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On a new generalization of Fibonacci and Lucas p-triangles

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Abstract. In this study, we introduce a new generalization of Fibonacci and Lucas p-triangles, which also provides a novel extension of the well-known Pascal's and Lucas triangles. The primary motivation for this investigation is to derive explicit formulas for the bi-periodic Fibonacci and Lucas p-numbers. To achieve this, a generalization of binomial coefficients is derived and several of their properties, including recurrence relations, the generating function, and convolution identity, are presented. Additionally, as an application of these triangles, we define bi-periodic incomplete Fibonacci and Lucas p-numbers and state several of their properties.

1. Introduction

The Fibonacci and Lucas sequences are widely used in both art and science, so their generalizations have been the focus of extensive study by numerous authors over many years. In particular, one interesting generalization of the Fibonacci sequence is the Fibonacci p-sequence, introduced by Stakhov and Rozin [18], and defined by the following lacunary recurrence relation of order $p+1$, where $p \geq 1$:

$$F_{p,n} = F_{p,n-1} + F_{p,n-p-1},\ n \geq p+1,$$

with initial values $F_{p,0} = 0$, $F_{p,i} = 1$ for $i = 1, 2, \ldots, p$. Similarly, the Lucas p-sequence is defined by the recurrence relation

$$L_{p,n} = L_{p,n-1} + L_{p,n-p-1},\ n \geq p+1,$$

but begins with initial values $L_{p,0} = p+1$, $L_{p,i} = 1$ for $i = 1, 2, \ldots, p$. It is clear to see that when $p = 1$, the Fibonacci and Lucas p-sequences reduce to the classical Fibonacci sequence $\{F_n\}$ and Lucas sequence

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$\{L_n\}$, respectively. For $p = 2$, we obtain the Fibonacci-Narayana sequence $\{N_n\}$ in [2], and Lucas-Narayana sequence $\{\mathcal{N}_n\}$.

Another interesting generalization of Fibonacci and Lucas sequences is the incomplete Fibonacci and incomplete Lucas sequences introduced by Filipponi [9]. Many authors have studied and generalized the incomplete Fibonacci and Lucas sequences, particularly Tasci and Firengiz [23] introduced the incomplete Fibonacci p-sequence $\{F_{p,n}(k)\}$ and incomplete Lucas p-sequence $\{L_{p,n}(k)\}$ by using the following combinatorial sums:

$$F_{p,n}(k) = \sum_{j=0}^{k} \binom{n-jp-1}{j} \quad \left(n = 1,2,3,\ldots; 0 \le k \le \left\lfloor \frac{n-1}{p+1} \right\rfloor\right)$$

and

$$L_{p,n}(k) = \sum_{j=0}^{k} \frac{n}{n-jp} \binom{n-jp}{j} \quad \left(n = 1,2,3,\ldots; 0 \le k \le \left\lfloor \frac{n}{p+1} \right\rfloor\right).$$

It is clear that when $k = \left\lfloor \frac{n-1}{p+1} \right\rfloor$, the incomplete Fibonacci p-numbers reduces to the Fibonacci p-numbers and when $k = \left\lfloor \frac{n}{p+1} \right\rfloor$, the incomplete Lucas p-numbers reduces to the Lucas p-numbers.

In this work, we consider a recent generalization of Fibonacci and Lucas sequences, previously studied by Ait-Amrane and Belbachir [1], and Yazlik et al. [25], which can be viewed as a broader extension of Fibonacci and Lucas p-sequences. In particular, for nonzero real numbers a, b, c and positive integer p, the bi-periodic Fibonacci p-sequence $\left\{U_{p,n}\right\}$ is defined by

$$U_{p,n} = a^{\xi(n+1)} b^{\xi(n)} U_{p,n-1} + c U_{p,n-p-1},\ n \ge p+1, \tag{1}$$

with the initial conditions $U_{p,0} = 0, U_{p,i} = a^{\lfloor i/2 \rfloor} b^{\lfloor (i-1)/2 \rfloor}$ for $i = 1, 2, \ldots, p$. A companion sequence related to the bi-periodic Fibonacci p-sequence is the bi-periodic Lucas p-sequence $\left\{V_{p,n}\right\}$ defined by

$$V_{p,n} = a^{\xi(n)} b^{\xi(n+1)} V_{p,n-1} + c V_{p,n-p-1},\ n \ge p+1, \tag{2}$$

with the initial conditions $V_{p,0} = p+1, V_{p,i} = a^{\lfloor (i+1)/2 \rfloor} b^{\lfloor i/2 \rfloor}$ for $i = 1, 2, \ldots, p$. Here $\xi(n) = \frac{1-(-1)^n}{2}$; that is, $\xi(n) = 0$ when n is even, and $\xi(n) = 1$ when n is odd. When $a = b = c = 1$ and $p = 1$, the bi-periodic Fibonacci and Lucas p-sequences reduce to the classical Fibonacci and Lucas sequences, respectively. For $p = c = 1$, the bi-periodic Fibonacci and Lucas p-sequences reduce to the bi-periodic Fibonacci and Lucas sequences in [8, 24] and [6], respectively. For $p = 2$ and $c = 1$, we obtain the bi-periodic Fibonacci-Narayana and bi-periodic Lucas-Narayana sequences. Also, if $c = 1$, they reduce to the generalized Fibonacci and Lucas p-sequences in [25], in addition, if $a = b = 1$ they reduce to the Fibonacci p-sequence and Lucas p-sequence in [18]. For related studies on these sequences, we refer to [3, 4, 7, 11, 16, 19–22].

These sequences are fundamental in mathematics and have numerous important applications in combinatorics, number theory, numerical analysis, and other fields, see [12]. Therefore, finding an explicit formula for a unified approach to dealing with them is essential, and this problem will be one of the focuses of the present paper.

On the other hand, Kuhapatanakul defined the Fibonacci p-triangle [13] and Lucas p-triangle [14] to derive several properties of Fibonacci p-numbers and Lucas p-numbers. For $p = 1$, they reduce to the classical Pascal's triangle and Lucas triangle, respectively, see Table 1 and Table 2. It is well-known that the sum of the elements along a rising diagonal of Pascal's and Lucas triangles is given by the Fibonacci and Lucas numbers. For more on these triangles, we refer to [5, 15, 17] and references therein. Since Pascal's and Lucas triangles are rich in mathematical properties and patterns, they are important tools in various areas of mathematics. Therefore, finding a generalization of these triangles would be interesting in its own right, and this will be another aim of this paper.

	0	1	2	3	4	5	
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 1: Pascal triangle

	0	1	2	3	4	5	
0	2						
1	1	2					
2	1	3	2				
3	1	4	5	2			
4	1	5	9	7	2		
5	1	6	14	16	9	2	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 2: Lucas triangle

The outline of this paper is as follows: In Section 2, we introduce a generalization of Pascal's triangle. We define the generalized binomial coefficients and explore several properties of them such as generating function, recurrence relations, convolution identity, etc. In Section 3, we construct the bi-periodic Fibonacci p-triangle and give a relation between the coefficients of the bi-periodic Fibonacci p-triangle and generalized binomial coefficients. Then, we derive an explicit formula for the bi-periodic Fibonacci p-sequences by using the Fibonacci p-triangle. Analogously, we define the bi-periodic Lucas p-triangle and state a link between the coefficients of the bi-periodic Fibonacci p-triangle and bi-periodic Lucas p-triangle. In Section 4, by using these triangles we introduce the bi-periodic incomplete Fibonacci and Lucas p-numbers and derive several properties of them.

2. A Generalization of Pascal's triangle

In this section, we introduce a generalization of Pascal's triangle. We define the generalized binomial coefficients and present some basic properties of them, such as the generating function, recurrence relations, symmetric relation, and convolution identity. This generalization allows us to provide an explicit formula for the bi-periodic Fibonacci p-numbers, which will be discussed in the next section.

The generalized Pascal's triangle is a triangular array of numbers where the entry in the n-th row and k-th column is called the generalized binomial coefficient, denoted by $\mathcal{B}_p(n,k;a,b)$, $\mathcal{B}_p(n,k)$ for short, and is defined as follows.

Definition 2.1. *Let p be a positive integer. For nonnegative integers $0 \le k \le n$, the generalized binomial coefficient $\mathcal{B}_p(n,k)$, is defined by the recurrence relation*

$$\mathcal{B}_p(n+1,k) = a^{\xi(n+pk+1)} b^{\xi(n+pk)} \mathcal{B}_p(n,k) + \mathcal{B}_p(n,k-1) \tag{3}$$

with initial conditions $\mathcal{B}_p(n,0) = a^{\lfloor (n+1)/2 \rfloor} b^{\lfloor n/2 \rfloor}$ and $\mathcal{B}_p(n,k) = 0$ for $n < k$.

For $a = b = p = 1$, the coefficient $\mathcal{B}_p(n,k)$ is reduced to the binomial coefficient. Using the recurrence relation, we give the first few values of the coefficients $\mathcal{B}_p(n,i)$ as shown in Table 3 and Table 4.

n	$\mathcal{B}_p(n,0)$	$\mathcal{B}_p(n,1)$	$\mathcal{B}_p(n,2)$	$\mathcal{B}_p(n,3)$	$\mathcal{B}_p(n,4)$	$\mathcal{B}_p(n,5)$
0	1					
1	a	1				
2	ab	$2a$	1			
3	a^2b	$3ab$	$3a$	1		
4	a^2b^2	$4a^2b$	$6ab$	$4a$	1	
5	a^3b^2	$5a^2b^2$	$10a^2b$	$10ab$	$5a$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3: The first values of $\mathcal{B}_p(n,k)$ for p odd.

n	$\mathcal{B}_p(n,0)$	$\mathcal{B}_p(n,1)$	$\mathcal{B}_p(n,2)$	$\mathcal{B}_p(n,3)$	$\mathcal{B}_p(n,4)$	$\mathcal{B}_p(n,5)$
0	1					
1	a	1				
2	ab	$a+b$	1			
3	a^2b	a^2+2ab	$2a+b$	1		
4	a^2b^2	$2a^2b+2ab^2$	$a^2+4ab+b^2$	$2a+2b$	1	
5	a^3b^2	$2a^3b+3a^2b^2$	$a^3+6a^2b+3ab^2$	$3a^2+6ab+b^2$	$3a+2b$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4: The first values of $\mathcal{B}_p(n,k)$ for p even.

Theorem 2.2. *The generating function of the sequence $\{\mathcal{B}_p(n,k)\}$ is given by*

$$\sum_{k\geq 0}\mathcal{B}_p(n,k)x^k=\begin{cases}\frac{1}{2}\left[\left(1+\frac{a}{\sqrt{ab}}\right)\left(x+\sqrt{ab}\right)^n+\left(1-\frac{a}{\sqrt{ab}}\right)\left(x-\sqrt{ab}\right)^n\right], & \text{if } p \text{ is odd,}\\ (x+a)^{\lfloor\frac{n}{2}\rfloor+\xi(n)}(x+b)^{\lfloor\frac{n}{2}\rfloor}, & \text{if } p \text{ is even.}\end{cases}$$

Proof. We will use the induction proof to show the above formula. It is clear that the result is true for $n=1$. Assume that it is true for any k such that $1\leq k\leq n$. Using the inductive hypothesis, we get the desired result as follows:

For p odd, we have

$$\begin{aligned}&\frac{1}{2}\left[\left(1+\frac{a}{\sqrt{ab}}\right)\left(x+\sqrt{ab}\right)^{n+1}+\left(1-\frac{a}{\sqrt{ab}}\right)\left(x-\sqrt{ab}\right)^{n+1}\right]\\ &\quad=\frac{x}{2}\left[\left(1+\frac{a}{\sqrt{ab}}\right)\left(x+\sqrt{ab}\right)^{n}+\left(1-\frac{a}{\sqrt{ab}}\right)\left(x-\sqrt{ab}\right)^{n}\right]\\ &\quad\quad+\frac{\sqrt{ab}}{2}\left[\left(1+\frac{a}{\sqrt{ab}}\right)\left(x+\sqrt{ab}\right)^{n}-\left(1-\frac{a}{\sqrt{ab}}\right)\left(x-\sqrt{ab}\right)^{n}\right]\\ &\quad=x\sum_{k\geq 0}\mathcal{B}_p(n,k)x^k+\sum_{k\geq 0}a^{\xi(n-k+1)}b^{\xi(n-k)}\mathcal{B}_p(n,k)x^k\\ &\quad=\sum_{k\geq 0}\left(\mathcal{B}_p(n,k-1)+a^{\xi(n-k+1)}b^{\xi(n-k)}\mathcal{B}_p(n,k)\right)x^k\\ &\quad=\sum_{k\geq 0}\mathcal{B}_p(n+1,k)x^k.\end{aligned}$$

For p even, we have

$$\begin{aligned}(x+a)^{\lfloor\frac{n+1}{2}\rfloor+\xi(n+1)}(x+b)^{\lfloor\frac{n+1}{2}\rfloor} &= \left(x+a^{\xi(n+1)}b^{\xi(n)}\right)(x+a)^{\lfloor\frac{n}{2}\rfloor+\xi(n)}(x+b)^{\lfloor\frac{n}{2}\rfloor}\\ &= \left(x+a^{\xi(n+1)}b^{\xi(n)}\right)\sum_{k\geq 0}\mathcal{B}_p(n,k)x^k\\ &= \sum_{k\geq 0}\left(\mathcal{B}_p(n,k-1)+a^{\xi(n+1)}b^{\xi(n)}\mathcal{B}_p(n,k)\right)x^k\\ &= \sum_{k\geq 0}\mathcal{B}_p(n+1,k)x^k.\end{aligned}$$

Thus the theorem is proved. □

In the following result, we give an expression of the coefficient $\mathcal{B}_p(n,k)$ using the binomial coefficient.

Theorem 2.3. *Let p be a positive integer. For nonnegative integers n and k with $0 \leq k \leq n$, we have*

$$\mathcal{B}_p(n,k) = \begin{cases} a^{\xi(n-k)}\binom{n}{k}(ab)^{\lfloor\frac{n-k}{2}\rfloor}, & \text{if } p \text{ is odd,}\\[2ex] a^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor}\displaystyle\sum_{j=0}^{k}\binom{\lfloor\frac{n}{2}\rfloor+\xi(n)}{j}\binom{\lfloor\frac{n}{2}\rfloor}{k-j}a^{-j}b^{-k+j}, & \text{if } p \text{ is even.}\end{cases}$$

Proof. Using Theorem 2.2, we obtain the desired results as follows:

For p odd, we have

$$\begin{aligned}\sum_{k\geq 0}\mathcal{B}_p(n,k)x^k &= \frac{1}{2}\left[\left(1+\frac{a}{\sqrt{ab}}\right)\left(x+\sqrt{ab}\right)^n+\left(1-\frac{a}{\sqrt{ab}}\right)\left(x-\sqrt{ab}\right)^n\right]\\ &= \frac{1}{2}\left(\sum_{k\geq 0}\binom{n}{k}\left(\sqrt{ab}\right)^k x^{n-k}+\sum_{k\geq 0}\binom{n}{k}(-1)^k\left(\sqrt{ab}\right)^k x^{n-k}\right)\\ &\quad+\frac{a}{2\sqrt{ab}}\left(\sum_{k\geq 0}\binom{n}{k}\left(\sqrt{ab}\right)^k x^{n-k}-\sum_{k\geq 0}\binom{n}{k}(-1)^k\left(\sqrt{ab}\right)^k x^{n-k}\right)\\ &= \sum_{k\geq 0}\binom{n}{2k}\left(\sqrt{ab}\right)^{2k}x^{n-2k}+\frac{a}{\sqrt{ab}}\sum_{k\geq 0}\binom{n}{2k+1}\left(\sqrt{ab}\right)^{2k+1}x^{n-(2k+1)}\\ &= \sum_{k\geq 0}\binom{n}{k}\left(\frac{a}{\sqrt{ab}}\right)^{\xi(n-k)}\left(\sqrt{ab}\right)^{n-k}x^k\\ &= \sum_{k\geq 0}a^{\xi(n-k)}\binom{n}{k}(ab)^{\lfloor\frac{n-k}{2}\rfloor}x^k.\end{aligned}$$

For p even, we have

$$\begin{aligned}\sum_{k\geq 0}\mathcal{B}_p(n,k)x^k &= (x+a)^{\lfloor\frac{n}{2}\rfloor+\xi(n)}(x+b)^{\lfloor\frac{n}{2}\rfloor}\\ &= \sum_{k\geq 0}\binom{\lfloor\frac{n}{2}\rfloor+\xi(n)}{k}a^{\lfloor\frac{n}{2}\rfloor+\xi(n)-k}x^k\sum_{k\geq 0}\binom{\lfloor\frac{n}{2}\rfloor}{k}b^{\lfloor\frac{n}{2}\rfloor-k}x^k\\ &= \sum_{k\geq 0}\left(\sum_{j=0}^{k}\binom{\lfloor\frac{n}{2}\rfloor+\xi(n)}{j}\binom{\lfloor\frac{n}{2}\rfloor}{k-j}a^{\lfloor\frac{n}{2}\rfloor+\xi(n)-j}b^{\lfloor\frac{n}{2}\rfloor-k+j}\right)x^k\end{aligned}$$

Then, by identification, we get the desired result. □

Corollary 2.4. *For p even, we have*

$$\mathcal{B}_p(2n,n)=\sum_{k\geq 0}\binom{n}{k}^2 a^{n-k}b^k, \tag{4}$$

and

$$\mathcal{B}_p(2n+1,n)=a\sum_{k\geq 0}\binom{n+1}{k}\binom{n}{k}a^{n-k}b^k. \tag{5}$$

Taking $a=b=1$ in (4) and (5), we obtain the following identities, respectively, which are given in [10]:

$$\binom{2n}{n}=\sum_{k\geq 0}\binom{n}{k}^2$$

and

$$\binom{2n+1}{n}=\sum_{k\geq 0}\binom{n+1}{k}\binom{n}{k}.$$

Proposition 2.5. *The following properties hold:*

(i) If p is odd, then $\mathcal{B}_p(n,k;a,b)=\left(\frac{a}{b}\right)^{\xi(n-k)}\mathcal{B}_p(n,k;b,a)$.

(ii) If p and n are even, then $\mathcal{B}_p(n,k;a,b)=\mathcal{B}_p(n,k;b,a)$.

Proof. The result follows from the definition of generalized binomial coefficients. □

Proposition 2.6. *For* $0\leq k\leq n$*, we have the following symmetric relation:*

$$\mathcal{B}_p(n,k;a,b)=\begin{cases} a^{\xi(n)\xi(k+1)}b^{\xi(n)\xi(k)}(ab)^{\lfloor\frac{n}{2}\rfloor-k}\mathcal{B}_p(n,n-k;b,a), & \text{if } p \text{ is odd},\\ a^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor-k}\mathcal{B}_p(n,n-k;b,a), & \text{if } p \text{ is even}.\end{cases}$$

Proof. For p odd, the result follows immediately from Theorem 2.3. For p even, we have

$$\begin{aligned} & a^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor-k}\mathcal{B}_p(n,n-k;b,a)\\ =\ & a^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor-k}b^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor}\sum_{j\geq 0}\binom{\lfloor\frac{n}{2}\rfloor+\xi(n)}{j}\binom{\lfloor\frac{n}{2}\rfloor}{n-k-j}b^{-j}a^{-n+k+j}\\ =\ & a^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor}\sum_{j\geq 0}\binom{\lfloor\frac{n}{2}\rfloor+\xi(n)}{\lfloor\frac{n}{2}\rfloor+\xi(n)-j}\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor-n+k+j}b^{\lfloor\frac{n}{2}\rfloor-k-j+\xi(n)}a^{\lfloor\frac{n}{2}\rfloor-n+j}. \end{aligned}$$

By taking $j:=\left\lfloor\frac{n}{2}\right\rfloor+\xi(n)-j$ in the last equality, we obtain

$$\begin{aligned} a^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor-k}\mathcal{B}_p(n,n-k;b,a) &= a^{\xi(n)}(ab)^{\lfloor\frac{n}{2}\rfloor}\sum_{j\geq 0}\binom{\lfloor\frac{n}{2}\rfloor+\xi(n)}{j}\binom{\lfloor\frac{n}{2}\rfloor}{k-j}b^{k-j}a^{-j}\\ &= \mathcal{B}_p(n,k;a,b). \end{aligned}$$

□

In the next theorem, we provide a generalization of the Chu-Vandermonde convolution:

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}.$$

Theorem 2.7. *For $n \geq 0$, $m \geq 0$, we have*

$$\mathcal{B}_p(n+m,k) = \begin{cases} \displaystyle\sum_{j=0}^{k} \left(\frac{b}{a}\right)^{\xi(n-j)\xi(m-k+j)} \mathcal{B}_p(n,j)\mathcal{B}_p(m,k-j), & \text{if } p \text{ is odd,} \\ \displaystyle\frac{1}{a^{\xi(m)}} \sum_{j=0}^{\xi(m)k} \left(\frac{-1}{a}\right)^{j} \sum_{i=0}^{k-j} \mathcal{B}_p(n+\xi(m),i)\mathcal{B}_p(m,k-j-i), & \text{if } p \text{ is even.} \end{cases}$$

Proof. For p odd, we have

$$\begin{aligned} \sum_{k\geq 0} \mathcal{B}_p(n+m,k)x^k &= a^{\xi(n+m-k)} \sum_{k\geq 0} \binom{n+m}{k} (ab)^{\lfloor \frac{n+m-k}{2} \rfloor} x^k \\ &= \sum_{k\geq 0} \left(\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} \right) \left(\frac{a}{\sqrt{ab}}\right)^{\xi(n+m-k)} (ab)^{n+m-k} x^k \\ &= \sum_{k\geq 0} \left(\sum_{j=0}^{k} \binom{n}{j} (\sqrt{ab})^{n-j} \binom{m}{k-j} (\sqrt{ab})^{m-k+j} \right) \left(\frac{a}{\sqrt{ab}}\right)^{\xi(n+m-k)} x^k \\ &= \sum_{k\geq 0} \left(\sum_{j=0}^{k} \binom{n}{j} (\sqrt{ab})^{n-j} \left(\frac{a}{\sqrt{ab}}\right)^{\xi(n-j)} \binom{m}{k-j} (\sqrt{ab})^{m-k+j} \left(\frac{a}{\sqrt{ab}}\right)^{\xi(m-k+j)} \left(\frac{a}{\sqrt{ab}}\right)^{-2\xi(n-j)\xi(m-k+j)} \right) x^k \\ &= \sum_{k\geq 0} \left(\sum_{j=0}^{k} \left(\frac{b}{a}\right)^{\xi(n-j)\xi(m-k+j)} \mathcal{B}_p(n,j)\mathcal{B}_p(m,k-j) \right) x^k. \end{aligned}$$

For p even, by using Theorem 2.2, and the fact that $\lfloor \frac{n+m}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \xi(n)\xi(m)$, and $\xi(n+m) = \xi(n) + \xi(m) - 2\xi(n)\xi(m)$, we get

$$\begin{aligned} \sum_{k\geq 0} \mathcal{B}_p(n+m,k)x^k &= (x+a)^{\lfloor \frac{n+m}{2} \rfloor + \xi(n+m)} (x+b)^{\lfloor \frac{n+m}{2} \rfloor} \\ &= \left(\frac{1}{x+a}\right)^{\xi(m)} (x+a)^{\lfloor \frac{n+\xi(m)}{2} \rfloor + \xi(n+\xi(m))} (x+b)^{\lfloor \frac{n+\xi(m)}{2} \rfloor} (x+a)^{\lfloor \frac{m}{2} \rfloor + \xi(m)} (x+b)^{\lfloor \frac{m}{2} \rfloor} \end{aligned}$$

- If m is even, then

$$\begin{aligned} \sum_{k\geq 0} \mathcal{B}_p(n+m,k)x^k &= \sum_{k\geq 0} \mathcal{B}_p(n,k)x^k \sum_{k\geq 0} \mathcal{B}_p(m,k)x^k \\ &= \sum_{k\geq 0} \left(\sum_{j=0}^{k} \mathcal{B}_p(n,j)\mathcal{B}_p(m,k-j) \right) x^k. \end{aligned}$$

- If m is odd, then

$$
\begin{aligned}
\sum_{k\geq 0} \mathcal{B}_p(n+m,k)x^k &= \frac{1}{a}\sum_{k\geq 0}\left(\frac{-x}{a}\right)^k \sum_{k\geq 0}\mathcal{B}_p(n+1,k)x^k \sum_{k\geq 0}\mathcal{B}_p(m,k)x^k \\
&= \frac{1}{a}\sum_{k\geq 0}\left(\frac{-x}{a}\right)^k \sum_{k\geq 0}\left(\sum_{j=0}^{k}\mathcal{B}_p(n+1,j)\mathcal{B}_p(m,k-j)\right)x^k \\
&= \sum_{k\geq 0}\left(\frac{1}{a}\sum_{j=0}^{k}\left(\frac{-1}{a}\right)^j \sum_{i=0}^{k-j}\mathcal{B}_p(n+1,i)\mathcal{B}_p(m,k-j-i)\right)x^k.
\end{aligned}
$$

Thus, by identification, we obtain the desired result. □

Corollary 2.8. *For $1 \leq m \leq n$ with $0 \leq r \leq n$ and $0 \leq s \leq m$, we have*

$$
\mathcal{B}_p(n-r+p(m-s), n-m) = \sum_{j=m}^{n}\left(\frac{b}{a}\right)^{\xi(p(j+s))\xi(r+j)} \mathcal{B}_p(n-r,n-j)\mathcal{B}_p(p(m-s), j-m). \tag{6}
$$

Proof. The result follows from Theorem 2.7. □

3. Bi-periodic Fibonacci *p*-triangle and bi-periodic Lucas *p*-triangle

In this section, we first define the bi-periodic Fibonacci p-triangle. Then, we establish a relationship between the coefficients of the bi-periodic Fibonacci p-triangle and the generalized binomial coefficients, which allows us to derive an explicit formula for the bi-periodic Fibonacci p-sequence. Next, we define the bi-periodic Lucas p-triangle and obtain a relation between the coefficients of the bi-periodic Fibonacci p-triangle and those of the bi-periodic Lucas p-triangle.

The bi-periodic Fibonacci p-triangle is a triangular array of numbers where the entry in the n-th row and k-th column of this array is denoted by $\mathcal{F}_p(n,k;a,b)$, and is defined as follows.

Definition 3.1. *Let p be a fixed positive integer. For nonnegative integers n and k with $0 \leq k \leq n$, we define the coefficient $\mathcal{F}_p(n,k;a,b)$, $\mathcal{F}_p(n,k)$ for short, by the recurrence relation*

$$
\mathcal{F}_p(n,k) = a^{\xi(n+k)}b^{\xi(n+k+1)}\mathcal{F}_p(n-1,k) + \mathcal{F}_p(n-p,k-1) \tag{7}
$$

with initial conditions $\mathcal{F}_p(n,0) = a^{\lfloor (n+1)/2\rfloor}b^{\lfloor n/2\rfloor}$ and

$$
\mathcal{F}_p(n,k) = \begin{cases} 0, & \text{if } n < pk, \\ 1, & \text{if } n = pk. \end{cases}
$$

For convenience, we define $\mathcal{F}_p(n,k) = 0$ for $n < 0$ or $k < 0$. The bi-periodic Fibonacci p-triangles for $p = 1,2,3$ are given in Tables 5, 6, 7, respectively.

For $a = b = 1$, they reduce to the Fibonacci p-triangles in [13].

Note that each k-th column of the bi-periodic Fibonacci p-triangle is derived from the k-th column of the generalized binomial triangle by shifting it down $(p-1)k$ places, as expressed below:

$$
\mathcal{F}_p(n,k) = \mathcal{B}_p(n-pk+k,k). \tag{8}
$$

Now, we give an expression of the bi-periodic Fibonacci p-numbers using the generalized binomial coefficient.

n	$\mathcal{F}_p(n,0)$	$\mathcal{F}_p(n,1)$	$\mathcal{F}_p(n,2)$	$\mathcal{F}_p(n,3)$	$\mathcal{F}_p(n,4)$	$\mathcal{F}_p(n,5)$	$\cdots$
0	1						
1	a	1					
2	ab	$2a$	1				
3	a^2b	$3ab$	$3a$	1			
4	a^2b^2	$4a^2b$	$6ab$	$4a$	1		
5	a^3b^2	$5a^2b^2$	$10a^2b$	$10ab$	$5a$	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 5: The first values of $\mathcal{F}_p(n,k)$ for $p = 1$

n	$\mathcal{F}_p(n,0)$	$\mathcal{F}_p(n,1)$	$\mathcal{F}_p(n,2)$	$\mathcal{F}_p(n,3)$	$\cdots$
0	1				
1	a				
2	ab	1			
3	a^2b	$a+b$			
4	a^2b^2	a^2+2ab	1		
5	a^3b^2	$2a^2b+2ab^2$	$2a+b$		
6	a^3b^3	$2a^3b+3a^2b^2$	$a^2+4ab+b^2$	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 6: The first values of $\mathcal{F}_p(n,k)$ for $p = 2$.

n	$\mathcal{F}_p(n,0)$	$\mathcal{F}_p(n,1)$	$\mathcal{F}_p(n,2)$	$\cdots$
0	1			
1	a			
2	ab			
3	a^2b	1		
4	a^2b^2	$2a$		
5	a^3b^2	$3ab$		
6	a^3b^3	$4a^2b$	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 7: The first values of $\mathcal{F}_p(n,k)$ for $p = 3$.

Theorem 3.2. *Let n and p be positive integers. Then the bi-periodic Fibonacci p-numbers satisfy the following formula:*

$$U_{p,n+1} = \sum_{k=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \mathcal{F}_p(n-k,k)c^k.$$

Proof. It can be proven by using (1), (7), and induction on n. □

Using Theorem 2.3, Theorem 3.2, and equation (8), we can express an explicit formula for the bi-periodic

Fibonacci p-sequence as follows:

$$U_{p,n+1}=\begin{cases}\displaystyle\sum_{k=0}^{\left\lfloor\frac{n}{p+1}\right\rfloor}a^{\xi(n-pk-k)}\binom{n-pk}{k}(ab)^{\left\lfloor\frac{n-pk-k}{2}\right\rfloor}c^{k}, & \text{if } p \text{ is odd,}\\[2ex] \displaystyle\sum_{k=0}^{\left\lfloor\frac{n}{p+1}\right\rfloor}\sum_{j=0}^{k}a^{\left\lfloor\frac{n-pk+1}{2}\right\rfloor}b^{\left\lfloor\frac{n-pk}{2}\right\rfloor}\binom{\left\lfloor\frac{n-pk+1}{2}\right\rfloor}{j}\binom{\left\lfloor\frac{n-pk}{2}\right\rfloor}{k-j}a^{-j}b^{-k+j}c^{k}, & \text{if } p \text{ is even.}\end{cases}\tag{9}$$

Note that for $p = c = 1$ we get the following identity for bi-periodic Fibonacci numbers given in [24]:

$$U_{1,n+1}=a^{\xi(n)}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-k}{k}(ab)^{\left\lfloor\frac{n}{2}\right\rfloor-k}.$$

Corollary 3.3. *For $p = 2$ and $c = 1$, the bi-periodic Narayana-Fibonacci numbers satisfy the following explicit formula*

$$U_{2,n+1}=a^{\xi(n)}\sum_{k=0}^{\left\lfloor\frac{n}{3}\right\rfloor}(ab)^{\left\lfloor\frac{n}{2}\right\rfloor-k}\sum_{j=0}^{k}\binom{\left\lfloor\frac{n}{2}\right\rfloor-k+\xi(n)}{j}\binom{\left\lfloor\frac{n}{2}\right\rfloor-k}{k-j}a^{-j}b^{-k+j}.$$

Now, we define the bi-periodic Lucas p-triangle. The bi-periodic Lucas p-triangle is a triangular array of numbers where the entry in the n-th row and k-th column of this array is denoted by $\mathcal{L}_p(n,k)$, and is defined as follows.

Definition 3.4. *Let p be a fixed positive integer. For nonnegative integers n and k with $0 \le k \le n$, we define the coefficient $\mathcal{L}_p(n,k;a,b)$, $\mathcal{L}_p(n,k)$ for short, by the recurrence relation*

$$\mathcal{L}_p(n,k)=a^{\xi(n+k)}b^{\xi(n+k+1)}\mathcal{L}_p(n-1,k)+\mathcal{L}_p(n-p,k-1)\tag{10}$$

with initial conditions $\mathcal{L}_p(n,0)=a^{\lfloor (n+1)/2\rfloor}b^{\lfloor n/2\rfloor}$ for $n>0$ and

$$\mathcal{L}_p(n,k)=\begin{cases}0, & \text{if } n<pk,\\ p+1, & \text{if } n=pk.\end{cases}$$

For convenience, we define $\mathcal{L}_p(n,k)=0$ for $n<0$ or $k<0$. The bi-periodic Lucas p-triangles for $p = 1, 2, 3$ are given in Tables 8, 9, 10, respectively.

n	$\mathcal{L}_p(n,0)$	$\mathcal{L}_p(n,1)$	$\mathcal{L}_p(n,2)$	$\mathcal{L}_p(n,3)$	$\mathcal{L}_p(n,4)$	$\mathcal{L}_p(n,5)$	$\cdots$
0	2						
1	a	2					
2	ab	$3a$	2				
3	a^2b	$4ab$	$5a$	2			
4	a^2b^2	$5a^2b$	$9ab$	$7a$	2		
5	a^3b^2	$6a^2b^2$	$14a^2b$	$16ab$	$9a$	2	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 8: The first values of $\mathcal{L}_p(n,k)$ for $p = 1$.

For $a = b = p = 1$, it gives the classical Lucas triangle in [5, 17]. For $a = b = 1$, they reduce to the Lucas p-triangle in [14].

n	$\mathcal{L}_p(n,0)$	$\mathcal{L}_p(n,1)$	$\mathcal{L}_p(n,2)$	$\mathcal{L}_p(n,3)$	$\cdots$
0	3				
1	a				
2	ab	3			
3	a^2b	$a+3b$			
4	a^2b^2	a^2+4ab	3		
5	a^3b^2	$2a^2b+4ab^2$	$4a+3b$		
6	a^3b^3	$2a^3b+5a^2b^2$	$a^2+8ab+3b^2$	3	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 9: The first values of $\mathcal{L}_p(n,k)$ for $p=2$.

n	$\mathcal{L}_p(n,0)$	$\mathcal{L}_p(n,1)$	$\mathcal{L}_p(n,2)$	$\cdots$
0	4			
1	a			
2	ab			
3	a^2b	4		
4	a^2b^2	$5a$		
5	a^3b^2	$6ab$		
6	a^3b^3	$7a^2b$	4	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 10: The first values of $\mathcal{L}_p(n,k)$ for $p=3$.

Theorem 3.5. *Let n and p be positive integers. Then the bi-periodic Lucas p-numbers satisfy the following formula*

$$V_{p,n}=\sum_{k=0}^{\left\lfloor\frac{n}{p+1}\right\rfloor}\mathcal{L}_p(n-k,k)c^k.$$

Proof. It can be proven using (2), (10), and induction on n. For $n=1$, it is clear to see that the identity holds. Assume that the identity holds for $n>1$. Now we show that it holds for $n+1$.

$$\begin{aligned}
V_{p,n+1} &= a^{\xi(n+1)}b^{\xi(n)}V_{p,n}+cV_{p,n-p}\\
&= a^{\xi(n+1)}b^{\xi(n)}\sum_{k=0}^{\left\lfloor\frac{n}{p+1}\right\rfloor}\mathcal{L}_p(n-k,k)c^k+\sum_{k=0}^{\left\lfloor\frac{n}{p+1}\right\rfloor}\mathcal{L}_p(n-k,k)c^{k+1}\\
&= a^{\xi(n+1)}b^{\xi(n)}\sum_{k=0}^{\left\lfloor\frac{n}{p+1}\right\rfloor}\mathcal{L}_p(n-k,k)c^k+\sum_{k=1}^{\left\lfloor\frac{n}{p+1}\right\rfloor+1}\mathcal{L}_p(n-k+1,k-1)c^k\\
&= a^{\xi(n+1)}b^{\xi(n)}\sum_{k=0}^{\left\lfloor\frac{n}{p+1}\right\rfloor}\mathcal{L}_p(n-k,k)c^k+\sum_{k=1}^{\left\lfloor\frac{n+1}{p+1}\right\rfloor}\mathcal{L}_p(n-p-k+1,k-1)c^k
\end{aligned}$$

Since $\mathcal{L}_p(n-k,k)=0$ for $k=\left\lfloor\frac{n+1}{p+1}\right\rfloor$, we have

$$V_{p,n+1} = a^{\xi(n+1)}b^{\xi(n)}\mathcal{L}_p(n,0)+\sum_{k=1}^{\left\lfloor\frac{n+1}{p+1}\right\rfloor}\left(a^{\xi(n+1)}b^{\xi(n)}\mathcal{L}_p(n-k,k)+\mathcal{L}_p(n-p-k+1,k-1)\right)c^k$$

$$= \sum_{k=0}^{\left\lfloor \frac{n+1}{p+1} \right\rfloor} \left(a^{\xi(n+1)} b^{\xi(n)} \mathcal{L}_p(n-k,k) + \mathcal{L}_p(n-p-k+1,k-1) \right) c^k$$
$$= \sum_{k=0}^{\left\lfloor \frac{n+1}{p+1} \right\rfloor} \mathcal{L}_p(n-k+1,k) c^k.$$

□

Next, we give a relation between the coefficients of the bi-periodic Fibonacci p-triangle and those of the bi-periodic Lucas p-triangle.

Theorem 3.6. *For the coefficients of the bi-periodic Fibonacci and Lucas p-triangles, we have*

$$\mathcal{L}_p(n,k) = \begin{cases} \mathcal{F}_p(n,k) + p\mathcal{F}_p(n-p,k-1), & \text{if } p \text{ is odd and } n \geq p, \\ \mathcal{F}_p(n,k) + pb\mathcal{F}_p(n-p-1,k-1) + p\mathcal{F}_p(n-2p,k-2), & \text{if } p \text{ is even and } n \geq 2p+1. \end{cases}$$

Proof. Let $n \geq p$ and p be odd. From [1], we have $V_{p,n} = U_{p,n+1} + pcU_{p,n-p}$.
Thus, we obtain

$$V_{p,n} = U_{p,n+1} + pcU_{p,n-p}$$
$$= \sum_{k=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \mathcal{F}_p(n-k,k)c^k + p \sum_{k=0}^{\left\lfloor \frac{n-p-1}{p+1} \right\rfloor} \mathcal{F}_p(n-p-1-k,k)c^{k+1}$$
$$= \sum_{k=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \mathcal{F}_p(n-k,k)c^k + p \sum_{k=1}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \mathcal{F}_p(n-p-k,k-1)c^k$$
$$= \mathcal{F}_p(n,0) + \sum_{k=1}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \left(\mathcal{F}_p(n-k,k) + p\mathcal{F}_p(n-p-k,k-1) \right) c^k$$
$$= \sum_{k=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \left(\mathcal{F}_p(n-k,k) + p\mathcal{F}_p(n-p-k,k-1) \right) c^k.$$

From Theorem 3.5, we get the desired result.
Now let $n \geq 2p+1$ and p be even. From [1], we have $V_{p,n} = U_{p,n+1} + pcbU_{p,n-p-1} + pc^2U_{p,n-2p-1}$.
Thus, we obtain

$$V_{p,n} = U_{p,n+1} + pcbU_{p,n-p-1} + pc^2U_{p,n-2p-1}$$
$$= \sum_{k=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \mathcal{F}_p(n-k,k)c^k + pb \sum_{k=0}^{\left\lfloor \frac{n-1}{p+1} \right\rfloor - 1} \mathcal{F}_p(n-p-1-(k+1),k)c^{k+1}$$
$$+p \sum_{k=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor - 2} \mathcal{F}_p(n-2p-(k+2),k)c^{k+2}$$
$$= \sum_{k=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \mathcal{F}_p(n-k,k)c^k + pb \sum_{k=1}^{\left\lfloor \frac{n-1}{p+1} \right\rfloor} \mathcal{F}_p(n-p-1-k,k-1)c^k + p \sum_{k=2}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \mathcal{F}_p(n-2p-k,k-2)c^k$$

$$= \sum_{k=0}^{\lfloor \frac{n}{p+1} \rfloor} \mathcal{F}_p(n-k,k)c^k + pb \sum_{k=1}^{\lfloor \frac{n}{p+1} \rfloor} \mathcal{F}_p(n-p-k-1,k-1)c^k + p \sum_{k=2}^{\lfloor \frac{n}{p+1} \rfloor} \mathcal{F}_p(n-2p-k,k-2)c^k$$

$$= \mathcal{F}_p(n,0) + c\mathcal{F}_p(n-1,1) + pb\mathcal{F}_p(n-p-2,0)c$$

$$+ \sum_{k=2}^{\lfloor \frac{n}{p+1} \rfloor} \Big(\mathcal{F}_p(n-k,k) + pb\mathcal{F}_p(n-p-k-1,k-1) + p\mathcal{F}_p(n-2p-k,k-2)\Big) c^k$$

$$= \sum_{k=0}^{\lfloor \frac{n}{p+1} \rfloor} \Big(\mathcal{F}_p(n-k,k) + pb\mathcal{F}_p(n-p-k-1,k-1) + p\mathcal{F}_p(n-2p-k,k-2)\Big) c^k.$$

From Theorem 3.5, we obtain the desired result. □

Theorem 3.7. *For $n \geq 1$, we have*

$$\mathcal{L}_p(n,k) = \begin{cases} a^{\xi(n+k)} b^{\xi(n+k+1)} \mathcal{F}_p(n-1,k) + (p+1)\mathcal{F}_p(n-p,k-1), & \text{if } p \text{ is odd} \\ a\mathcal{F}_p(n-1,k;b,a) + (p+1)\mathcal{F}_p(n-p,k-1;b,a), & \text{if } p \text{ is even.} \end{cases}$$

Proof. For p odd, let $\{g(n,k)\}$ be the sequence defined by the relation

$$g(n,k) = a^{\xi(n+k)} b^{\xi(n+k+1)} \mathcal{F}_p(n-1,k) + (p+1)\mathcal{F}_p(n-p,k-1).$$

Now we show that the sequence $\{g(n,k)\}$ satisfies the same recurrence relation and has the same initial conditions as $\{\mathcal{L}_p(n,k)\}$. It is clear to see that $g(n,0) = a^{\xi(n)} b^{\xi(n+1)} \mathcal{F}_p(n-1,0) = a^{\lfloor \frac{n+1}{2} \rfloor} b^{\lfloor \frac{n}{2} \rfloor}$ and

$$g(n,k) = \begin{cases} 0, & \text{if } n < pk, \\ (p+1)\mathcal{F}_p(pk-p,k-1) = p+1, & \text{if } n = pk. \end{cases}$$

For nonnegative integers n and k with $0 \leq k \leq n$, we have

$$\begin{aligned} g(n+1,k) &= a^{\xi(n+k+1)} b^{\xi(n+k)} \mathcal{F}_p(n,k) + (p+1)\mathcal{F}_p(n+1-p,k-1) \\ &= a^{\xi(n+k+1)} b^{\xi(n+k)} \Big(a^{\xi(n+k)} b^{\xi(n+k+1)} \mathcal{F}_p(n-1,k) + \mathcal{F}_p(n-p,k-1)\Big) \\ &\quad +(p+1)\Big(a^{\xi(n-p+k)} b^{\xi(n-p+k+1)} \mathcal{F}_p(n-p,k-1) + \mathcal{F}_p(n+1-2p,k-2)\Big) \\ &= \Big(ab\mathcal{F}_p(n-1,k) + (p+1)a^{\xi(n-p+k)} b^{\xi(n-p+k+1)} \mathcal{F}_p(n-p,k-1)\Big) \\ &\quad + \Big(a^{\xi(n-p+k)} b^{\xi(n-p+k+1)} \mathcal{F}_p(n-p,k-1) + (p+1)\mathcal{F}_p(n+1-2p,k-2)\Big) \\ &= a^{\xi(n+k+1)} b^{\xi(n+k)} \Big(a^{\xi(n+k)} b^{\xi(n+k+1)} \mathcal{F}_p(n-1,k) + (p+1)\mathcal{F}_p(n-p,k-1)\Big) \\ &\quad + \Big(a^{\xi(n-p+k)} b^{\xi(n-p+k+1)} \mathcal{F}_p(n-p,k-1) + (p+1)\mathcal{F}_p(n+1-2p,k-2)\Big) \\ &= a^{\xi(n+k+1)} b^{\xi(n+k)} g(n,k) + g(n-p+1,k-1). \end{aligned}$$

Thus, $g(n,k) = \mathcal{L}_p(n,k)$.

For p even, the proof is similar.

□

Theorem 3.8. *For $n \geq 1$, we have*

$$\mathcal{L}_p(n,k) = (p+1)\mathcal{F}_p(n,k;a,b) - pa\mathcal{F}_p(n-1,k;b,a).$$

Proof. For p odd, from Theorem 3.7 we have

$$\begin{aligned}\mathcal{L}_p(n,k) &= a^{\xi(n+k)}b^{\xi(n+k+1)}\mathcal{F}_p(n-1,k) + (p+1)\mathcal{F}_p(n-p,k-1)\\ &= a^{\xi(n+k)}b^{\xi(n+k+1)}\mathcal{F}_p(n-1,k) + \mathcal{F}_p(n-p,k-1) + p\mathcal{F}_p(n-p,k-1)\\ &= \mathcal{F}_p(n,k) + p\mathcal{F}_p(n-p,k-1)\\ &= \mathcal{F}_p(n,k) + p\left(\mathcal{F}_p(n,k) - a^{\xi(n+k)}b^{\xi(n+k+1)}\mathcal{F}_p(n-1,k)\right)\\ &= (p+1)\mathcal{F}_p(n,k) - pa^{\xi(n+k)}b^{\xi(n+k+1)}\mathcal{F}_p(n-1,k).\end{aligned}$$

By using Proposition 2.5, we have

$$a^{\xi(n+k)}b^{\xi(n+k+1)}\mathcal{F}_p(n-1,k) = a^{\xi(n+k)}b^{\xi(n+k+1)}\left(\frac{a}{b}\right)^{\xi(n-k-1)}\mathcal{B}_p(n-1-pk+k,k;b,a) = a\mathcal{F}_p(n-1,k;b,a).$$

Thus, we obtain

$$\mathcal{L}_p(n,k) = (p+1)\mathcal{F}_p(n,k) - pa\mathcal{F}_p(n-1,k;b,a).$$

For p even, by using the Theorem 2.3 and relation (8) we can easily verify the following relation

$$\mathcal{F}_p(n,k;a,b) = a\mathcal{F}_p(n-1,k;b,a) + \mathcal{F}_p(n-p,k-1;b,a).$$

Then, from Theorem 3.7 and above relation, we have

$$\begin{aligned}\mathcal{L}_p(n,k) &= a\mathcal{F}_p(n-1,k;b,a) + (p+1)\mathcal{F}_p(n-p,k-1;b,a)\\ &= a\mathcal{F}_p(n-1,k;b,a) + (p+1)\mathcal{F}_p(n,k;a,b) - (p+1)a\mathcal{F}_p(n-1,k;b,a)\\ &= (p+1)\mathcal{F}_p(n,k;a,b) - pa\mathcal{F}_p(n-1,k;b,a).\end{aligned}$$

□

Based on Theorem 2.3, Theorem 3.5, and Theorem 3.8, we can now provide an explicit formula of bi-periodic Lucas p-sequence as follows:

Theorem 3.9. *The bi-periodic Lucas p-numbers satisfy the following formula:*

$$V_{p,n} = \begin{cases} \displaystyle\sum_{k=0}^{\lfloor\frac{n}{p+1}\rfloor} a^{\xi(n-pk-k)}(ab)^{\lfloor\frac{n-pk-k}{2}\rfloor}\frac{n}{n-pk}\binom{n-pk}{k}c^k, & \text{if } p \text{ is odd}\\ \displaystyle\sum_{k=0}^{\lfloor\frac{n}{p+1}\rfloor} a^{\xi(n-pk)}(ab)^{\lfloor\frac{n-pk}{2}\rfloor}\sum_{j=0}^{k}\frac{\lfloor\frac{n-pk+1}{2}\rfloor+pj}{\lfloor\frac{n-pk+1}{2}\rfloor}\binom{\lfloor\frac{n-pk+1}{2}\rfloor}{j}\binom{\lfloor\frac{n-pk}{2}\rfloor}{k-j}a^{-j}b^{-k+j}c^k, & \text{if } p \text{ is even.}\end{cases}$$

Proof. For p odd, we have

$$\begin{aligned}\mathcal{L}_p(n-k,k) &= (p+1)\mathcal{F}_p(n-k,k;a,b) - pa\mathcal{F}_p(n-k-1,k;b,a)\\ &= (p+1)\mathcal{B}_p(n-pk,k;a,b) - pa\mathcal{B}_p(n-pk-1,k;b,a)\\ &= (p+1)a^{\xi(n-pk-k)}\binom{n-pk}{k}(ab)^{\lfloor\frac{n-pk-k}{2}\rfloor}\\ &\quad -pab^{\xi(n-pk-k-1)}\binom{n-pk-1}{k-1}(ab)^{\lfloor\frac{n-pk-k-1}{2}\rfloor}\\ &= a^{\xi(n-pk-k)}\binom{n-pk}{k}(ab)^{\lfloor\frac{n-pk-k}{2}\rfloor}\left((p+1)\binom{n-pk}{k} - p\binom{n-pk-1}{k}\right)\end{aligned}$$

$$= a^{\xi(n-pk-k)}(ab)^{\lfloor \frac{n-pk-k}{2} \rfloor}\frac{n}{n-pk}\binom{n-pk}{k}.$$

For even p, we have

$$\begin{aligned}
\mathcal{L}_p(n-k,k) &= (p+1)\mathcal{F}_p(n-k,k;a,b) - pa\mathcal{F}_p(n-k-1,k;b,a) \\
&= (p+1)\mathcal{B}_p(n-pk,k;a,b) - pa\mathcal{F}_p(n-pk-1,k;b,a) \\
&= (p+1)a^{\xi(n-pk)}(ab)^{\lfloor \frac{n-pk}{2} \rfloor}\sum_{j=0}^{k}\binom{\lfloor \frac{n-pk+1}{2} \rfloor}{j}\binom{\lfloor \frac{n-pk}{2} \rfloor}{k-j}a^{-j}b^{-k+j} \\
&\quad -pab^{\xi(n-pk-1)}(ab)^{\lfloor \frac{n-pk-1}{2} \rfloor}\sum_{j=0}^{k}\binom{\lfloor \frac{n-pk}{2} \rfloor}{j}\binom{\lfloor \frac{n-pk-1}{2} \rfloor}{k-j}b^{-j}a^{-k+j}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_p(n-k,k) &= (p+1)a^{\xi(n-pk)}(ab)^{\lfloor \frac{n-pk}{2} \rfloor}\sum_{j=0}^{k}\binom{\lfloor \frac{n-pk+1}{2} \rfloor}{j}\binom{\lfloor \frac{n-pk}{2} \rfloor}{k-j}a^{-j}b^{-k+j} \\
&\quad -pa^{\xi(n-pk)}(ab)^{\lfloor \frac{n-pk}{2} \rfloor}\sum_{j=0}^{k}\binom{\lfloor \frac{n-pk}{2} \rfloor}{k-j}\binom{\lfloor \frac{n-pk-1}{2} \rfloor}{j}a^{-j}b^{-k+j} \\
&= a^{\xi(n-pk)}(ab)^{\lfloor \frac{n-pk}{2} \rfloor}\sum_{j=0}^{k}\left((p+1)\binom{\lfloor \frac{n-pk+1}{2} \rfloor}{j} - p\binom{\lfloor \frac{n-pk+1}{2} \rfloor - 1}{j}\right) \\
&\quad \cdot\binom{\lfloor \frac{n-pk}{2} \rfloor}{j}a^{-j}b^{-k+j} \\
&= a^{\xi(n-pk)}(ab)^{\lfloor \frac{n-pk}{2} \rfloor}\sum_{j=0}^{k}\frac{\lfloor \frac{n-pk+1}{2} \rfloor + pj}{\lfloor \frac{n-pk+1}{2} \rfloor}\binom{\lfloor \frac{n-pk+1}{2} \rfloor}{j}\binom{\lfloor \frac{n-pk}{2} \rfloor}{k-j}a^{-j}b^{-k+j}.
\end{aligned}$$

Thus, we get the desired result from Theorem 3.5. □

Remark 3.10. *It should be noted that Ait-Amrane and Belbachir [1] derived other explicit formulas for the bi-periodic Fibonacci and Lucas p-numbers using multinomial coefficients. However, in this paper, we derive explicit formulas using bi-periodic Fibonacci and Lucas p-triangles, which provide a more convenient way to define incomplete versions of bi-periodic Fibonacci and Lucas p-numbers as will be discussed in the next section.*

Next, we show a relation between the generalized binomial coefficients and the bi-periodic Fibonacci p-numbers. We will also show a similar relation for the generalized binomial coefficients and the bi-periodic Lucas p-numbers.

Proposition 3.11. *For any nonnegative integer n, we have*

$$\sum_{k=0}^{n}\mathcal{B}_p(n-1,n-k)c^{n-k}U_{p,pk+1} = U_{p,(p+1)n}, \tag{11}$$

Proof. From (8) and Theorem 3.2, we have

$$\begin{aligned}
U_{p,(p+1)n} &= \sum_{j\geq 0}\mathcal{F}_p((p+1)n-j-1,j)c^j \\
&= \sum_{j\geq 0}\mathcal{B}_p(pj+n-1,n-j)c^{n-j},
\end{aligned}$$

by using Corollary 2.8, we get

$$\begin{aligned} U_{p,(p+1)n} &= \sum_{j\geq 0}\sum_{k\geq 0}\mathcal{B}_p(n-1,n-k)\mathcal{B}_p(pj,k-j)c^{n-j} \\ &= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k)\sum_{j\geq 0}\mathcal{B}_p(p(k-j),j)c^{n-k+j} \\ &= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k)c^{n-k}\sum_{j=0}^{k}\mathcal{B}_p(pk-pj,j)c^{j} \\ &= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k)c^{n-k}U_{p,pk+1}. \end{aligned}$$

□

The relation (11) can be expressed as follows.

$$\sum_{k=0}^{n}\left(\frac{a}{b}\right)^{\xi(p)\xi(k)}\mathcal{B}_p(n,n-k;b,a)c^{n-k}U_{p,pk} = U_{p,(p+1)n}. \tag{12}$$

Note that for $c = p = 1$ in (12), we obtain the following identity for bi-periodic Fibonacci numbers given in [8, Theorem 10]:

$$\sum_{k=0}^{n}\binom{n}{k}(a)^{\xi(k)}(ab)^{\lfloor \frac{k}{2}\rfloor}q_k = q_{2n}.$$

Proposition 3.12. *For any nonnegative integer n, we have*

$$\sum_{k=0}^{n}\mathcal{B}_p(n-1,n-k;b,a)c^{n-k}V_{p,pk+1} = V_{p,(p+1)n}. \tag{13}$$

Proof. For p odd, by using Theorem 3.5 and Theorem 3.7, we have

$$\begin{aligned} V_{p,(p+1)n} &= \sum_{j\geq 0}\mathcal{L}_p((p+1)n-j,j)c^{j} \\ &= \sum_{j\geq 0}\left(a^{\xi((p+1)n)}b^{\xi((p+1)n)+1}\mathcal{F}_p((p+1)n-j-1,j) + (p+1)\mathcal{F}_p((p+1)n-j-p,j-1)\right)c^{j} \\ &= \sum_{j\geq 0}\left(a\mathcal{F}_p((p+1)n-j-1,j;b,a) + (p+1)\mathcal{F}_p((p+1)n-j-p,j-1)\right)c^{j} \\ &= \sum_{j\geq 0}\left(a\mathcal{B}_p(n+p(n-j)-1,j;b,a) + (p+1)\mathcal{B}_p(n+p(n-j)-1,j-1)\right)c^{j} \\ &= \sum_{j\geq 0}\left(a\mathcal{B}_p(n+pj-1,n-j;b,a) + c(p+1)\mathcal{B}_p(n+p(j-1)-1,n-j)\right)c^{n-j} \end{aligned}$$

By using Corollary 2.8, we get

$$\begin{aligned} V_{p,(p+1)n} = \sum_{j\geq 0}\sum_{k\geq 0}\Big(& a\mathcal{B}_p(n-1,n-k;b,a)\mathcal{B}_p(pj,k-j;b,a) \\ &+c(p+1)\left(\frac{b}{a}\right)^{\xi(p(j+1))}\mathcal{B}_p(n-1,n-k)\mathcal{B}_p(p(j-1),k-j)\Big)c^{n-j} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\geq 0}\sum_{k\geq 0}\Big(a\mathcal{B}_p(n-1,n-k;b,a)\mathcal{B}_p(pj,k-j;b,a)\\
&\quad +c(p+1)\left(\frac{b}{a}\right)^{\xi(p(j+1))}\left(\frac{a}{b}\right)^{\xi(j+1)}\mathcal{B}_p(n-1,n-k;b,a)\mathcal{B}_p(p(j-1),k-j)\Big)c^{n-j}\\
&= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k;b,a)\sum_{j\geq 0}\Big(a\mathcal{B}_p(pk-pj,j;b,a)+c(p+1)\mathcal{B}_p(p(k-j-1),j)\Big)c^{n-k+j}\\
&= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k;b,a)c^{n-k}\sum_{j\geq 0}\Big(a\mathcal{B}_p(pk-pj,j;b,a)+(p+1)\mathcal{B}_p(pk-pj,j-1)\Big)c^{j}\\
&= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k;b,a)c^{n-k}\sum_{j\geq 0}\Big(a\mathcal{F}_p(pk-j,j;b,a)+(p+1)\mathcal{F}_p(pk-j-p+1,j-1)\Big)c^{j}\\
&= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k;b,a)c^{n-k}\sum_{j\geq 0}\mathcal{L}_p(pk-j+1,j)c^{j}\\
&= \sum_{k\geq 0}\mathcal{B}_p(n-1,n-k;b,a)c^{n-k}V_{p,pk+1}.
\end{aligned}
$$

For p odd, the proof is similar. □

4. Some applications of bi-periodic Fibonacci and Lucas p-triangles

In this section, we introduce the bi-periodic incomplete Fibonacci p-numbers by using Theorem 3.2, then we give some recurrence relations and sum formulas involving bi-periodic incomplete Fibonacci p-numbers. Analogous results will be presented for the Lucas case.

Definition 4.1. *Let n and k be positive integers such that $0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor$. The bi-periodic incomplete Fibonacci p-numbers are defined as*

$$U_{p,n+1}(k) = \sum_{i=0}^{k} \mathcal{F}_p(n-i,i)c^i.$$

Some special cases of Definition 4.1 are given as follows:

- $U_{p,n+1}(0) = a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}$,
- $U_{p,n}(1) = \begin{cases} a^{\frac{n+\xi(n)-p-1}{2}} b^{\frac{n-\xi(n)-p-1}{2}} c\,(n-p), & p \text{ is odd},\\ a^{\frac{n+\xi(n)-p}{2}} b^{\frac{n-\xi(n)-p}{2}} c\left(\frac{a\left(n-\xi(n)-p\right)+b\left(n+\xi(n)-p\right)}{2a}\right), & p \text{ is even}.\end{cases}$
- $U_{p,n+1}(\left\lfloor \frac{n}{p+1} \right\rfloor) = U_{p,n+1}$.

Now, we give some recurrence relations involving the bi-periodic incomplete Fibonacci p-numbers.

Proposition 4.2. *For any nonnegative integer n and k with $0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor$, we have*

$$U_{p,n}(k) = a^{\xi(n+1)}b^{\xi(n)}U_{p,n-1}(k) + cU_{p,n-p-1}(k-1). \tag{14}$$

Proof. From Theorem 3.2 and Definition 3.1, we have

$$
\begin{aligned}
&a^{\xi(n+1)}b^{\xi(n)}U_{p,n-1}(k) + cU_{p,n-p-1}(k-1)\\
&= a^{\xi(n+1)}b^{\xi(n)}\sum_{i=0}^{k}\mathcal{F}_p(n-2-i,i)\,c^i + \sum_{i=0}^{k-1}\mathcal{F}_p(n-p-2-i,i)\,c^{i+1}
\end{aligned}
$$

$$
\begin{aligned}
&= a^{\xi(n+1)}b^{\xi(n)}\sum_{i=0}^{k}\mathcal{F}_p\left(n-2-i,i\right)c^i+\sum_{i=1}^{k}\mathcal{F}_p\left(n-p-1-i,i-1\right)c^i\\
&= \sum_{i=0}^{k}\left(a^{\xi(n+1)}b^{\xi(n)}\mathcal{F}_p\left(n-2-i,i\right)+\mathcal{F}_p\left(n-p-1-i,i-1\right)\right)c^i\\
&= \sum_{i=0}^{k}\mathcal{F}_p\left(n-1-i,i\right)c^i=U_{p,n}(k).
\end{aligned}
$$

□

The relation (14) can be transformed into a nonhomogeneous relation as follows:

$$
\begin{aligned}
&U_{p,n}(k)=a^{\xi(n+1)}b^{\xi(n)}U_{p,n-1}(k)+cU_{p,n-p-1}(k-1)\\
&=a^{\xi(n+1)}b^{\xi(n)}U_{p,n-1}(k)+cU_{p,n-p-1}(k)+c\left(U_{p,n-p-1}(k-1)-U_{p,n-p-1}(k)\right)\\
&=a^{\xi(n+1)}b^{\xi(n)}U_{p,n-1}(k)+cU_{p,n-p-1}(k)-\mathcal{F}_p\left(n-p-2-k,k\right)c^{k+1}.
\end{aligned}
\tag{15}
$$

Proposition 4.3. *For any nonnegative integer n and k with $0\le k\le\left\lfloor\frac{n}{p+1}\right\rfloor$, we have*

$$\sum_{i=0}^{h}\mathcal{B}_p\left(h,h-i;a^{\xi(n+pi)}b^{\xi(n+pi+1)},a^{\xi(n+pi+1)}b^{\xi(n+pi)}\right)c^{h-i}U_{p,n+pi}(k+i)=U_{p,n+(p+1)h}(k+h).$$

Proof. We will use the induction on h to show the above formula. It is clear that the result is true for $h=1$, since

$$a^{\xi(n+p)}b^{\xi(n+p+1)}U_{p,n+p}(k+1)+cU_n(k)=U_{n+p+1}(k+1).$$

Assume that it is true for any r such that $1\le r\le h$. Then by the inductive hypothesis and Definition 2.1, we have

$$
\begin{aligned}
&\sum_{i=0}^{h+1}\mathcal{B}_p\left(h+1,h+1-i;a^{\xi(n+pi)}b^{\xi(n+pi+1)},a^{\xi(n+pi+1)}b^{\xi(n+pi)}\right)c^{h-i}U_{p,n+pi}(k+i)\\
&= a^{\xi(n+h+p(h+1))}b^{\xi(n+(p+1)(h+1))}\sum_{i=0}^{h+1}\mathcal{B}_p\left(h,h+1-i;a^{\xi(n+pi)}b^{\xi(n+pi+1)},a^{\xi(n+pi+1)}b^{\xi(n+pi)}\right)c^{h+1-i}\\
&\quad U_{p,n+pi}(k+i)+\sum_{i=0}^{h+1}\mathcal{B}_p\left(h,h-i;a^{\xi(n+pi)}b^{\xi(n+pi+1)},a^{\xi(n+pi+1)}b^{\xi(n+pi)}\right)c^{h+1-i}U_{p,n+pi}(k+i)\\
&= a^{\xi(n+h+p(h+1))}b^{\xi(n+(p+1)(h+1))}\sum_{i=0}^{h}\mathcal{B}_p\left(h,h-i;a^{\xi(n+p+pi)}b^{\xi(n+p+pi+1)},a^{\xi(n+p+pi+1)}b^{\xi(n+p+pi)}\right)c^{h-i}\\
&\quad U_{p,n+p+pi}(k+i+1)+c\sum_{i=0}^{h}\mathcal{B}_p\left(h,h-i;a^{\xi(n+pi)}b^{\xi(n+pi+1)},a^{\xi(n+pi+1)}b^{\xi(n+pi)}\right)c^{h-i}U_{p,n+pi}(k+i)\\
&= a^{\xi(n+h+p(h+1))}b^{\xi(n+(p+1)(h+1))}U_{p,n+p+(p+1)h}(k+h+1)+cU_{p,n+(p+1)h}(k+h)\\
&= U_{p,n+(h+1)(p+1)}(k+h+1).
\end{aligned}
$$

□

Now we introduce the bi-periodic incomplete Lucas p-numbers and obtain several of their properties.

Definition 4.4. *Let n and k be positive integers such that $0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor$. The bi-periodic incomplete Lucas p-numbers are defined as*

$$V_{p,n}(k) = \sum_{i=0}^{k} \mathcal{L}_p(n-i,i)c^i.$$

Some special cases of Definition 4.4 are given as follows:

- $V_{p,n}(0) = a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}$,
- $V_{p,n}(1) = \begin{cases} a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor} + a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor - \frac{p+1}{2}} nc, & p \text{ is odd}, \\ a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor} + a^{\xi(n)-1}(ab)^{\lfloor \frac{n}{2} \rfloor} c(\left\lfloor \frac{n-p+1}{2} \right\rfloor + p), & p \text{ is even}. \end{cases}$
- $V_{p,n}(\left\lfloor \frac{n}{p+1} \right\rfloor) = V_{p,n}$.

Proposition 4.5. *For any nonnegative integer n and k with $0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor$, we have*

$$V_{p,n}(k) = a^{\xi(n)}b^{\xi(n+1)}V_{p,n-1}(k) + cV_{p,n-p-1}(k-1). \tag{16}$$

Proof. From relation (10) and Definition 4.4, we get the desired result. □

The relation (16) can be transformed into a nonhomogeneous relation as follows:

$$V_{p,n}(k) = a^{\xi(n)}b^{\xi(n+1)}V_{p,n-1}(k) + cV_{p,n-p-1}(k) - \mathcal{L}_p\left(n-p-1-k,k\right)c^{k+1}. \tag{17}$$

Proposition 4.6. *For any nonnegative integer n and k with $0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor$, we have*

$$\sum_{i=0}^{h} \mathcal{B}_p\left(h, h-i; a^{\xi(n+pi+1)}b^{\xi(n+pi)}, a^{\xi(n+pi)}b^{\xi(n+pi+1)}\right)c^{h-i}V_{p,n+pi}(k+i) = V_{p,n+(p+1)h}(k+h).$$

Proof. The proof can be carried out using Definition 2.1 and relation (16). □

Finally, we give a connection between the bi-periodic incomplete Fibonacci p-numbers and bi-periodic incomplete Lucas p-numbers.

Theorem 4.7. *For bi-periodic incomplete Fibonacci and Lucas p-numbers, we have*

$$V_{p,n}(k) = \begin{cases} U_{p,n+1}(k) + pcU_{p,n-p}(k-1), & \text{if } p \text{ is odd and } n \geq p, \\ U_{p,n+1}(k) + pcbU_{p,n-p-1}(k-1) + pc^2U_{p,n-2p-1}(k-2), & \text{if } p \text{ is even and } n \geq 2p+1. \end{cases}$$

Proof. From Definition 4.4 and Theorem 3.6, we get the desired result. □

5. Concluding remarks

In this paper, we introduced a new generalization of binomial coefficients, which are one of the fundamental tools used to construct the generalized Pascal's triangle. We explored some basic properties, such as recurrence relations, the generating function, convolution identity, and symmetry relation. Additionally, we defined generalized Fibonacci and Lucas p-triangles, which extend the classical Pascal's and Lucas triangles. By examining the relationship between the generalized binomial coefficients and the coefficients of the Fibonacci p-triangles, we derived an explicit formula for bi-periodic Fibonacci and Lucas p-numbers, one of the main goals of this paper. Furthermore, we introduced an incomplete version of these numbers and established several of their properties. These results provide insight into the properties of this new family of triangles, but many areas remain unexplored. For instance, future work could investigate combinatorial interpretations of bi-periodic incomplete Fibonacci and Lucas p-numbers.

Declarations

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