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A note on the independent bondage number of planar graphs

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Abstract. A vertex subset S of a graph G is an independent set if no two vertices in S are adjacent and is a dominating set if every vertex not in S is adjacent to a vertex in S. If S is both independent and dominating in G, then S is said to be an independent dominating set. The independent domination number of G is the minimum cardinality among all independent dominating sets of G. In this paper, we investigate the independent bondage number of G defined as the minimum number of edges whose removal from G results in a graph with a greater independent domination number. We prove that the independent bondage number is at most 5 (respectively, 6, 7) for planar graphs with minimum degree at least 3 without cycles of lengths 4 and 5 (respectively, without cycles of length 4, without intersecting triangles). All these results improve two earlier bounds for planar graphs.

1. Introduction

All graphs considered are finite and simple. For a graph G, we denote by $V(G)$ and $E(G)$ its *vertex set* and *edge set*, respectively. The *open neighborhood* of a vertex $v \in V(G)$ is the set of vertices adjacent to v, denoted $N_G(v)$. The *degree* of vertex v of G, denoted by $d_G(v)$, is the cardinality of its open neighborhood. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ is the minimum cardinality of a maximal independent set in G.

In this paper, we are interested in studying the *independent bondage number* $b_i(G)$ of a graph G defined as the cardinality of a smallest set of edges $F \subseteq E(G)$ for which $i(G - F) > i(G)$. The concept of bondage number was first introduced and defined for the domination number in 1983 by Bauer et al. [1] and it was later studied for several domination parameters. We refer the reader to the survey [7].

To our knowledge, the independent bondage number has only been studied in three papers, which explains the few results obtained to date. Indeed, in 2017, Nader Jafari and Kamarulhaili [4] showed that the associated decision problem for independent bondage is NP-hard, even for bipartite graphs. In 2018, Priddy, Wang and Wei [6] gave the exact values of $b_i(G)$ for some classes of graphs including paths, cycles, complete graphs and complete t-partite graphs. Moreover, they gave an upper bound on $b_i(G)$, which will be useful to us in the remainder, expressed in terms of the degree sum of any two adjacent vertices of G.

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Theorem 1.1 (Priddy, Wang and Wei, [6]). *If G is a nonempty graph, then*

$$b_i(G) \leq \min\{d_G(u) + d_G(v) - |N_G(u) \cap N_G(v)| - 1 : \ uv \in E(G)\}.$$

Restricted to planar graphs, Priddy et al. [6] provided the following sharp upper bound in terms of the maximum degree

Theorem 1.2 (Priddy, Wang and Wei, [6]). *If G is a connected planar graph, then $b_i(G) \leq \Delta(G) + 2$.*

One of the questions raised in [6] was whether there was a constant C such that $b_i(G) \leq C$ for any planar graph G. This question has been addressed by Pham and Wei [5] for planar graphs with minimum degree at least three by showing the following upper bound.

Theorem 1.3 (Pham and Wei, [5]). *If G is a planar graph with $\delta(G) \geq 3$, then $b_i(G) \leq \min\{9, \Delta(G) + 2\}$.*

It is worth mentioning that Pham and Wei were unable to provide an example for the sharpness of the upper bound in Theorem 1.3. However, they provided a class of planar graphs G with minimum degree at least 3 and $b_i(G) = 6$.

More recently, Gamlath *et al.* [2] improved the bound of Theorem 1.3 by showing the following

Theorem 1.4 (Gamlath, Wei and Reid, [2]). *If G is a planar graph with $\delta(G) \geq 3$, then $b_i(G) \leq \min\{8, \Delta(G) + 2\}$.*

Our aim in this paper is to improve the upper bounds in Theorems 1.2 and 1.4 for some classes of planar graphs. Recall that, for any integer $k \geq 3$, C_k denote a cycle of length k. Moreover, when we say that a graph has no cycle meaning that such a cycle is not necessarily induced. Also, we say that two cycles are adjacent if they share a common edge. In the rest of the paper, we shall prove the following.

Theorem 1.5. *Let G be a planar graph with $\delta(G) \geq 3$.*

1. *If G is without C_4, then $b_i(G) \leq 6$.*
2. *If G is without intersecting triangles, then $b_i(G) \leq 7$.*
3. *If G is without a C_4 adjacent to a C_3, then $b_i(G) \leq 7$.*

Before proving our results, we give some additional definitions and notations.

Let G be a planar graph. We use $F(G)$ to denote the set of *faces* of G. Let $r_G(f)$ denote the degree of a face f in G. A vertex of degree k is called a *k-vertex*. A *k^+-vertex* (respectively, *k^--vertex*) is a vertex of degree at least k (respectively, at most k). We use the same notations for faces, more precisely, a *k-face* (respectively, *k^+-face*, *k^--face*) is a face of degree k (respectively, at least k, at most k). A k-face having the boundary vertices x_1, x_2, ..., x_k in the cyclic order is denoted by $[x_1x_2...x_k]$. A *(k_1, k_2, k_3)-triangle* is a 3-face $[xyz]$ with $d_G(x) = k_1$, $d_G(y) = k_2$ and $d_G(z) = k_3$. For a vertex $v \in V(G)$, let $n_i(v)$ denote the number of i-vertices adjacent to v for $i \geq 1$, and $m_i(v)$ the number of i-faces incident to v.

2. Independent bondage of planar graphs without small cycles

2.1. *Proof of Theorem 1.5-(1)*

We proceed by contradiction. Let H be a planar graph with $\delta(H) \geq 3$ and without C_4 such that $b_i(H) > 6$. By Theorem 1.1, H satisfies the following properties:

Claim 2.1. *H does not contain a 3-vertex adjacent to a 4^--vertex.*

Claim 2.2. *H does not contain a $(4^-, 4^-, \Delta^-(H))$-triangle.*

We apply a discharging procedure. Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$ can be rewritten as $(6|E(H)| - 10|V(H)|) + (4|E(H)| - 10|F(H)|) = -20$. Using the relation $\sum_{v \in V(H)} d_H(v) = \sum_{f \in F(H)} r_H(f) = 2|E(H)|$ we get that:

$$\sum_{v \in V(H)} (3d_H(v) - 10) + \sum_{f \in F(H)} (2r_H(f) - 10) = -20 \tag{1}$$

We define the weight function $\omega : V(H) \cup F(H) \longrightarrow \mathbb{R}$ by $\omega(x) = 3d_H(x) - 10$ if $x \in V(H)$ and $\omega(x) = 2r_H(x) - 10$ if $x \in F(H)$. It follows from Equation (1) that the total sum of weights equals -20. In what follows, we will define discharging rules (R1) to (R3) and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is finished. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(H) \cup F(H)$, leading us to the following contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) = \sum_{x \in V(H) \cup F(H)} \omega(x) = -20 < 0$$

and hence will demonstrate that such a counterexample cannot exist.

The discharging rules are defined as follows:

(R1) Every 5^+-vertex gives $\frac{1}{3}$ to each adjacent 3-vertex.
(R2) Every 4-vertex gives 1 to each incident 3-face.
(R3) Every 5^+-vertex gives 2 to each incident 3-face.

Since H does not contain C_4, by hypothesis, the following fact is easy to observe and will be frequently used throughout the proof without further notice.

Observation 2.3. *A k-vertex v is incident to at most $\lfloor \frac{k}{2} \rfloor$ 3-faces.*

Let $v \in V(H)$ be a k-vertex and recall that $\delta(H) \geq 3$. Consider the following cases:

1. Case $k = 3$. Observe that $\omega(v) = -1$. By Claim 2.1, v has three neighbors of degree at least 5. Then, by (R1) we have $\omega^*(v) = -1 + 3 \times \frac{1}{3} = 0$.
2. Case $k = 4$. Observe that $\omega(v) = 2$. By Claim 2.1, v has four neighbors of degree at least 4. By Observation 2.3, v is incident to at most two 3-faces and thus by (R2) $\omega^*(v) \geq 2 - 2 \times 1 = 0$.
3. Case $k = 5$. Observe that $\omega(v) = 5$. By Claim 2.2 and Observation 2.3, v is incident to at most two $(3, 5, 5^+)$-triangles. Hence, by (R1) and (R3), $\omega^*(v) \geq 5 - \max\{2\times(2+\frac{1}{3})+1\times\frac{1}{3}, 1\times(2+\frac{1}{3})+4\times\frac{1}{3}, 5\times\frac{1}{3}\} \geq 0$.
4. Case $k \geq 6$. Observe that $\omega(v) = 3k - 10$. By (R1) and (R3), we have :

$$\begin{aligned}
\omega^*(v) &= 3k - 10 - 2 \times m_3(v) - \frac{1}{3} \times n_3(v) \\
&\geq 3k - 10 - 2 \times \left\lfloor \frac{k}{2} \right\rfloor - \frac{1}{3} \times \left(k - \left\lfloor \frac{k}{2} \right\rfloor\right) \\
&\geq \frac{11}{6}k - 10 \geq 0
\end{aligned}$$

Let $f \in F(H)$ be a k-face.

1. Case $k = 3$. Observe that $\omega(f) = -4$. Suppose $f = [rst]$ and consider the following cases:
   (a) Suppose $d_H(r) = 3$. Hence, by Claim 2.1, r is the unique 3-vertex and $d_H(s) \geq 5$ and $d_H(t) \geq 5$. Therefore, by (R3), we have $\omega^*(f) = -4 + 2 \times 2 \geq 0$

(b) Suppose now $d_H(r) \geq 4$, $d_H(s) \geq 4$ and $d_H(t) \geq 4$. By Claim 2.2, at least one of the three vertices r, s and t is a 5^+-vertex, say $d_H(t) \geq 5$.
Hence, by (R2) and (R3), $\omega^*(f) \geq -4 + \min\{2 \times 1 + 1 \times 2, 1 \times 1 + 2 \times 2, 3 \times 2\} \geq 0$.

2. Case $k \geq 5$. The initial charge of f is $\omega(f) = 2k - 10 \geq 0$ and it remains unchanged during the discharging process. Hence, $\omega^*(f) = \omega(f) = 2k - 10 \geq 0$.

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore, H cannot exist. □

2.2. Proof of Theorem 1.5-(2)

We proceed by contradiction. Let H be a planar graph with $\delta(H) \geq 3$ and without intersecting triangles, such that $b_i(H) > 7$. By Theorem 1.1, we have the following properties:

Claim 2.4. *H does not contain a 3-vertex adjacent to a 5^--vertex.*

Claim 2.5. *H does not contain a $(3^-, 6^-, \Delta^-(H))$-triangle.*

Claim 2.6. *H does not contain a $(4^-, 5^-, \Delta^-(H))$-triangle.*

Now, we apply a discharging procedure. Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$ can be rewritten as $(2|E(H)| - 4|V(H)|) + (2|E(H)| - 4|F(H)|) = -8$. Using the relation $\sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} r(f) = 2|E(H)|$ we get that:

$$\sum_{v \in V(H)} (d(v) - 4) + \sum_{f \in F(H)} (r(f) - 4) = -8 \tag{2}$$

We define the weight function $\omega : V(H) \cup F(H) \longrightarrow \mathbb{R}$ by $\omega(x) = d_H(x) - 4$ if $x \in V(H)$ and $\omega(x) = r_H(x) - 4$ if $x \in F(H)$. It follows from Equation (2) that the total sum of weights equals -8.

As before, we will define discharging rules (R1) to (R3) as follows :

(R1) Every k-vertex, for $k \geq 6$, gives $\frac{1}{3}$ to each adjacent 3-vertex.
(R2) Every 5-vertex gives $\frac{1}{3}$ to each incident 3-face.
(R3) Every k-vertex, for $k \geq 6$, gives $\frac{1}{2}$ to each incident 3-face.

Since H does not contain intersecting triangles, by hypothesis, the following fact is easy to observe and will be frequently used throughout the proof without further notice.

Observation 2.7. *Let v be a k-vertex with $k \geq 3$. Then $m_3(v) \leq 1$.*

Let $v \in V(H)$ be a k-vertex. Recall that $\delta(H) \geq 3$ and consider the following cases:

1. Case $k = 3$. Observe that $\omega(v) = -1$. By Claim 2.4, v has three neighbors of degree at least 6. Hence, by (R1), we have $\omega^*(v) = -1 + 3 \times \frac{1}{3} = 0$.
2. Case $k = 4$. Observe that $\omega(v) = 0$ and v gives nothing. Hence, we have $\omega^*(v) = \omega(v) = 0$.
3. Case $k = 5$. Observe that $\omega(v) = 1$. By Claim 2.4, v has five neighbors of degree at least 4. By Observation 2.7, we have the following :
   (a) If $m_3(v) = 1$, then by (R2), $\omega^*(v) \geq 1 - \frac{1}{3} > 0$.
   (b) If $m_3(v) = 0$, then v gives nothing. Hence, we have $\omega^*(v) = \omega(v) = 1 > 0$.
4. Case $k = 6$. Observe that $\omega(v) = 2$. By Observation 2.7, we have the following:

(a) If $m_3(v) = 1$,. By Claim 2.4 and Claim 2.5, v is adjacent to at most four 3-vertices. Hence, by (R1) and (R3), $\omega^*(v) \geq 2 - 4 \times \frac{1}{3} - 1 \times \frac{1}{2} \geq 0$.

(b) If $m_3(v) = 0$, then by (R1) , $\omega^*(v) \geq 2 - 6 \times \frac{1}{3} = 0$.

5. Case $k \geq 7$. Observe that $\omega(v) = k - 4$. By Observation 2.7, we have the following :

(a) If $m_3(v) = 1$, then by (R1) and (R3):

$$\begin{aligned}\omega^*(v) &= k - 4 - \frac{1}{2} \times m_3(v) - \frac{1}{3} \times n_3(v)\\ &\geq k - 4 - \frac{1}{2} \times 1 - \frac{1}{3} \times k\\ &\geq \frac{4k - 27}{6} > 0\end{aligned}$$

(b) If $m_3(v) = 0$, then by (R1), $\omega^*(v) \geq k - 4 - k \times \frac{1}{3} = \frac{2}{3}k - 4 > 0$.

Let $f \in F(H)$ be a k-face.

1. Case $k = 3$. Observe that $\omega(f) = -1$. Suppose $f = [rst]$ and consider the following situations:

(a) Suppose $d_H(r) = 3$. Then, by Claim 2.5, r is the unique 3-vertex and $d_H(s) \geq 7$ and $d_H(t) \geq 7$. Hence, by (R3), we have $\omega^*(f) = -1 + 2 \times \frac{1}{2} = 0$

(b) Suppose now $d_H(r) = 4$. Then by Claim 2.6, r is the unique 4-vertex and $d_H(s) \geq 6$ and $d_H(t) \geq 6$. Hence, by (R3), $\omega^*(f) = -1 + 2 \times \frac{1}{2} = 0$.

(c) Assume that $d_H(r) \geq 5$. Then by Claim 2.6, $d_H(s) \geq 5$ and $d_H(t) \geq 5$. Hence, by (R2) and (R3), $\omega^*(f) \geq -1 + \min\{\frac{1}{3} \times 3, \frac{1}{3} \times 2 + \frac{1}{2} \times 1, \frac{1}{3} \times 1 + \frac{1}{2} \times 2, \frac{1}{2} \times 3\} \geq 0$.

2. Case $k \geq 4$. The initial charge of f is $\omega(f) = k - 4 \geq 0$ and it remains unchanged during the discharging process. Hence $\omega^*(f) = \omega(f) = k - 4 \geq 0$.

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore, H cannot exist. □

2.3. *Proof of Theorem 1.5-(3)*

We proceed by contradiction. Let H be a planar graph with $\delta(H) \geq 3$ and without a C_4 adjacent to a C_3, such that $b_i(H) > 7$. Note that H satisfies the reducible configurations given in Claim 2.4, Claim 2.5 and Claim 2.6.

Now, we apply a discharging procedure.

We define the weight function $\omega : V(H) \cup F(H) \longrightarrow \mathbb{R}$ by $\omega(x) = 3d_H(x) - 10$ if $x \in V(H)$ and $\omega(x) = 2r_H(x) - 10$ if $x \in F(H)$. It follows from Equation (1) that the total sum of weights equals -20. As before, we will define discharging rules (R1) to (R5) as follows :

(R1) Every k-vertex, for $k \geq 6$, gives $\frac{1}{3}$ to each adjacent 3-vertex.
(R2) Every 4-vertex gives 1 to each incident 3-face.
(R3) Every k-vertex, for $k \geq 5$, gives 2 to each incident 3-face.
(R4) Every 4-vertex gives $\frac{1}{2}$ to each incident 4-face.
(R5) Every k-vertex, for $k \geq 5$, gives 1 to each incident 4-face.

Since H does not contain a C_4 adjacent to a C_3, it is easy to observe that H does not contain adjacent 3-cycles. One can easily derive the following observation.

Observation 2.8. *Let v be a k-vertex with $k \geq 3$. Then $m_3(v) \leq \lfloor \frac{k}{2} \rfloor$. Moreover :*

1. *If* $m_3(v) = \lfloor \frac{k}{2} \rfloor$, *then* $m_4(v) = 0$.
2. *If* $1 \leq m_3(v) < \lfloor \frac{k}{2} \rfloor$, *then* $m_4(v) \leq d_H(v) - 2 \times m_3(v) - 1$.
3. *If* $m_3(v) = 0$, *then* $m_4(v) \leq d_H(v)$.

Let $v \in V(H)$ be a k-vertex. Recall that $\delta(H) \geq 3$ and consider the following cases:

1. Case $k = 3$. Observe that $\omega(v) = -1$. By Claim 2.4, v has three neighbors of degree at least 6. Hence, by (R1), we have $\omega^*(v) = -1 + 3 \times \frac{1}{3} = 0$.
2. Case $k = 4$. Observe that $\omega(v) = 2$. By Claim 2.4, v has four neighbors of degree at least 4. By Observation 2.8, we have the following :
   (a) If $m_3(v) = 2$, then $m_4(v) = 0$, and hence, by (R2), $\omega^*(v) \geq 2 - 2 \times 1 = 0$.
   (b) If $m_3(v) = 1$, then $m_4(v) \leq 1$, and hence, by (R2) and (R4), $\omega^*(v) \geq 2 - 1 \times 1 - 1 \times \frac{1}{2} > 0$.
   (c) If $m_3(v) = 0$, then $m_4(v) \leq 4$, and hence, by (R4), $\omega^*(v) \geq 2 - 4 \times \frac{1}{2} = 0$.
3. Case $k = 5$. Observe that $\omega(v) = 5$. By Claim 2.4, v has five neighbors of degree at least 4. By Observation 2.8, we have the following :
   (a) If $m_3(v) = 2$, then $m_4(v) = 0$, and hence, by (R3), $\omega^*(v) \geq 5 - 2 \times 2 > 0$.
   (b) If $m_3(v) = 1$, then $m_4(v) \leq 2$, and hence, by (R3) and (R5), $\omega^*(v) \geq 5 - 2 \times 1 - 2 \times 1 > 0$.
   (c) If $m_3(v) = 0$, then $m_4(v) \leq 5$, and hence, by (R5), $\omega^*(v) \geq 5 - 5 \times 1 = 0$.
4. Case $k \geq 6$. Observe that $\omega(v) = 3k - 10$. By Observation 2.8, we have the following :
   (a) If $m_3(v) = \lfloor \frac{k}{2} \rfloor$, then $m_4(v) = 0$. Hence, by (R1) and (R3):

$$\begin{aligned}\omega^*(v) &= 3k - 10 - 2 \times m_3(v) - \frac{1}{3} \times n_3(v) \\ &\geq 3k - 10 - 2 \times \left\lfloor \frac{k}{2} \right\rfloor - \frac{1}{3} \times k \\ &\geq \frac{5}{3}k - 10 > 0\end{aligned}$$

   (b) If $1 \leq m_3(v) \leq \lfloor \frac{k}{2} \rfloor - 1$, then $m_4(v) \leq k - 3$. Moreover, v has at most $k - m_3(v)$ neighbors of degree 3. Hence, by (R1), (R3) and (R5):

$$\begin{aligned}\omega^*(v) &= 3k - 10 - 2 \times m_3(v) - 1 \times m_4(v) - \frac{1}{3} \times n_3(v) \\ &\geq 3k - 10 - 2 \times (\lfloor \frac{k}{2} \rfloor - 1) - (k-3) \times 1 - \frac{1}{3} \times (k - m_3(v)) \\ &\geq 3k - 10 - 2 \times (\lfloor \frac{k}{2} \rfloor - 1) - (k-3) \times 1 - \frac{1}{3} \times (k - (\lfloor \frac{k}{2} \rfloor - 1)) \\ &\geq \frac{5}{6}k - 5 \geq 0\end{aligned}$$

   (c) If $m_3(v) = 0$, then $m_4(v) \leq k$. Hence, by (R1) and (R5),

$$\omega^*(v) \geq 3k - 10 - k \times 1 - k \times \frac{1}{3} = \frac{5}{3}k - 10 \geq 0.$$

Let $f \in F(H)$ be a k-face.

1. Case $k = 3$. Observe that $\omega(f) = -4$. Suppose $f = [rst]$ and consider the following situations:
   (a) Suppose $d_H(r) = 3$. Then, by Claim 2.4, r is the unique 3-vertex and $d_H(s) \geq 6$ and $d_H(t) \geq 6$. Hence, by (R3), we have $\omega^*(f) = -4 + 2 \times 2 \geq 0$

(b) Suppose now $d_H(r) \geq 4$, $d_H(s) \geq 4$ and $d_H(t) \geq 4$. By Claim 2.6, at least one of the three vertices r, s and t is a 5^+-vertex. Say $d_H(t) \geq 5$.
Then, by (R2) and (R3), $\omega^*(f) \geq -4 + \min\{2 \times 1 + 1 \times 2, 1 \times 1 + 2 \times 2, 3 \times 2\} \geq 0$.

2. Case $k = 4$. The initial charge of f is $\omega(f) = -2$. By Claim 2.4, at most two 3-vertices are incident to the 4-face. Suppose $f = [rstu]$ and consider the following situations:
   (a) Suppose $d_H(r) = 3 = d_H(t)$. By Claim 2.4, $d_H(s) \geq 6$ and $d_H(u) \geq 6$. Hence, by (R5), we have $\omega^*(f) = -2 + 2 \times 1 \geq 0$
   (b) Suppose now $d_H(r) = 3$. By Claim 2.4, $d_H(s) \geq 6$ and $d_H(u) \geq 6$. Moreover, assume $d_H(t) \geq 4$. Then, by (R4) and (R5), $\omega^*(f) \geq -2 + \min\{2 \times 1 + 1 \times \frac{1}{2}, 3 \times 1\} \geq 0$.
   (c) Assume $d_H(r) \geq 4$, $d_H(s) \geq 4$, $d_H(t) \geq 4$ and $d_H(u) \geq 4$. Then, by (R4) and (R5), $\omega^*(f) \geq -2 + \min\{4 \times \frac{1}{2}, 3 \times \frac{1}{2} + 1 \times 1, 2 \times \frac{1}{2} + 2 \times 1, 1 \times \frac{1}{2} + 3 \times 1, 4 \times 1\} \geq 0$.
3. Case $k \geq 5$. The initial charge of f is $\omega(f) = 2k - 10 \geq 0$ and it remains unchanged during the discharging process. Hence, $\omega^*(f) = \omega(f) = 2k - 10 \geq 0$.

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore, H cannot exist. □

We conclude this section by giving an upper bound on the independent bondage number for another specific class of planar graphs, namely planar graphs with no C_4 and C_5. Since the proof uses the same reducible configurations and discharging rules as in the proof of Theorem 5.2 [3], we omit the proof of our result.

Theorem 2.9. *If G is a planar graph with $\delta(G) \geq 3$ and without C_4 and C_5, then $b_i(G) \leq 5$.*

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