Filomat 39:13 (2025), 4579–4583
https://doi.org/10.2298/FIL2513579W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remarks on strong parity factors in graphs

Sufang Wang[a]

[a]*School of Public Management, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China*

Abstract. A graph G has a strong parity factor F if for any subset $X \subseteq V(G)$ with $|X|$ even, G contains a spanning subgraph F satisfying $\delta(F) \geq 1$, $d_F(u) \equiv 1 \pmod 2$ for any $u \in X$, and $d_F(v) \equiv 0 \pmod 2$ for any $v \in V(G) \setminus X$. In this paper, we first establish a neighborhood condition for the existence of a strong parity factor in a graph. Then we show an independence number and minimum degree condition to ensure that a graph has a strong parity factor.

1. Introduction

In this paper, we deal only with finite undirected graphs which have neither loops nor multiple edges. Let G be a graph. Then we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. The order of G is the number $n = |V(G)|$ of its vertices. For a vertex x of G, let $N_G(x)$ denote the neighborhood of x in G. Then $d_G(x) = |N_G(x)|$ is the degree of x in G. The minimum degree of G is denoted by $\delta(G)$ (δ for short). For any $X \subseteq V(G)$, we let $N_G(X) = \bigcup_{x \in X} N_G(x)$. We denote by $G[X]$ the subgraph induced in G by X, and by $G - X$ the subgraph obtained from G by deleting the vertices in X and their incident edges. An independent set is a set of vertices whose induced subgraph has no edge. Let $\alpha(G)$, $\omega(G)$ and $i(G)$ denote the independence number, the number of connected components and the number of isolated vertices of G, respectively.

Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ and $g(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$. A spanning subgraph F of G with $d_F(x) \equiv f(x) \pmod 2$ and $g(x) \leq d_F(x) \leq f(x)$ for any $x \in V(G)$ is called a (g, f)-parity factor. Let a and b be two integers with $1 \leq a \leq b$ and $a \equiv b \pmod 2$. If $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$, then a (g, f)-parity factor is called an (a, b)-parity factor. If $a = 1$, then an (a, b)-parity factor is a $(1, b)$-odd factor. Notice that a $(1, b)$-odd factor is an extension of a 1-factor (or a perfect matching). A (g, f)-parity factor is an f-factor if $g(v) = f(v)$ for every $v \in V(G)$. If $f(v) = k$ for any $v \in V(G)$, then an f-factor is a k-factor. For any real function φ defined on $V(G)$ and any subset $S \subseteq V(G)$, let $\varphi(S) = \sum_{x \in S} \varphi(x)$.

Cleemput and Zamfirescu [3], Gross, Kahl and Saccoman [8], Enomoto, Plummer and Saito [6] investigated the existence of 1-factors in graphs. Zhou, Zhang and Sun [33] showed an A_α-spectral radius condition for the existence of $(1, 2)$-factors in graphs. Zhou, Sun and Liu [27] established a connection between distance signless Laplacian spectral radius and $(1, 2)$-factors in graphs. Bazgan, Benhamdine, Li and

2020 *Mathematics Subject Classification*. Primary 05C70; Secondary 90B99.
Keywords. graph; neighborhood; independence number; minimum degree; strong parity factor.
Received: 01 December 2024; Accepted: 11 March 2025
Communicated by Paola Bonacini
Email address: wangsufangjust@163.com (Sufang Wang)
ORCID iD: https://orcid.org/0009-0007-5527-5713 (Sufang Wang)

Woźniak [1] claimed a toughness condition for a graph with a $(1,2)$-factor. Kaneko [9] obtained a criterion for a graph having a $(1,2)$-factor. Kano, Lu and Yu [10] gave a sufficient condition for a graph to possess a $(1,2)$-factor via the number of isolated vertices. Zhou [21–23], Zhou, Sun and Liu [28] presented some sufficient conditions for graphs to possess $(1,2)$-factors. Wu [19], Dai [5], Liu and Pan [13], Liu [11] provided some sufficient conditions for graphs to have $(1,2)$-factors with given properties. Wang and Zhang [17], Zhou, Xu and Sun [30] obtained some results for the existence of $(1,b)$-factors in graphs. Cui and Kano [4] studied the existence of $(1,b)$-odd factors in graphs. Furuya and Yashima [7], Zhou, Zhang and Liu [32], Zhou, Pan and Xu [25] showed some sufficient conditions for a graph to have an $[a,b]$-factor. Liu and Lu [12] posed a degree condition for a graph to contain an (a,b)-parity factor. Yang, Zhang, Lu and Liu [20] provided an independence number and connectivity condition for the existence of an (a,b)-parity factor in a graph. Some other results on graph factors were obtained by Lv [15], Wang and Zhang [16], Wu [18], Zhou and Liu [24], Zhou and Wu [29], Zhou, Sun and Liu [26].

A graph G has a strong parity factor F if for any subset $X \subseteq V(G)$ with $|X|$ even, G contains a spanning subgraph F satisfying $\delta(F) \geq 1$, $d_F(u) \equiv 1 \pmod 2$ for any $u \in X$, and $d_F(v) \equiv 0 \pmod 2$ for any $v \in V(G) \setminus X$. Bujtás, Jendrol' and Tuza [2] introduced the concept of strong parity factor and put forward some sufficient conditions for graphs to have strong parity factors. Lu, Yang and Zhang [14] gave a necessary and sufficient condition for the existence of a strong parity factor in a graph. Zhou and Zhang [31] provided two sufficient conditions for graphs to contain strong parity factors.

Lu, Yang and Zhang [14] presented a characterization of a graph having a strong parity factor, which is shown in the following.

Theorem 1.1 (Lu, Yang and Zhang [14]). A graph G has a strong parity factor if and only if

$$\omega(G - S) \leq \sum_{x \in S} d_G(x) - 2|S| + 1$$

for any $S \subseteq V(G)$.

Zhou and Zhang [31] presented the following problem on strong parity factors of graphs.

Problem. Investigate other sufficient conditions for the existence of strong parity factors in graphs.

In this paper, we continue to investigate the existence of strong parity factors in graphs, and obtain some results on strong parity factors in graphs which partially settles the above problem.

Theorem 1.2. Let G be a graph with minimum degree $\delta \geq 3$. If G satisfies

$$N_G(X) = V(G) \quad \text{or} \quad |N_G(X)| > \left(1 + \frac{1}{2(\delta - 2)}\right)|X| - \frac{1}{\delta - 2} \tag{1}$$

for any $X \subseteq V(G)$, then G contains a strong parity factor.

Theorem 1.3. Let $t \geq 1$ be an integer, and let G be a t-connected graph with minimum degree $\delta \geq 3$. If G satisfies

$$\alpha(G) < (\delta - 2)t + 2, \tag{2}$$

then G has a strong parity factor.

2. The proof of Theorem 1.2

Proof of Theorem 1.2. Suppose to the contrary that G contains no strong parity factor. Then by Theorem 1.1, we conclude

$$\omega(G - S) \geq \sum_{x \in S} d_G(x) - 2|S| + 2 \geq (\delta - 2)|S| + 2 \tag{3}$$

for some subset $S \subseteq V(G)$.

Claim 1. $\omega(G) = 1$.

Proof. Assume that $\omega(G) \geq 2$, that is, G is not connected. Then for any connected component C of G, we have $V(C) \neq \emptyset$ and $N_G(V(C)) \neq V(G)$. According to $\delta \geq 3$, we deduce $|V(C)| > 2$. Combining this with (1) and $N_G(V(C)) \neq V(G)$, we get

$$\begin{aligned} |V(C)| \geq & |N_G(V(C))| \\ > & \left(1 + \frac{1}{2(\delta - 2)}\right)|V(C)| - \frac{1}{\delta - 2} \\ = & |V(C)| + \frac{|V(C)|}{2(\delta - 2)} - \frac{1}{\delta - 2} \\ > & |V(C)| + \frac{2}{2(\delta - 2)} - \frac{1}{\delta - 2} \\ = & |V(C)|, \end{aligned}$$

which is a contradiction. Claim 1 is proved. □

Claim 2. $|S| \geq 1$.

Proof. Assume that $|S| = 0$. Together with (3), we obtain

$$\omega(G) \geq 2,$$

which contradicts Claim 1. We completes the proof of Claim 2. □

According to (3) and Claim 2, we have $\omega(G - S) \geq (\delta - 2)|S| + 2 \geq \delta \geq 3$. Let a denote the number of isolated vertices of $G - S$. We shall consider two cases by the value of a.

Case 1. $a \geq 1$.

In this case, we easily see $N_G(V(G) \setminus S) \neq V(G)$. Together with (1), we conclude

$$\begin{aligned} |N_G(V(G) \setminus S)| > & \left(1 + \frac{1}{2(\delta - 2)}\right)|V(G) \setminus S| - \frac{1}{\delta - 2} \\ = & \left(1 + \frac{1}{2(\delta - 2)}\right)|V(G)| - \left(1 + \frac{1}{2(\delta - 2)}\right)|S| - \frac{1}{\delta - 2}. \end{aligned} \tag{4}$$

Notice that $|V(G)| - a \geq |N_G(V(G) \setminus S)|$. Combining this with (4), we possess

$$\begin{aligned} |V(G)| - a \geq & |N_G(V(G) \setminus S)| \\ > & \left(1 + \frac{1}{2(\delta - 2)}\right)|V(G)| - \left(1 + \frac{1}{2(\delta - 2)}\right)|S| - \frac{1}{\delta - 2}, \end{aligned}$$

which implies that

$$|V(G)| < (2\delta - 3)|S| - 2(\delta - 2)a + 2. \tag{5}$$

On the other hand, $G - S$ has at least $(\delta - 2)|S| + 2 - a$ nontrivial connected components, and thus $a + 2((\delta - 2)|S| + 2 - a) \leq |V(G)| - |S|$. Hence, we deduce

$$|V(G)| \geq (2\delta - 3)|S| - a + 4. \tag{6}$$

It follows from (5) and (6) that $a < 0$, which contradicts $a \geq 1$.

Case 2. $a = 0$.

In this case, every connected component has at least two vertices. Take one connected component C in $G - S$, and let $X = V(G) - S - V(C)$. It is clear that $N_G(X) \neq V(G)$ and $|N_G(X)| \leq |X| + |S|$. Combining these with (1), we obtain

$$|X| + |S| \geq |N_G(X)| > \left(1 + \frac{1}{2(\delta - 2)}\right)|X| - \frac{1}{\delta - 2},$$

which yields that

$$|X| < 2(\delta - 2)|S| + 2. \tag{7}$$

Recall that $G - S$ has at least $(\delta - 2)|S| + 2$ nontrivial connected components. Thus, we deduce

$$|X| = |V(G) - S - V(C)| \geq 2((\delta - 2)|S| + 1) = 2(\delta - 2)|S| + 2,$$

which is a contradiction to (7). Consequently, the theorem is verified. □

3. The proof of Theorem 1.3

Proof of Theorem 1.3. Suppose to the contrary that a t-connected graph G has no strong parity factor. In terms of Theorem 1.1, there exists some subset S of $V(G)$ such that

$$\omega(G - S) \geq \sum_{x \in S} d_G(x) - 2|S| + 2 \geq (\delta - 2)|S| + 2. \tag{8}$$

Claim 1. $|S| \geq 1$.
Proof. If $|S| = 0$, then it follows from (8) that

$$\omega(G) \geq 2,$$

which contradicts that G is t-connected. This completes the proof of Claim 1. □
Claim 2. $|S| \geq t$.
Proof. Let $|S| \leq t - 1$. Since G is t-connected, $G - S$ is connected. Thus, we obtain $\omega(G - S) = 1$. Combining this with (8), $\delta \geq 3$ and Claim 1, we conclude

$$1 = \omega(G - S) \geq (\delta - 2)|S| + 2 \geq \delta \geq 3,$$

which is a contradiction. This completes the proof of Claim 2. □

Notice that $\alpha(G) \geq \omega(G - S)$. Together with (2), (8), $\delta \geq 3$ and Claim 2, we possess

$$(\delta - 2)t + 2 > \alpha(G) \geq \omega(G - S) \geq (\delta - 2)|S| + 2 \geq (\delta - 2)t + 2,$$

which is a contradiction. We verify Theorem 1.3. □

4. Concluding remark

Bujtás, Jendrol' and Tuza [2] introduced the concept of strong parity factor and conjectured that each 2-edge-connected graph with minimum degree at least three contains a strong parity factor. Lu, Yang and Zhang [14] presented a characterization for the existence of a strong parity factor in a graph and confirmed the above conjecture for 3-edge-connected graph. Zhou and Zhang [31] put forward two sufficient conditions for a graph to possess a strong parity factor with respect to the size or the toughness. In this paper, we also study a strong parity factor of a graph and provide two sufficient conditions for a graph to contain a strong parity factor via the neighborhood, or the independence number and the minimum degree. Indeed, there are very few results on strong parity factors of graphs. Hence, it is natural and interesting to establish some new sufficient conditions to guarantee that a graph has a strong parity factor.

Data availability statement

My manuscript has no associated data.

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

Acknowledgments

The author would like to express her deepest gratitude to the anonymous referees for their helpful comments and suggestions.

References

[1] C. Bazgan, A. Benhamdine, H. Li, M. Woźniak, Partitioning vertices of 1-tough graph into paths, Theoret. Comput. Sci. 263 (2001) 255–261.
[2] C. Bujtás, S. Jendrol', Z. Tuza, On specific factors in graphs, Graphs Combin. 36 (2020) 1391–1399.
[3] N. Cleemput, C. Zamfirescu, Hamiltonian cycles and 1-factors in 5-regular graphs, J. Combin. Thcory Ser. B 154 (2022) 239–261.
[4] Y. Cui, M. Kano, Some results on odd factors of graphs, J. Graph Theory 12 (1988) 327–333.
[5] G. Dai, Degree sum conditions for path-factor uniform graphs. Indian J. Pure Appl. Math. 55(4) (2024) 1409–1415,
[6] H. Enomoto, M. Plummer, A. Saito, Neighborhood unions and factor critical graphs, Discrete Math. 205 (1999) 217–220.
[7] M. Furuya, T. Yashima, Neighborhood-union condition for an $[a, b]$-factor avoiding a specified Hamiltonian cycle, Discrete Math. 340 (2017) 1419–1425.
[8] D. Gross, N. Kahl, J. Saccoman, Graphs with the maximum or minimum number of 1-factors, Discrete Math. 310 (2010) 687–691.
[9] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, J. Combin. Thcory Ser. B 88 (2003) 195–218.
[10] M. Kano, H. Lu, Q. Yu, Component factors with large components in graphs, Appl. Math. Lett. 23 (2010) 385–389.
[11] H. Liu, On path-factor critical uniform graphs, Indian J. Pure Appl. Math. 55(4) (2024) 1222–1230.
[12] H. Liu, H. Lu, A degree condition for a graph to have (a, b)-parity factors, Discrete Math. 341 (2018) 244–252.
[13] H. Liu, X. Pan, Independence number and minimum degree for path-factor critical uniform graphs, Discrete Appl. Math. 359 (2024) 153–158.
[14] H. Lu, Z. Yang, X. Zhang, A characterization for graphs having strong parity factors, Graphs Combin. 37 (2021) 945–949.
[15] X. Lv, A degree condition for graphs being fractional (a, b, k)-critical covered, Filomat 37(10) (2023) 3315–3320.
[16] S. Wang, W. Zhang, An A_α-spectral radius for a spanning tree with constrained leaf distance in a graph, Filomat 39(2) (2025) 639–648.
[17] S. Wang, W. Zhang, Some results on star-factor deleted graphs, Filomat 38(3) (2024) 1101–1107.
[18] J. Wu, Characterizing spanning trees via the size or the spectral radius of graphs, Aequationes Math. 98(6) (2024) 1441–1455.
[19] J. Wu, Path-factor critical covered graphs and path-factor uniform graphs, RAIRO Oper. Res. 56(6) (2022) 4317–4325.
[20] Z. Yang, X. Zhang, H. Lu, Y. Lin, Sufficient conditions for a graph to have all $[a, b]$-factors and (a, b)-parity factors, Bull. Malays. Math. Sci. Soc. 45 (2022) 1657–1667.
[21] S. Zhou, Degree conditions and path factors with inclusion or exclusion properties, Bull. Math. Soc. Sci. Math. Roumanie 66(1) (2023) 3–14.
[22] S. Zhou, Path factors and neighborhoods of independent sets in graphs, Acta Math. Appl. Sin. Engl. Ser. 39(2) (2023) 232–238.
[23] S. Zhou, Some spectral conditions for star-factors in bipartite graphs, Discrete Appl. Math. 369 (2025) 124-130.
[24] S. Zhou, H. Liu, Star-factors with large components, fractional k-extendability and spectral radius in graphs, Discrete Appl. Math. 361 (2025) 402–411.
[25] S. Zhou, Q. Pan, Y. Xu, A new result on orthogonal factorizations in networks, Filomat 38(20) (2024) 7235–7244.
[26] S. Zhou, Z. Sun, H. Liu, $\mathcal{D}$-index and Q-index for spanning trees with leaf degree at most k in graphs, Discrete Math. 347(5) (2024) 113927.
[27] S. Zhou, Z. Sun, H. Liu, Distance signless Laplacian spectral radius for the existence of path-factors in graphs, Aequationes Math. 98(3) (2024) 727–737.
[28] S. Zhou, Z. Sun, H. Liu, Some sufficient conditions for path-factor uniform graphs, Aequationes Math. 97(3) (2023) 489–500.
[29] S. Zhou, J. Wu, Spanning k-trees and distance spectral radius in graphs, J. Supercomput. 80(16) (2024) 23357–23366.
[30] S. Zhou, Y. Xu, Z. Sun, Some results about star-factors in graphs, Contrib. Discrete Math. 19(3) (2024) 154–162.
[31] S. Zhou, Y. Zhang, Sufficient conditions for graphs to have strong parity factors, RAIRO Oper. Res. 57(5) (2023) 2465–2471.
[32] S. Zhou, Y. Zhang, H. Liu, Some properties of (a, b, k)-critical graphs, Filomat 38(16) (2024) 5885–5894.
[33] S. Zhou, Y. Zhang, Z. Sun, The A_α-spectral radius for path-factors in graphs, Discrete Math. 347(5) (2024) 113940.