ is a partial isometry.*

Proof. Since $(A^\dagger A^*)^\dagger = AA^*\left(A^\dagger A^*AA^*\right)^\dagger$, applying property of conjugate operator to this equality and using ((1) $\Leftrightarrow$ (12)) of Theorem 2.1, we have

$$(A(A^\dagger)^*)^\dagger = \left(AA^*A(A^\dagger)^*\right)^\dagger AA^* = \left(A^2\right)^\dagger AA^*.$$

Since A is also bi-dagger, it implies that

$$(A(A^\dagger)^*)^\dagger = \left(A^2\right)^\dagger AA^* = \left(A^\dagger\right)^2 AA^* = A^\dagger A^* = (A(A^\dagger)^*)^*,$$

that is, $A(A^\dagger)^*$ is a partial isometry. □

In the following example, we illustrate previous result. Precisely, we give a SD operator such that $(A^\dagger A^*)^\dagger = AA^*\left(A^\dagger A^* AA^*\right)^\dagger$ which is not bi-dagger and $A(A^\dagger)^*$ is not partial isometry.

Example 2.6. *Consider*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

on $\mathscr{H} = \mathbb{C}^2$. Then, by

$$A^\dagger = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

we have $A^*A^\dagger = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} = A^\dagger A^*$ such that

$$(A^\dagger A^*)^\dagger = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = AA^*\left(A^\dagger A^* AA^*\right)^\dagger.$$

Since $A(A^\dagger)^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$, notice that

$$\left(A(A^\dagger)^*\right)^\dagger = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} = \left(A(A^\dagger)^*\right)^*$$

and

$$(A^2)^\dagger = A^\dagger = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & 0 \end{bmatrix} = (A^\dagger)^2.$$

Hence, A is SD satisfying $(A^\dagger A^*)^\dagger = AA^*\left(A^\dagger A^* AA^*\right)^\dagger$, but A is not bi-dagger and $A(A^\dagger)^*$ is not partial isometry.

Theorem 2.7. *Every SD operator $A \in \mathscr{B}(\mathscr{H})$ is bi-dagger, if $(AA^*A(A^\dagger)^*)^\dagger = A^*(AA^*A)^\dagger$.*

Proof. Since A is SD, then applying ((1) ⇔ (12)) and ((1) ⇔ (28)) of Theorem 2.1 we have

$$A^2 = AA^*A(A^\dagger)^* \text{ and } \left(A^\dagger\right)^2 = A^*(AA^*A)^\dagger.$$

These equalities show that A is bi-dagger, if $(AA^*A(A^\dagger)^*)^\dagger = A^*(AA^*A)^\dagger$ establishes. □

Notice that, for the operator A given in Example 2.6, we can check that condition $(AA^*A(A^\dagger)^*)^\dagger = A^*(AA^*A)^\dagger$ of Theorem 2.7 is not satisfied.

Several necessary and sufficient conditions for a SD operator to be bi-dagger, are developed too.

Theorem 2.8. *If $A \in \mathscr{B}(\mathscr{H})$ is a SD operator, the following are equivalent.*

(1) *A is bi-dagger*
(2) $A^*A^2 = A^\dagger A^2 A^* A^\dagger A^2$
(3) $(A^*)^2 A = A^* A^\dagger A^2 A^* A^\dagger A$
(4) $A^2 = (A^\dagger)^* AA^* A^\dagger A^2$
(5) $(A^*)^2 = A^* A^\dagger A^2 A^* A^\dagger$

(6) $(A^*)^2AA^*A = (A^*)^3A^2$
(7) $(A^*)^2AA^* = (A^*)^3A^2A^\dagger$
(8) $A^*AA^*A^2 = (A^*)^2A^3$
(9) $AA^*A^2 = AA^\dagger A^*A^3$
(10) $A^*A^2 = A^\dagger A^*A^3$
(11) $(A^*)^2A = (A^*)^3A(A^\dagger)^*$
(12) $(A^*)^2 = (A^*)^3A(A^\dagger)^*A^\dagger$
(13) $A^2 = (A^\dagger)^*A^\dagger A^*A^3$.

Proof. (1) ⇔ (2) Let A be represented by (2). For $E = (A_1^*)^2A_1^2 + (A_2A_1)^*A_2A_1 = A_1^*DA_1$, we observe that

$$\begin{bmatrix} (D^{-1}A_1^*)^2 & D^{-1}A_1^*D^{-1}A_2^* \\ 0 & 0 \end{bmatrix} = (A^\dagger)^2 = (A^2)^\dagger = \begin{bmatrix} E^{-1}(A_1^*)^2 & E^{-1}(A_2A_1)^* \\ 0 & 0 \end{bmatrix}$$

is equivalent to

$$D^{-1}A_1^*D^{-1}A_1^* = E^{-1}(A_1^*)^2 \tag{10}$$

$$D^{-1}A_1^*D^{-1}A_2^* = E^{-1}A_1^*A_2^*. \tag{11}$$

Multiplying (10) and (11) from the right hand side by A_1 and A_2, respectively, and then adding obtained equalities, we get $D^{-1}A_1^* = E^{-1}A_1^*D$. Applying property of conjugate operator to this equality, we have $A_1D^{-1} = DA_1E^{-1}$. Since A is SD, (3) gives $DA_1^* = A_1^*D$. Thus, $E = A_1^*DA_1 = DA_1^*A_1$, which yields

$$DA_1 = A_1D^{-1}E = A_1D^{-1}DA_1^*A_1 = A_1A_1^*A_1.$$

We deduce that A is bi-dagger if and only if $DA_1 = A_1A_1^*A_1$. From the equalities

$$A^*A^2 = \begin{bmatrix} A_1^* & A_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_1^2 & 0 \\ A_2A_1 & 0 \end{bmatrix} = \begin{bmatrix} DA_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$A^\dagger A^2A^*A^\dagger A^2 = \begin{bmatrix} A_1A_1^*A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

notice that $A^*A^2 = A^\dagger A^2A^*A^\dagger A^2$ if and only if $DA_1 = A_1A_1^*A_1$ if and only if A is bi-dagger.

(2) ⇒ (4) If we multiply $A^*A^2 = A^\dagger A^2A^*A^\dagger A^2$ from the left side by $(A^\dagger)^*$, we observe that $A^2 = (A^\dagger)^*AA^*A^\dagger A^2$.

(4) ⇒ (2) Multiplyin $A^2 = (A^\dagger)^*AA^*A^\dagger A^2$ from the left side by A^*, we get $A^*A^2 = A^\dagger A^2A^*A^\dagger A^2$.

(2) ⇔ (3) and (4) ⇔ (5) This equivalences follow by properties of conjugate operator.

(1) ⇔ (6) As in part (1) ⇔ (2) of this proof, from (10) and (11), A is bi-dagger if and only if $A_1^*D^{-1} = E^{-1}A_1^*D$ which is equivalent to $EA_1^* = A_1^*D^2$. Since $E = A_1^*DA_1 = DA_1^*A_1$, we get $EA_1^* = A_1^*(DA_1A_1^*) = A_1^*E$. Notice that

$$\begin{bmatrix} A_1^*D^2 & 0 \\ 0 & 0 \end{bmatrix} = (A^*)^2AA^*A = (A^*)^3A^2 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

if and only if $A_1^*D^2 = EA_1^*$ which is equivalent to A is bi-dagger.

(6) ⇒ (7) Using $(A^*)^2AA^*A = (A^*)^3A^2$, we obtain $(A^*)^2AA^* = (A^*)^2AA^*AA^\dagger = (A^*)^3A^2A^\dagger$.

(7) ⇒ (6) Multiplying $(A^*)^2AA^* = (A^*)^3A^2A^\dagger$ by A from the right side, we get (6).

(6) ⇔ (8) and (7) ⇔ (9) Applying properties of conjugate operator, we check these equivalence.

(9) ⇔ (10) If we multiply $AA^*A^2 = AA^\dagger A^*A^3$ by $A^\dagger$ from the left side, it follows that $A^*A^2 = A^\dagger A^*A^3$. The converse is evident.

(10) ⇔ (11) and (12) ⇔ (13) It follows by conjugate operator properties.

(11) ⇔ (12) Similarly as ((9) ⇔ (10)), we verify this equivalence. □

3. Generalized Aluthge transformation of SD operators

In this section, we obtain generalized Aluthge transformation, generalized $*$-Aluthge transformation and generalized $\dagger$ -Aluthge transformation of SD operators.

Definition 3.1. *(Generalized $\dagger$ -Aluthge transformation). Let $A = U|A|$ be the polar decomposition of a closed range operator $A \in \mathscr{B}(\mathscr{H})$. Generalized $\dagger$- Aluthge transformation of A for all $s, t > 0$ is defined by*

$$\widetilde{A}^{(\dagger)}(s,t) := \left(\widetilde{A^{\dagger}}(s,t)\right)^{\dagger} = \left(\left|A^{\dagger}\right|^{s} U^{*} \left|A^{\dagger}\right|^{t}\right)^{\dagger}.$$

Theorem 3.2. *Let $A = U|A|$ be the polar decomposition of a SD operator $A \in \mathscr{B}(\mathscr{H})$. Then the following properties hold:*

(1) $\widetilde{A}(s,t) = |A|^{s+t-1}A$ *for any $s, t > 0$ such that $s + t > 1$.*
(2) $\widetilde{A}(s,t) = (|A|^{\dagger})^{1-s-t}A$ *for any $s, t > 0$ such that $1 > s + t > 0$.*
(3) $\widetilde{A}(s,t) = \widetilde{A}(s', t')$ *for any $s, t > 0$ and $s', t' > 0$ such that $s + t = s' + t'$.*
(4) $\widetilde{A}(s, 2 - s) = |A|A$ *for any $s \in (0, 2)$.*
(5) $\widetilde{A}(s, 1 - s) = A^{\dagger}A^2$ *for any $s \in (0, 1)$.*
(6) $\widetilde{A}^{(*)}(s,t) = A^2|A|^{s+t-3}A^*$ *for any $s, t > 0$ such that $s + t > 3$.*
(7) $\widetilde{A}^{(*)}(s,t) = A^2(|A|^{\dagger})^{3-s-t}A^*$ *for any $s, t > 0$ such that $s + t < 3$.*
(8) $\widetilde{A}^{(*)}(s,t) = \widetilde{A}^{(*)}(s', t')$ *for any $s, t > 0$ and $s', t' > 0$ such that $s + t = s' + t'$*
(9) $\widetilde{A}^{(*)}(s, 4 - s) = A^2|A|A^*$ *for any $s \in (0, 4)$.*
(10) $\widetilde{A}^{(*)}(s, 1 - s) = A^2A^{\dagger}$ *for any $s \in (0, 1)$.*
(11) $\widetilde{A}^{\dagger}(s,t) = A(|A|^{\dagger})^{s+t+3}(A^*)^2$ *for any $s, t > 0$.*
(12) $\widetilde{A}^{\dagger}(s, 1 - s) = A(A^{\dagger})^2$ *for any $s \in (0, 1)$.*

Proof. (1) By matrix form (2) of A and Theorem 1.2, generalized Alutghe transform accepts the following matrix representations

$$\begin{aligned}
\widetilde{A}(s,t) &= |A|^{s}U|A|^{t} = \begin{bmatrix} D^{\frac{s}{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 D^{-\frac{1}{2}} & 0 \\ A_2 D^{-\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} D^{\frac{t}{2}} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} D^{\frac{s}{2}} A_1 D^{\frac{t-1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \\
\text{by} \quad (5) \quad &= \begin{bmatrix} A_1 D^{\frac{s+t-1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} D^{\frac{s+t-1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} \\
(\text{for} \quad s + t > 1) \quad &= |A|^{s+t-1}A.
\end{aligned}$$

(2) Note that

$$\begin{aligned}
\widetilde{A}(s,t) &= \begin{bmatrix} A_1 D^{\frac{s+t-1}{2}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D^{\frac{-1}{2}(1-s-t)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} \\
(\text{for} \quad 1 > s + t > 0) \quad &= (|A|^{\dagger})^{1-s-t}A.
\end{aligned}$$

(3) The statements (1) and (2) yield that

$$\widetilde{A}(s,t) = \begin{bmatrix} D^{\frac{s+t-1}{2}} A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D^{\frac{s'+t'-1}{2}} A_1 & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{A}(s', t').$$

(4) It is evident by part (1).
(5) By statement (2), we have

$$\widetilde{A}(s,1-s)=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}A_1&0\\A_2&0\end{bmatrix}=A^{\dagger}A^2.$$

(6) We recall the following relation [16, Lemma 3.12]:

$$|A^*|^{\alpha} \quad = \quad U\,|A|^{\alpha}\,U^* \quad \text{holds for } \alpha>0.$$

Therefore

$$\widetilde{A}^{(*)}(s,t) \quad = \quad |A^*|^sU|A^*|^t=U\,|A|^s\,U^*U^2\,|A|^t\,U^*. \tag{12}$$

Straightforward computation implies that $U^*U^2=\begin{bmatrix}D^{\frac{-1}{2}}A_1&0\\0&0\end{bmatrix}$. Hence, we have

$$\begin{aligned}\widetilde{A}^{(*)}(s,t) &= \begin{bmatrix}A_1D^{-\frac{1}{2}}&0\\A_2D^{-\frac{1}{2}}&0\end{bmatrix}\begin{bmatrix}A_1D^{\frac{s+t-1}{2}}&0\\0&0\end{bmatrix}\begin{bmatrix}D^{-\frac{1}{2}}A_1^*&D^{-\frac{1}{2}}A_2^*\\0&0\end{bmatrix}\\
&= \begin{bmatrix}A_1D^{-\frac{1}{2}}A_1D^{\frac{s+t-1}{2}}&0\\A_2D^{-\frac{1}{2}}A_1D^{\frac{s+t-1}{2}}&0\end{bmatrix}\begin{bmatrix}D^{-\frac{1}{2}}A_1^*&D^{-\frac{1}{2}}A_2^*\\0&0\end{bmatrix}\\
&= \begin{bmatrix}A_1D^{-\frac{1}{2}}A_1D^{\frac{s+t-2}{2}}A_1^*&A_1D^{-\frac{1}{2}}A_1D^{\frac{s+t-2}{2}}A_2^*\\A_2D^{-\frac{1}{2}}A_1D^{\frac{s+t-2}{2}}A_1^*&A_2D^{-\frac{1}{2}}A_1D^{\frac{s+t-2}{2}}A_2^*\end{bmatrix}\\
&= A^2(A^*A)^{\frac{s+t-3}{2}}A^*.\end{aligned}$$

(7) and (8) By matrix forms presented in the proof of part (6), we obtain the desired results.
(9) It follows by (6).
(10) From matrix forms given in the proof of (6) and matrix representation of $A^{\dagger}$, we conclude that $\widetilde{A}^{(*)}(s,1-s)=A^2A^{\dagger}$ for any $s\in(0,1)$.
(11) By applying Lemma 3.1 and Corollary 5.3 of [16] and obtained matrix forms of operators, we conclude that

$$\begin{aligned}\widetilde{A^{\dagger}}(s,t) &= |A^{\dagger}|^sU^*|A^{\dagger}|^t\\
&= U|(A^*)^{\dagger}|^sU^*U^*U|(A^*)^{\dagger}|^tU^*\\
&= U|(A^*)^{\dagger}|^sU^*|(A^*)^{\dagger}|^tU^*\\
&= U(|A|^s)^{\dagger}U^*(|A|^t)^{\dagger}U^*\\
&= \begin{bmatrix}A_1D^{-\frac{1}{2}}&0\\A_2D^{-\frac{1}{2}}&0\end{bmatrix}\begin{bmatrix}D^{\frac{-s}{2}}A_1^*D^{\frac{-1}{2}}D^{\frac{-t}{2}}&0\\0&0\end{bmatrix}\begin{bmatrix}D^{-\frac{1}{2}}A_1^*&D^{-\frac{1}{2}}A_2^*\\0&0\end{bmatrix}\\
&= \begin{bmatrix}A_1D^{-\frac{1}{2}}&0\\A_2D^{-\frac{1}{2}}&0\end{bmatrix}\begin{bmatrix}A_1^*D^{\frac{-s-t-1}{2}}&0\\0&0\end{bmatrix}\begin{bmatrix}D^{-\frac{1}{2}}A_1^*&D^{-\frac{1}{2}}A_2^*\\0&0\end{bmatrix}\\
&= \begin{bmatrix}A_1D^{\frac{-s-t-3}{2}}A_1^*A_1^*&A_1D^{\frac{-s-t-3}{2}}A_1^*A_2^*\\A_2D^{\frac{-s-t-3}{2}}A_1^*A_1^*&A_2D^{\frac{-s-t-3}{2}}A_1^*A_2^*\end{bmatrix}\\
&= A(|A|^{\dagger})^{s+t+3}(A^*)^2.\end{aligned}$$

(12) Using matrix forms presented in the proof of (11) and matrix representation $A^{\dagger}$, we prove that $\widetilde{A^{\dagger}}(s,1-s)=A(A^{\dagger})^2$ for any $s\in(0,1)$.
□

Corollary 3.3. *If $A\in\mathscr{B}(\mathscr{H})$ is an idempotent operator, then, for any $s\in(0,1)$, the following properties hold:*

(1) $\widetilde{A}(s,1-s)=P_{\mathscr{R}(A^*)}$.

(2) $\widetilde{A}^{(*)}(s,1-s)=P_{\mathscr{R}(A)}$.

(3) $\widetilde{A}^{(\dagger)}(s, 1-s) = AA^*$.

Proof. (1) and (2) follow by parts (5) and (10) of Theorem 3.2.
(3) By (12) of Theorem 3.2 and (30) of Theorem 2.1, we have

$$\widetilde{A}^{\dagger}(s, 1-s) = A(A^{\dagger})^2 = \left(A^{\dagger}\right)^* A^{\dagger}A^*.$$

On the other hand,

$$\begin{aligned}(A^*)^2 - A^* = 0 \quad &\Leftrightarrow \quad A^*(A^* - 1) = 0 \\ &\Leftrightarrow \quad \mathscr{R}((A^* - 1)) \subset \mathscr{N}(A^*) \\ &\Leftrightarrow \quad \mathscr{R}((A^* - 1)) \subset \mathscr{N}(A^{\dagger}) \\ &\Leftrightarrow \quad A^{\dagger}A^* = A^{\dagger}.\end{aligned}$$

Therefore,

$$\widetilde{A}^{\dagger}(s, 1-s) = \left(A^{\dagger}\right)^* A^{\dagger} = (AA^*)^{\dagger}.$$

Then $\widetilde{A}^{(\dagger)}(s, 1-s) = AA^*$.
□

Corollary 3.4. *If $A \in \mathscr{B}(\mathscr{H})$ is an idempotent operator, then, for any $s, s', s'' \in (0, 1)$, the following assertions hold:*

(1) $\widetilde{A}^{(*)}(s, 1-s)\widetilde{A}^{(\dagger)}(s', 1-s') = \widetilde{A}^{(\dagger)}(s', 1-s')\widetilde{A}^{(*)}(s, 1-s) = AA^*$
(2) $\widetilde{A}(s, 1-s)\widetilde{A}^{(*)}(s', 1-s')\widetilde{A}^{(\dagger)}(s'', 1-s'') = A^*$
(3) $\widetilde{A}^{(*)}(s', 1-s')\widetilde{A}^{(\dagger)}(s'', 1-s'')\widetilde{A}(s, 1-s) = A$.

Theorem 3.5. *If $A \in \mathscr{B}(\mathscr{H})$ is a* SD *and bi-dagger operator, then $\widetilde{A}^{(*)}(s', 1-s')\widetilde{A}(s, 1-s) = A^2$ for any $s', s \in (0, 1)$.*

Proof. Parts (10) and (5) of Theorem 3.2 impliy that

$$\widetilde{A}^{(*)}(s', 1-s')\widetilde{A}(s, 1-s) = A^2(A^{\dagger})^2A^2 = A^2.$$

□

Recall that the **Dixmier angle** between two closed subspaces $\mathscr{M}$ and $\mathscr{N}$ of $\mathscr{H}$ is the angle $\theta_0(\mathscr{M}, \mathscr{N}) \in \left[0, \frac{\pi}{2}\right]$ whose cosine is defined by

$$c_0(\mathscr{M}, \mathscr{N}) = \sup\{|\langle \xi, \eta \rangle| : \xi \in \mathscr{M}, \eta \in \mathscr{N} \text{ and } \|\xi\| \leqslant 1, \|\eta\| \leqslant 1\}.$$

See [9].

Corollary 3.6. *If $A \in \mathscr{B}(\mathscr{H})$ is an idempotent operator, then, for any $s', s \in (0, 1)$, the following assertions hold:*

(1) $c_0(\mathscr{R}(\widetilde{A}(s, 1-s)), \mathscr{R}(\widetilde{A}^{(*)}(s', 1-s'))) = \|A^{\dagger}\|$
(2) $c_0(\mathscr{R}(\widetilde{A}^{(\dagger)}(s', 1-s')), \mathscr{R}(\widetilde{A}(s, 1-s))) = \|A^{\dagger}\|$
(3) $c_0(\mathscr{R}(\widetilde{A}^{(*)}(s', 1-s')), \mathscr{R}(\widetilde{A}^{(\dagger)}(s, 1-s))) = 1$.

Proof. (1) By Corollary 3.3 and [9, Proposition 2.1.], we conclude that

$$c_0(\mathscr{R}(\widetilde{A}(s, 1-s)), \mathscr{R}(\widetilde{A}^{(*)}(s', 1-s'))) = \| P_{\mathscr{R}(A^*)}P_{\mathscr{R}(A)} \| = \|A^{\dagger}A^2A^{\dagger}\| = \|A^{\dagger}\|.$$

Similar to (1), (2) and (3) are also proved. □

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