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# Durrmeyer type Lototsky-Chlodowsky operators

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**Abstract.** In this paper, we present a introduction to Durrmeyer type Lototsky-Chlodowsky operators. We explore their approximation properties within a weighted function space. Subsequently, we derive the rates of convergence utilizing the second modulus of continuity. Moreover, we investigate the convergence rates in the $L_1$ space. In addition, we establish Voronovskaja-type theorem for the considered Durrmeyer type Lototsky-Chlodowsky operators, thereby contributing significant insights into their asymptotic behaviour.

## 1. Introduction

Let $I \subset \mathbb{R}$. As usual, we denote with $B(I)$ the space of all bounded functions and with $C(I)$ the space of all continuous functions defined on $I$, endowed with the usual sup-norm.
In 1966, King [4] introduced Lototsky-Bernstein operators as follows:
$L_n : B\,[0,1] \to C\,[0,1]$ for $n \in \mathbb{N}$

$$L_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) a_{n,k}(x), \quad x \in [0,1]. \tag{1}$$

Using the basis function $a_{n,k}(x)$ derived from the following relation

$$\prod_{i=1}^{n} (h_i(x)y + 1 - h_i(x)) = \sum_{k=0}^{n} a_{n,k}(x) y^k, \; y \in \mathbb{R}. \tag{2}$$

$$a_{n,k}(x) = \sum_{\substack{J \cup \bar{J} = \mathbb{N}_n \\ Card(J)=k}}^{n} \prod_{i \in \bar{J}} (1 - h_i(x)) \prod_{i \in J} h_i(x)$$

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where $h_i : [0,1] \to [0,1]$ is a sequence of continuous function and $a_{0,0}(x) = 1,\ a_{0,k}(x) = 0$ for $k > 0$. King has provided a sufficient condition for the sequence $(h_i)_{i\in\mathbb{N}}$ to ensure that $(L_n)_{n\geq 1}$ acts as an approximation process on $C[0,1]$. This result can be expressed as follows.
Assume that,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} h_i(x) = x \text{ uniformly in } x \text{ on } [0,1],$$

then, there holds

$$\lim_{n\to\infty} (L_n f)(x) = f(x) \text{ uniformly in } x \text{ on } [0,1],$$

for every $f \in C([0,1])$.

In recent years, the study of these operators has been significantly advanced. Notable contributions to this field can be found in the works of Ron Goldman, Xiao-Wei Xu, Dumitru Popa, Ulrich Abel, Octavian Agratini and Xiao-Ming Zeng ([9], [1], [8], [7]). In this article, the Lototsky-Chlodowsky operators, which are based on Lototsky-Bernstein operators are defined, and then their Durrmeyer type generalization are introduced.

The classical Bernstein-Chlodowsky polynomials are expressed in the following form

$$(C_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k},\ 0 \leq x \leq b_n \tag{3}$$

where $b_n$ is a sequence of positive number such that $\lim\limits_{n\to\infty} b_n = \infty,\ \lim\limits_{n\to\infty}\frac{b_n}{n} = 0$. Chlodowsky introduced these polynomials in 1932 as a generalization of Bernstein polynomials on an unbounded set. In 2005, Ibikli and Karsli [2] introduced the Bernstein-Chlodowsky operators within the framework of Durrmeyer type operators as follows:

$$(D_n f)(x) = \frac{n+1}{b_n}\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} f(t)p_{n,k}\left(\frac{t}{b_n}\right)dt, \quad 0 \leq x \leq b_n \tag{4}$$

where $(b_n)$ is a positive increasing sequence with the following properties,

$$\lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty}\frac{b_n}{n} = 0$$

and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ \ 0 \leq x \leq 1$$

is the Bernstein basis. Recently in [6] Serin, Karsli and Tasdelen constructed the Lototsky-Chlodowsky operators given by
$L_n^* : B[0,\infty) \to C[0,\infty)$ for $n \in \mathbb{N}$,

$$(L_n^* f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right) a_{n,k}\left(\frac{x}{b_n}\right),\ \ x \in [0, b_n] \tag{5}$$

where $a_{n,k}\left(\frac{x}{b_n}\right)$ are Lototsky-Bernstein basis functions satisfying

$$\prod_{i=1}^{n}\left(h_i\left(\frac{x}{b_n}\right)y + 1 - h_i\left(\frac{x}{b_n}\right)\right) = \sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)y^k, y \in \mathbb{R} \tag{6}$$

$$a_{n,k}\left(\frac{x}{b_n}\right)=\sum_{\substack{J\cup \bar{J}=\mathbb{N}_n\\ Card(J)=k}}^{n}\prod_{i\in\bar{J}}\left(1-h_i\left(\frac{x}{b_n}\right)\right)\prod_{i\in J}h_i\left(\frac{x}{b_n}\right).$$

The approximation properties of the operators (5) can be found in [6]. Based on this Lototsky-Chlodowsky operators and Durrmeyer type Bernstein-Chlodowsky operator $D_n$ , we construct sequence of operators $D_n^*$ which approximate functions $f \in L_1[0,\infty)$. This paper aims to define and establish the approximation properties for Durrmeyer-type extensions of $L_n^*$ operators. $L_1([0,b_n])$ denotes the Banach space consisting of all real valued Lebesque integrable function on $[0,b_n]$ with the norm denoted by $\|.\|_1$ , $\left\|f\right\|_1=\int_0^{b_n}\left|f(x)\right|dx$. In this paper, we introduced Durrmeyer type modification of the operators (5) as follows:
$D_n^*: L_1[0,\infty)\to C[0,\infty)$ for $n\in\mathbb{N}$

$$(D_n^* f)(x)=\frac{n+1}{b_n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)f(t)dt,\ \ 0\le x\le b_n. \tag{7}$$

**Remark 1.1.** *Let $x\in[0,b_n]$. For all $f,g\in L_1[0,\infty)$ and $a,b\in\mathbb{R}$, obviously $(D_n^*(af+bg))(x)=a(D_n^*f)(x)+b(D_n^*g)(x)$ holds true.*

**Remark 1.2.** *Utilizing a bivariate kernel, we can write $D_n^*f$ in a more compact form as follows:*

$$(D_n^* f)(x)=\int_0^{b_n}K_n^*(x,t)f(t)dt,\quad x\in[0,b_n] \tag{8}$$

*where*

$$K_n^*(x,t)=\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)p_{n,k}\left(\frac{t}{b_n}\right),\ \ (x,t)\in[0,b_n]\times[0,b_n]. \tag{9}$$

**Remark 1.3.** *By using beta function, for any $p\in\mathbb{N}_0$ , we deduce*

$$\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)t^p dt=\frac{(k+p)!}{k!}\frac{n!}{(n+p+1)!},k=\overline{0,n}. \tag{10}$$

**Lemma 1.4.** *For $f\in L_1[0,\infty)$ , we have $\left\|D_n^*f\right\|_{L_1[0,\infty)}\le\left\|f\right\|_{L_1[0,\infty)}$ .*

*Proof.* By the definition of (7), we have

$$\begin{aligned}\left|(D_n^* f)(x)\right| &= \left|\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)f(t)dt\right|\\ &\le \frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)\right|dt\end{aligned}$$

$$\int_0^{b_n}\left|(D_n^* f)(x)\right|dx \le \int_0^{b_n}\left(\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)\right|dt\right)dx$$

$$\leq \int_0^{b_n}\left(\int_0^{b_n}\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)\right|dt\right)dx$$

by applying the Generalized Minkowski inequality, we can write

$$\begin{aligned}
&\leq \int_0^{b_n}\left(\int_0^{b_n}\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)\right|dx\right)dt\\
&\leq \int_0^{b_n}\frac{n+1}{b_n}\left|f(t)\right|\left(\int_0^{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)p_{n,k}\left(\frac{t}{b_n}\right)dx\right)dt\\
&\leq \int_0^{b_n}\frac{n+1}{b_n}\left|f(t)\right|\left(\sum_{k=0}^{n}\int_0^{b_n}a_{n,k}\left(\frac{x}{b_n}\right)p_{n,k}\left(\frac{t}{b_n}\right)dx\right)dt\\
&\leq \int_0^{b_n}\left|f(t)\right|\left(\sum_{k=0}^{n}p_{n,k}\left(\frac{t}{b_n}\right)\frac{n+1}{b_n}\int_0^{b_n}a_{n,k}\left(\frac{x}{b_n}\right)dx\right)dt.
\end{aligned}$$

By applying Remark 1.3, we have

$$\frac{n+1}{b_n}\int_0^{b_n}a_{n,k}\left(\frac{x}{b_n}\right)dx=\frac{n+1}{b_n}\binom{n}{k}\int_0^{b_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=1.$$

Thus

$$\begin{aligned}
\left\|D_n^*f\right\|_{L_1[0,\infty)} &\leq \int_0^{b_n}\left|f(t)\right|\left(\sum_{k=0}^{n}p_{n,k}\left(\frac{t}{b_n}\right)\frac{n+1}{b_n}\int_0^{b_n}a_{n,k}\left(\frac{x}{b_n}\right)dx\right)dt\\
&\leq \int_0^{b_n}\left|f(t)\right|\sum_{k=0}^{n}p_{n,k}\left(\frac{t}{b_n}\right)dt\\
&\leq \int_0^{b_n}\left|f(t)\right|dt\\
&\leq \left\|f\right\|_{L_1[0,\infty)}.
\end{aligned}$$

In this study, we first present several lemmas to establish the covergence properties of the operators (7) using Korovkin's theorem, followed by a proof of the main theorem. We determine the rate of convergence of the operators to the target function $f$ by analyzing the modulus of continuity. Subsequently, we determine the rate of these operators with the assistance of the second modulus of continuity. Additionally, we provide a Voronovskaja-type theorem. □

## 2. Preliminary results

In this section, we present several lemmas that are necessary for proving our main theorems.

**Lemma 2.1.** *Let $x \in [0, b_n]$ and $e_i(t) = t^i, i = 0, 1, 2$. Then, the Durrmeyer type Lototsky-Chlodowsky operators $D_n^*$ satisfy*

$$\begin{aligned}
(D_n^* e_0)(x) &= 1 \\
(D_n^* e_1)(x) &= \frac{b_n}{n+2}\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + 1\right) \\
(D_n^* e_2)(x) &= \frac{b_n^2}{(n+2)(n+3)}\left[\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\right)^2 + \sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\left(1 - h_i\left(\frac{x}{b_n}\right)\right)\right] \\
&\quad + \frac{3b_n^2}{(n+2)(n+3)}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \frac{2b_n^2}{(n+2)(n+3)}.
\end{aligned}$$

*Proof.* It is clear that, $(D_n^* e_0)(x) = \sum\limits_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)$. Putting $y = 1$ in (6)

$$\prod_{i=1}^{n}\left(h_i\left(\frac{x}{b_n}\right) + 1 - h_i\left(\frac{x}{b_n}\right)\right) = \sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right) = 1$$

hence the result follows.
Moreover, we get from (7) and Remark 1.3

$$\begin{aligned}
(D_n^* e_1)(x) &= \frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right) t\,dt \\
&= \frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\left(\frac{b_n^2(k+1)}{(n+1)(n+2)}\right) \\
&= \frac{n+1}{b_n}\frac{b_n^2}{(n+1)(n+2)}\left(\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)k + \sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\right) \\
&= \frac{b_n}{n+2}\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + 1\right) \\
&= \frac{b_n}{n+2}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \frac{b_n}{n+2}.
\end{aligned}$$

Finally, in view of (7), we get

$$\begin{aligned}
(D_n^* e_2)(x) &= \frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right) t^2\,dt \\
&= \frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\left(\frac{b_n^3(k+2)(k+1)}{(n+3)(n+2)(n+1)}\right) \\
&= \frac{n+1}{b_n}\frac{b_n^3}{(n+3)(n+2)(n+1)}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)(k+2)(k+1)
\end{aligned}$$

$$= \frac{b_n^2}{(n+3)(n+2)}\left(\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)k^2 + 3\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)k + 2\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\right)$$

$$= \frac{b_n^2}{(n+3)(n+2)}\left[\begin{array}{c}\left(\sum\limits_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\right)^2+\sum\limits_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\left(1-h_i\left(\frac{x}{b_n}\right)\right)\\ +3\sum\limits_{i=1}^{n} h_i(\frac{x}{b_n})\ +2\end{array}\right]$$

$$= \frac{b_n^2}{(n+2)(n+3)}\left[\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\right)^2+\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\left(1-h_i\left(\frac{x}{b_n}\right)\right)\right] + \frac{3b_n^2}{(n+2)(n+3)}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)+\frac{2b_n^2}{(n+2)(n+3)}.$$

□

**Lemma 2.2.** *For each $x \in [0, b_n]$ , it can be deduced from the results presented Lemma 2.1 that*

$$(D_n^*(e_1 - x))(x) = \frac{b_n}{(n+2)}\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)+1\right)-x \tag{11}$$

$$\begin{aligned}(D_n^*(e_1 - x)^2)(x) &= (D_n^*e_2)(x) - 2x(D_n^*e_1)(x) + x^2(D_n^*e_0)(x)\\ &= \frac{b_n^2}{(n+2)(n+3)}\left[\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\right)^2+\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\left(1-h_i\left(\frac{x}{b_n}\right)\right)\right]\\ &\quad+\frac{3b_n^2}{(n+2)(n+3)}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)+\frac{2b_n^2}{(n+2)(n+3)}.\\ &\quad-\frac{2xb_n}{n+2}\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)+1\right)+x^2.\end{aligned} \tag{12}$$

**Lemma 2.3.** ([1]) *For every $\alpha > 0$, let $h_i\left(\frac{x}{b_n}\right) = \frac{i^\alpha \frac{x}{b_n}}{i^\alpha + \frac{x}{b_n}}$. Since $0 < \alpha < 1$, $\sum\limits_{i=1}^{n}\left(\frac{1}{i^\alpha + \frac{x}{b_n}}\right) \cong \frac{n^{1-\alpha}}{b_n(1-\alpha)}$ and thus*

$$\sum_{i=1}^{n}\left(h_i\left(\frac{x}{b_n}\right)-\frac{x}{b_n}\right) \cong -\frac{x^2}{b_n^3}\frac{n^{1-\alpha}}{1-\alpha}. \tag{13}$$

**Lemma 2.4.** ([5]) *For $q \in \mathbb{N}$ and fixed $x \in I$, let $A_n : L_\infty(I) \to C(I)$ be a sequence of positive linear operators with the property,*

$$(A_n(t-x)^p; x) = O(n^{-[(p+1)/2]}) \qquad (n \to \infty) \quad (p = 0, 1, ..., 2q + 2).$$

*Then we have for each $f \in L_\infty(I)$, which is $2q$ times differentiable at $x$ the asymptotic relation*

$$(A_n f)(x) = \sum_{p=0}^{2q} \frac{1}{p!}(A_n(t-x)^p; x) f^{(p)}(x) + O(n^{-q}) \qquad (n \to \infty). \tag{14}$$

*If, in addition, $f^{(2q+2)}(x)$ exists, the term of $O(n^{-q})$ in (12) can be replaced by $O(n^{-(q+1)})$.*

## 3. Convergence results

In this part, we study some approximation properties of the operator $(D_n^* f)(x)$ defined by (7). Let $[a, b]$ be a compact subset of $[0, +\infty)$ and consider the following type lattice homomorphism

$$H : C[0, \infty) \to C[a, b]$$

defined by $H(f) := f\ |_{[a,b]}$ for every $f \in L_1\ [0,\infty)$, where $f\ |_{[a,b]}$ is the restriction of the domain of $f$ to the interval $[a,b]$. Clearly, we have for each $i =\ 0,1,2$

$$\lim_{n\to+\infty} H\left(D_n^* e_i(x)\right) = H(e_i) \text{ uniformly on } [a,b].$$

Owing to the Korovkin property we have the following Korovkin type approximation result related to the uniform convergence. In this section, we explore the rate of convergence utilizing the modulus of continuity, Peetre's K-functional and elements of Lipschitz class.

**Theorem 3.1.** *For every $x \in [0,b_n]$ and $f \in L_1[0,\infty)$, let $\lim\limits_{n\to\infty} f(x) = k_f < \infty$ and $\lim\limits_{n\to\infty} \frac{b_n^2}{n} = 0$. Then, there holds*

$$\lim_{n\to\infty} \left\| D_n^* f - f \right\|_{L_1[0,\infty)} = 0.$$

Now, in order to get on uniform convergence result on the positive reel axis $[0,+\infty)$, we consider the following subspace

$$C_\rho([0,\infty)) = \{f \in C_\rho([0,\infty)) : \ for\ \forall x \in [0,\infty),\ \left|f(x)\right| \le M_f \rho(x)\}$$

endowed with the sup-norm. We have the following theorem.

**Theorem 3.2.** *For each $x \in [0,b_n]$ , let $\rho(x) =\ 1 + x^4$. Then for every $f \in C_\rho([0,\infty))$, one has*

$$\lim_{n\to\infty} \left\| D_n^* f - f \right\|_\rho = 0.$$

*Proof.* For $0 \le x \le b_n$ by using the (11) equality, we may write

$$\begin{aligned}
\left\|(D_n^* e_0)(x) - 1\right\|_\rho &= \sup_{0\le x\le b_n} \frac{\left|(D_n^* e_0)(x) - 1\right|}{\rho(x)} \\
&= \sup_{0\le x\le b_n} \frac{|1-1|}{\rho(x)} \\
&= 0
\end{aligned}$$

thus

$$\lim_{n\to\infty} \left\|(D_n^* e_0)(x) - 1\right\|_\rho = 0.$$

By using the (11) equality, we can write

$$\begin{aligned}
\left\|(D_n^* e_1)(x) - x\right\|_\rho &= \sup_{0\le x\le b_n} \frac{\left|(D_n^* e_1)(x) - x\right|}{\rho(x)} \\
&= \sup_{0\le x\le b_n} \frac{\left| \frac{b_n}{(n+2)} \left( \sum\limits_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + 1 \right) - x \right|}{1+x^4} \\
&= \sup_{0\le x\le b_n} \frac{\left| \frac{b_n}{(n+2)} \left( \frac{nx}{b_n} - \frac{x^2 n^{1-\alpha}}{b_n^3(1-\alpha)} \right) + \frac{b_n}{(n+2)} - x \right|}{1+x^4} \\
&= \sup_{0\le x\le b_n} \frac{\left| \frac{nx}{(n+2)} - \frac{x^2 n^{1-\alpha}}{b_n^2(1-\alpha)(n+2)} + \frac{b_n}{(n+2)} - x \right|}{1+x^4}
\end{aligned}$$

$$= \sup_{0\le x\le b_n} \frac{\left|\left(\frac{n}{(n+2)}-1\right)x - \frac{x^2 n^{1-\alpha}}{b_n^2(1-\alpha)(n+2)} + \frac{b_n}{(n+2)}\right|}{1+x^4}$$
$$\le \left|\frac{n}{(n+2)}-1\right| + \left|\frac{b_n}{(n+2)}\right| + \left|\frac{n^{1-\alpha}}{b_n^2(1-\alpha)(n+2)}\right|$$

$$\lim_{n\to\infty}\left|\frac{n}{(n+2)}-1\right| + \left|\frac{b_n}{(n+2)}\right| + \left|\frac{n^{1-\alpha}}{b_n^2(1-\alpha)(n+2)}\right| = 0$$

thus

$$\lim_{n\to\infty}\left\|(D_n^* e_1)(x) - x\right\|_\rho = 0.$$

By using the (11) equality, we can write

$$\left\|(D_n^* e_2)(x) - x^2\right\|_\rho = \sup_{0\le x\le b_n} \frac{\left| \frac{b_n^2}{(n+2)(n+3)}\left[\left(\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)\right)^2 + \sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)\left(1-h_i\left(\frac{x}{b_n}\right)\right)\right] + \frac{3b_n^2}{(n+2)(n+3)}\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right) + \frac{2b_n^2}{(n+2)(n+3)} - x^2 \right|}{\rho(x)}$$
$$\le \sup_{0\le x\le b_n} \frac{\left| \frac{b_n^2}{(n+2)(n+3)}\left(\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)\right)^2 + \frac{b_n^2}{(n+2)(n+3)}\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right) + \frac{3b_n^2}{(n+2)(n+3)}\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right) + \frac{2b_n^2}{(n+2)(n+3)} - x^2 \right|}{\rho(x)}$$

$$\le \sup_{0\le x\le b_n} \frac{\left| \frac{b_n^2}{(n+2)(n+3)}\left(\frac{nx}{b_n} - \frac{x^2 n^{1-\alpha}}{b_n^3(1-\alpha)}\right)^2 + \frac{b_n^2}{(n+2)(n+3)}\left(\frac{nx}{b_n} - \frac{x^2 n^{1-\alpha}}{b_n^3(1-\alpha)}\right) \frac{3b_n^2}{(n+2)(n+3)}\left(\frac{nx}{b_n} - \frac{x^2 n^{1-\alpha}}{b_n^3(1-\alpha)}\right) + \frac{2b_n^2}{(n+2)(n+3)} - x^2 \right|}{1+x^4}$$
$$\le \sup_{0\le x\le b_n} \frac{\left| \frac{b_n^2}{(n+2)(n+3)}\left(\frac{n^2x^2}{b_n^2} - \frac{2x^3 n^{2-\alpha}}{b_n^4(1-\alpha)} + \frac{x^4 n^{2-2\alpha}}{b_n^6(1-\alpha)^2}\right) + \frac{4b_n nx}{(n+2)(n+3)} - \frac{3x^2 n^{1-\alpha}}{b_n(n+2)(n+3)(1-\alpha)} + \frac{2b_n^2}{(n+2)(n+3)} - x^2 \right|}{1+x^4}$$
$$= \sup_{0\le x\le b_n} \frac{\left| \frac{n^2x^2}{(n+2)(n+3)} - \frac{2x^3 n^{2-\alpha}}{(n+2)(n+3)(1-\alpha)} + \frac{x^4 n^{2-2\alpha}}{b_n^2(n+2)(n+3)(1-\alpha)^2} \frac{4b_n nx}{(n+2)(n+3)} - \frac{3x^2 n^{1-\alpha}}{(n+2)(n+3)(1-\alpha)} + \frac{2b_n^2}{(n+2)(n+3)} - x^2 \right|}{1+x^4}$$
$$\le \left| \frac{n^2}{(n+2)(n+3)} - \frac{n^{2-\alpha}}{b_n(n+2)(n+3)(1-\alpha)} + \frac{n^{2-2\alpha}}{b_n^2(n+2)(n+3)(1-\alpha)^2} + \frac{4b_n n}{(n+2)(n+3)} - \frac{3n^{1-\alpha}}{(n+2)(n+3)(1-\alpha)} + \frac{2b_n^2}{(n+2)(n+3)} \right|$$

thus

$$\lim_{n\to\infty}\left\|D_n^* f - f\right\|_\rho = 0.$$

□

**Definition 3.3.** *For $\delta > 0$ and $f \in C[0,1]$, the modulus of continuity $\omega_f(\delta)$ of the function $f$ is defined by*

$$\omega_f(\delta) = \sup_{\substack{x,y\in[0,1]\\ |x-y|\le\delta}} \left|f(x) - f(y)\right|. \tag{15}$$

*Then, for any $\delta > 0$ and each $x \in [0,1]$, we have the following inequality*

$$\left|f(x) - f(y)\right| \le \omega_f(\delta)\left(\frac{\left|x-y\right|}{\delta} + 1\right). \tag{16}$$

**Definition 3.4.** *The second modulus of continuity of $f \in C[0,1]$ is defined by*

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \left\|f(x+2h) - 2f(x+h) + f(x)\right\|_{C[0,1]}.$$

**Definition 3.5.** ([10]) *Now, we consider the following Peetre's K-functional of the function $f \in C[0,1]$,*

$$K_2(f;\delta) = \inf_{g\in W^2[0,1]} \left\{\left\|f-g\right\|_{C[0,1]} + \delta\left\|g''\right\|_{C[0,1]} : g \in W^2\right\}$$

*where*

$$W^2[0,1] := \{g \in C[0,1] \mid g', g'' \in C[0,1]\}.$$

*The following inequality*

$$K_2(f;\delta) \le M\omega_2(f;\sqrt{\delta}) \tag{17}$$

*is valid for all $\delta > 0$. The positive constant $M$ is independent of $f$ and $\delta$.*

In the following theorem, the modulus of continuity is employed to evaluate the rate of approximation to the function $f$.

**Theorem 3.6.** *For each $x \in [0,b_n]$, let $f \in C[0,\infty)$. Then*

$$\left|(D_n^* f)(x) - f(x)\right| \le 2\omega(f;\delta), \tag{18}$$

*where $\delta = \sqrt{(D_n^*(e_1-x)^2)(x)}$.*

*Proof.* By applying the linearity of the operators $D_n^* f$, (11) and (16), we get

$$\begin{aligned}
\left|(D_n^* f)(x) - f(x)\right| &\le \frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)-f(x)\right|dt \\
&\le \frac{n+1}{b_n}\, a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)\omega(f;|t-x|)dt \\
&\le \frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)\left(1+\frac{|t-x|}{\delta}\right)\omega(f;\delta)dt \\
&\le \frac{n+1}{b_n}\omega(f;\delta)\left\{\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)\left(1+\frac{|t-x|}{\delta}\right)dt\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{n+1}{b_n}\omega(f;\delta)\left\{\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\left(\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)dt + \int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)\frac{|t-x|}{\delta}dt\right)\right\} \\
&\leq \omega(f;\delta)\left\{1 + \frac{n+1}{b_n\delta}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)|t-x|\,dt\right\}.
\end{aligned}
$$

By using the Cauchy-Schwarz inequality for the integral, we have

$$
\begin{aligned}
\left|(D_n^* f)(x) - f(x)\right| &\leq \left\{1 + \frac{n+1}{b_n\delta}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\left(\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)dt\right)^{1/2}\right. \\
&\quad \left. \times \left(\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)(t-x)^2\,dt\right)^{1/2}\right\}\omega(f;\delta). \qquad (19)
\end{aligned}
$$

By applying the Cauchy-Schwarz inequality, (19) leads to

$$
\begin{aligned}
&\leq \left\{1 + \frac{1}{\delta}\left(\frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)dt\right)^{1/2}\right. \\
&\quad \left. \times \left(\frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)(t-x)^2\,dt\right)^{1/2}\right\}\omega(f;\delta) \\
&= \left\{1 + \frac{1}{\delta}\left((D_n^* e_0)(x)\right)^{1/2}\left((D_n^*(e_1 - x)^2)(x)\right)^{1/2}\right\}\omega(f;\delta).
\end{aligned}
$$

In view of Lemma 2.1, we get the desired result for $\delta = \sqrt{(D_n^*(e_1 - x)^2)(x)}$. □

**Theorem 3.7.** *If $f \in C[0,\infty)$ and $x \in [0, b_n]$, then*

$$
\left|(D_n^* f)(x) - f(x)\right| \leq M\omega_2\left(f, \frac{1}{2}\sqrt{(D_n^*(e_1 - x)^2)(x) + \beta_n^2\left(\frac{x}{b_n}\right)}\right) + \omega\left(f; \beta_n\left(\frac{x}{b_n}\right)\right) \qquad (20)
$$

*where* $a_n\left(\frac{x}{b_n}\right) = \frac{\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)}{n}$, $\beta_n\left(\frac{x}{b_n}\right) = \left|\alpha_n\left(\frac{x}{b_n}\right) - x - \frac{2x - b_n}{n}\right|$ *and* $\gamma_n\left(\frac{x}{b_n}\right) = \left((D_n^*(e_1 - x)^2)(x) + \beta_n^2\left(\frac{x}{b_n}\right)\right)^{1/2}$. *M is a positive constant.*

*Proof.* Initially, we introduce the operators $E_n^*$, $n \geq 1$ as follows

$$
(E_n^* f)(x) = (D_n^* f)(x) - f\left((D_n^* e_1)(x)\right) + f(x), \; x \in [0, b_n]. \qquad (21)
$$

$$
\begin{aligned}
(E_n^*(e_1 - x))(x) &= (D_n^*(e_1 - x))(x) - ((D_n^* e_1)(x) - x) \\
&= (D_n^* e_1)(x) - x - (D_n^* e_1)(x) + x \\
&= 0.
\end{aligned}
$$

Let $x \in [0, b_n]$ and $g \in C^2[0, \infty)$. Using Taylor's expansion and the integral form of the remainder, we can write

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - u)g''(u)du, \ t \in [0, b_n].$$

Applying $E_n^*$ to Taylor's formula, we get

$$\begin{aligned}(E_n^* g)(x) &= g(x)(E_n^* e_0)(x) + g'(x)(E_n^*(e_1 - x))(x) + E_n^*\left(\int_x^{e_1}(e_1 - u)g''(u)du; x\right) \\ &= g(x) + E_n^*\left(\int_x^{e_1}(e_1 - u)g''(u)du; x\right).\end{aligned}$$

Therefore, we have

$$(E_n^* g)(x) - g(x) = D_n^*\left(\int_x^{e_1}(e_1 - u)g''(u)du; x\right) - \int_x^{(D_n^* e_1)(x)} ((D_n^* e_1)(x) - u)g''(u)du, \tag{22}$$

which implies

$$\begin{aligned}\left|(E_n^* g)(x) - g(x)\right| &\leq \left|D_n^*\left(\int_x^{e_1}(e_1 - u)g''(u)du; x\right)\right| + \left|\int_x^{(D_n^* e_1)(x)} ((D_n^* e_1)(x) - u)g''(u)du\right| \\ &\leq D_n^*\left(\int_x^{e_1} |(e_1 - u)| \left|g''(u)\right| du; x\right) + \int_x^{(D_n^* e_1)(x)} \left|(D_n^* e_1)(x) - u\right| \left|g''(u)\right| du.\end{aligned}$$

As a result, from (11) we get

$$\begin{aligned}\int_x^{(D_n^* e_1)(x)} \left|((D_n^* e_1)(x) - u)\right| \left|g''(u)\right| du &\leq ((D_n^* e_1)(x) - x)^2 \left\|g''\right\|_{C[0,\infty)} \\ &\leq \left\|g''\right\|_{C[0,\infty)} \left(\frac{b_n}{n+2}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \frac{b_n}{n+2} - x\right)^2 \\ &\leq \left\|g''\right\|_{C[0,\infty)} \left(\frac{n}{n+2}a_n\left(\frac{x}{b_n}\right) + \frac{b_n}{n+2} - x\right)^2 \\ &\leq \left\|g''\right\|_{C[0,\infty)} \beta_n^2\left(\frac{x}{b_n}\right).\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\left|(E_n^* g)(x) - g(x)\right| &\leq \left[(D_n^*(e_1 - x)^2)(x) + ((D_n^* e_1)(x) - x)^2\right] \left\|g''\right\|_{C[0,\infty)} \\ &\leq \left\|g''\right\|_{C[0,\infty)} (D_n^*(e_1 - x)^2)(x) + \left\|g''\right\|_{C[0,\infty)} \left(\frac{b_n}{n+2}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \frac{b_n}{n+2} - x\right)^2 \\ &\leq \left\|g''\right\|_{C[0,\infty)} \left[(D_n^*(e_1 - x)^2)(x) + \left(\frac{n}{n+2}a_n\left(\frac{x}{b_n}\right) + \frac{b_n}{n+2} - x\right)^2\right]\end{aligned}$$

$$\leq \quad \left\|g''\right\|_{C[0,\infty)} \left[ (D_n^*(e_1 - x)^2)(x) + \beta_n^2 \left(\frac{x}{b_n}\right)\right]. \tag{23}$$

In view of (21), we have

$$\left|(E_n^* f)(x)\right| = \left|(D_n^* f)(x)\right| + \left|f((D_n^* e_1)(x))\right| + \left|f(x)\right| \leq 3 \left\|f\right\|_{C[0,\infty)}.$$

Referring back to equation (21) and considering (23) and the definition of the modulus of smoothness $\omega(f;\delta)$, thus

$$\begin{aligned}
\left|(D_n^* f)(x) - f(x)\right| &\leq \left|(E_n^* f)(x) - f(x) + f\left((D_n^* e_1)(x)\right) - f(x) + (E_n^* g)(x) - (E_n^* g)(x) + g(x) - g(x)\right| \\
&\leq \left|(E_n^*(f-g))(x)\right| + \left|g(x) - f(x)\right| \\
&\quad + \left| f\left(\frac{b_n}{n+2}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \frac{b_n}{n+2}\right) - f(x)\right| + \left|(E_n^* g)(x) - g(x)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq 4\left\|f-g\right\|_{C[0,\infty)} + \left\|g''\right\|_{C[0,\infty)} \left( (D_n^*(e_1-x)^2)(x) + \beta_n^2\left(\frac{x}{b_n}\right)\right) + \omega\left(f; \left|\frac{b_n}{n+2}\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \frac{b_n}{n+2}\right|\right) \\
&\leq 4\left\|f-g\right\|_{C[0,\infty)} + \left\|g''\right\|_{C[0,\infty)} \left[\left( (D_n^*(e_1-x)^2)(x) + \beta_n^2\left(\frac{x}{b_n}\right)\right) + \omega\left(f;\beta_n\left(\frac{x}{b_n}\right)\right)\right] \\
&\leq 4\left(\left\|f-g\right\|_{C[0,\infty)} + \frac{1}{4}\left( (D_n^*(e_1-x)^2)(x) + \beta_n^2\left(\frac{x}{b_n}\right)\right)\left\|g''\right\|_{C[0,\infty)}\right) + \omega\left(f;\beta_n\left(\frac{x}{b_n}\right)\right).
\end{aligned}$$

Taking the infimum with in the right side over all $g \in C^2[0,\infty)$ , we have

$$\left|(D_n^* f)(x) - f(x)\right| \leq 4K_2\left(f, \frac{1}{4}\left( (D_n^*(e_1-x)^2)(x) + \beta_n^2\left(\frac{x}{b_n}\right)\right)\right) + \omega\left(f;\beta_n\left(\frac{x}{b_n}\right)\right).$$

In conclusion, by leveraging the relationship between the K-functional and the second modulus of continuity as illustrated in Definition 3.5, we attain the desired outcome. □

Furthermore, we delve into the rate of convergence utilizing functions from the Lipschitz class. To do so, we must first provide the following definiton:

**Definition 3.8.** *Let $f$ be a continuous real valued function defined on $[0,\infty)$. Then the following statement is valid for $f$ in the context of order $\gamma$ $(0 < \gamma \leq 1)$ on $[0,\infty)$,*

$$\left|f(x) - f(y)\right| \leq M\left|x-y\right|^{\gamma} \tag{24}$$

*for all $x, y \in [0,\infty)$ and $M > 0$. These set of Lipschitz continuous functions of order $\gamma$ with Lipschitz constant $M$ is denoted by $Lip_M(\gamma)$.*

**Theorem 3.9.** *Let $f \in Lip_M(\gamma)$ and $x \in [0, b_n]$. Then we have*

$$\left|(D_n^* f)(x) - f(x)\right| \leq M\left[D_n^*((t-x)^2;x)\right]^{\frac{\gamma}{2}}. \tag{25}$$

*Proof.* For $(D_n^* f)(x)$ and $f \in Lip_M(\gamma)$ , we can write

$$\begin{aligned}
\left|(D_n^* f)(x) - f(x)\right| &\leq D_n^*\left(\left|f(t) - f(x)\right|; x\right) \\
&= \frac{n+1}{b_n}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t) - f(x)\right| dt
\end{aligned}$$

$$\leq M\frac{(n+1)}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)|t-x|^{\gamma}\,dt. \tag{26}$$

By utilizing the Cauchy-Schwarz inequality for the integral, (26) leads to

$$\begin{aligned}
\left|(D_n^*f)(x)-f(x)\right| &\leq M\frac{(n+1)}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\left(\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)dt\right)^{1/2}\left(\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)|t-x|^{2\gamma}\,dt\right)^{1/2}\\
&\leq \left(M^2\frac{(n+1)}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)dt\right)^{1/2}\left(\frac{(n+1)}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\right.\\
&\quad\left.\times\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)|t-x|^{2}\,dt\right)^{\gamma/2}\\
&\leq \left(M^2(D_n^*e_0)(x)\right)^{1/2}\left((D_n^*(e_1-x)^2)(x)\right)^{\gamma/2}\\
&\leq M\left((D_n^*(e_1-x)^2)(x)\right)^{\gamma/2}.
\end{aligned}$$

Hence, the proof is completed. □

## 4. Rate of convergence in $L_1$

**Definition 4.1.** *For $\delta > 0$ and $f \in L_1(-\infty,\infty)$, the modulus of continuity $\omega_{L_1}(f;\delta)$ of the function $f$ is defined by*

$$\omega_{L_1}(f;\delta) = \sup_{|t|\leq\delta}\int_{-\infty}^{\infty}\left|f(x+t)-f(x)\right|dx. \tag{27}$$

*Then, for any $\delta > 0$ and $f \in L_1(-\infty,\infty)$*

$$\lim_{\delta\to 0}\omega_{L_1}(f;\delta) = 0. \tag{28}$$

**Definition 4.2.** *For $\delta > 0$ and $f \in L_1[0,\infty)$, the modulus of continuity $\omega_{L_1(b_n)}(f;\delta)$ of the function $f$ is defined by*

$$\omega_{L_1(b_n)}(f;\delta) = \sup_{|t-x|\leq\delta}\int_0^{b_n}\left|f(t)-f(x)\right|dx. \tag{29}$$

*Then, for any $\delta > 0$ and $f \in L_1[0,\infty)$*

$$\lim_{\delta\to 0}\omega_{L_1(b_n)}(f;\delta) = 0. \tag{30}$$

**Theorem 4.3.** *Let $f \in L_1[0,\infty)$. Then*

$$\left|(D_n^*f)(x)-f(x)\right| \leq 2\omega_{L_1(b_n)}(f;\delta) \tag{31}$$

*where $\delta = \int_0^{b_n}(t-x)^2K_n^*(x,t)dt = \sqrt{((D_n^*(e_1-x)^2)(x)}$.*

*Proof.*

$$\begin{aligned}\left|(D_n^*f)(x)-f(x)\right| &\leq \left|\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)f(t)-f(x)dt\right|\\ &\leq \frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)-f(x)\right|dt\end{aligned}$$

by the Hölder's inequality, we get

$$\left\|(D_n^*f)(x)-f(x)\right\|_{L_1[0,\infty)} \leq \left(\int_0^{b_n}\left(\int_0^{b_n}\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)-f(x)\right|\right)^2 dt\right)^{1/2} dx.$$

By applying the generalized Minkowski inequality, we have

$$\begin{aligned}\left\|(D_n^*f)(x)-f(x)\right\|_{L_1[0,\infty)} &\leq \int_0^{b_n}\left(\int_0^{b_n}\frac{n+1}{b_n}\sum_{k=0}^{n}a_{n,k}\left(\frac{x}{b_n}\right)p_{n,k}\left(\frac{t}{b_n}\right)\left|f(t)-f(x)\right|^2 dx\right)^{1/2} dt\\ &\leq \int_0^{b_n}\left(\sum_{k=0}^{n}\frac{n+1}{b_n}p_{n,k}\left(\frac{t}{b_n}\right)\int_0^{b_n}a_{n,k}\left(\frac{x}{b_n}\right)\left|f(t)-f(x)\right|^2 dx\right)^{1/2} dt\\ &\leq \int_0^{b_n}\left(\int_0^{b_n}\left|f(t)-f(x)\right|dx\right)K_n^*(x,t)dt.\end{aligned}$$

Thus, taking into account definition of the modulus of smoothness $\omega_{L_1(b_n)}(f;\delta)$

$$\begin{aligned}\left\|(D_n^*f)(x)-f(x)\right\|_{L_1[0,\infty)} &\leq \int_0^{b_n}\left(\sup_{|t-x|\leq\delta}\int_0^{b_n}\left|f(t)-f(x)\right|dx\right)K_n^*(x,t)dt\\ &\leq \int_0^{b_n}\omega_{L_1(b_n)}(f;|t-x|)K_n^*(x,t)dt.\end{aligned}$$

For positive sequence $\delta$, we have

$$\begin{aligned}\omega_{L_1(b_n)}(f;|t-x|) &= \omega_{L_1}\left(f,\frac{|t-x|\,\delta}{\delta}\right)\\ &\leq \left(\frac{|t-x|}{\delta}+1\right)\omega_{L_1(b_n)}(f;\delta).\end{aligned}$$

Thus

$$\left\|(D_n^*f)(x)-f(x)\right\|_{L_1[0,\infty)} \leq \omega_{L_1(b_n)}(f;\delta)\int_0^{b_n}K_n^*(x,t)dt$$

$$
\begin{aligned}
&\le \int_0^{b_n}\left(\frac{|t-x|}{\delta}+1\right)\omega_{L_1(b_n)}(f;\delta)K_n^*(x,t)dt \\
&\le \omega_{L_1(b_n)}(f;\delta)\int_0^{b_n}\left(\frac{|t-x|}{\delta}+1\right)K_n^*(x,t)dt \\
&\le \omega_{L_1(b_n)}(f;\delta)\left(\int_0^{b_n}\frac{|t-x|}{\delta}K_n^*(x,t)dt+\int_0^{b_n}K_n^*(x,t)dt\right) \\
&\le \omega_{L_1(b_n)}(f;\delta)\left(\int_0^{b_n}\frac{(t-x)^2}{\delta}K_n^*(x,t)dt+\int_0^{b_n}K_n^*(x,t)dt\right) \\
&\le \omega_{L_1(b_n)}(f;\delta)\left(\frac{1}{\delta}\int_0^{b_n}(t-x)^2K_n^*(x,t)dt+1\right). \qquad (32)
\end{aligned}
$$

If $\delta=\int_0^{b_n}(t-x)^2K_n^*(x,t)dt=\sqrt{(D_n^*(e_1-x)^2)(x)}$ is taken here

$$\left\|(D_n^*f)(x)-f(x)\right\|_{L_1[0,\infty)}\le 2\omega_{L_1(b_n)}(f;\delta).$$

Now, we obtain a Voronovskaja-type asymtotic estimate of the operators $D_n^*$. □

**Theorem 4.4.** *Let $f\in L_1[0,\infty)$ and $f'$, $f''$ exist at a fixed point $x\in(0,\infty)$. Then, we have*

$$\lim_{n\to\infty}\frac{n}{b_n}((D_n^*f)(x)-f(x))=xf''(x). \qquad (33)$$

*Proof.* By the Taylor expansion, we may write

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+(t-x)^2\lambda(t;x), \qquad (34)$$

where $\lambda(t;x)\in L_1[0,\infty)$ and $\lambda(t;x)\to 0$ as $t\to x$. Appyling our operator $D_n^*$ to the both side of (34) , we get

$$
\begin{aligned}
(D_n^*f)(x) &= f(x)+(D_n^*(t-x);x)f'(x)+D_n^*((t-x)^2;x)\frac{f''(x)}{2!} \\
&\quad +D_n^*((t-x)^2\lambda(t;x);x),
\end{aligned}
$$

$$
\begin{aligned}
\frac{n}{b_n}((D_n^*f)(x)-f(x)) &= \frac{n}{b_n}D_n^*((t-x);x)f'(x) \\
&\quad +\frac{n}{b_n}D_n^*((t-x)^2;x)\frac{f''(x)}{2!} \\
&\quad +\frac{n}{b_n}D_n^*((t-x)^2\lambda(t;x);x).
\end{aligned}
$$

In view of Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have

$$\frac{n}{b_n}((D_n^*f)(x)-f(x)) \le \frac{n}{b_n}f'(x)\left(\frac{b_n}{n+2}\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)+\frac{b_n}{n+2}-x\right) \qquad (35)$$

$$+\frac{n}{b_n}\frac{f''(x)}{2!}\left[\begin{array}{l}\frac{b_n^2}{(n+2)(n+3)}\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)^2\\ +\frac{4b_n^2}{(n+2)(n+3)}\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)\end{array}\right.$$
$$\left.+\frac{2b_n^2}{(n+2)(n+3)}-\frac{2xb_n}{n+2}\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)-\frac{2xb_n}{n+2}+x^2\right]$$
$$+\frac{n}{b_n}D_n^*((t-x)^2\lambda(t;x);x).$$

Now we shall show that

$$\lim_{n\to\infty}\frac{n}{b_n}D_n^*((t-x)^2\lambda(t;x);x)=0.$$

From the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{n}{b_n}\left|D_n^*((t-x)^2\lambda(t;x);x)\right| &\leq \frac{n}{b_n}D_n^*(\left|(t-x)^2\lambda(t;x)\right|;x)\\ &\leq \left(\frac{n^2}{b_n^2}D_n^*((t-x)^4;x)\right)^{1/2}(D_n^*(\lambda^2(t;x);x))^{1/2}.\end{aligned}$$

$\lim_{n\to\infty}((\frac{n}{b_n})^2D_n^*((t-x)^4;x)=0$ and since $\lambda(t;x)\in L_1[0,\infty)$ and $\lambda(t;x)\to 0$ as $t\to x$, it follows

$$\lim_{n\to\infty}D_n^*(\lambda^2(t;x);x)=\lambda^2(x;x)=0. \tag{36}$$

Thus we obtain

$$\lim_{n\to\infty}\frac{n}{b_n}D_n^*((t-x)^2\lambda(t;x);x)=0 \tag{37}$$

and then, by taking limit as $n\to\infty$ in (36) and using (37) , we completed the proof. □

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