Filomat 39:13 (2025), 4601–4612
https://doi.org/10.2298/FIL2513601B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bullen inequality for third differentiable functions

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Abstract. This paper develops a novel Bullen inequality for third-differentiable functions using Riemann integrals. Furthermore, new Bullen inequalities are proposed utilizing a summation parameter $p \geq 1$ for and s-convex functions ,convex functions and P-functions classes. Particular cases are studied when the third derivative functions are also bounded and Lipschitzian.

1. Introduction

It is important that the mathematical literature investigates numerical integration and defines error limits. Error bounds for functions with variable differentiability have been the focus of much research. The Bullen-type inequality is a helpful mathematical instrument for integral estimation. The well-known Hermite-Hadamard inequality is defined as follows [1], for a convex function:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

In [2], Bullen improved the right side of (1) by the following inequality, which is known as Bullen's inequality:

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

The estimation of Bullen-type inequalities for functions whose first derivative absolute values are convex is as follows. [3, Remark 4.2].

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]}{16}.$$

2020 *Mathematics Subject Classification*. Primary 26D10; Secondary 26A51, 26A33, 26D15.
Keywords. s-convex function, Bullen inequality, Hölder inequality, Riemann integral
Received: 28 September 2024; Revised: 04 December 2024; Accepted: 19 February 2025
Communicated by Miodrag Spalević
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The following is an estimation of Bullen-type inequality for functions whose second derivative absolute values are convex: [8, Proposition.4] and [9, Corollary 1.].

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{96}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2}$$

Bullen's inequality has been extensively studied in the literature, leading to numerous directions for extension and a rich mathematical literature (see [3]- [9]).

Convex functions have been used in a variety of mathematical areas as a result of their efforts and study, leading to the discovery of many mathematical inequalities. The author of [10] introduces a well-known class of functions called s-convex functions.

Definition 1.1. *Let $s \in (0,1]$. We say that $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a s-convex function in the second sense, if Φ is non-negative and for all $x_1, x_2 \in I$, $\tau \in (0,1)$ we have*

$$\Phi(\tau x_1 + (1-\tau)x_2) \leq \tau^s \Phi(x_1) + (1-\tau)^s \Phi(x_2). \tag{3}$$

If the inequality (3) is reversed, then Φ is said to be s-concave function in the second sense.

By setting

- $s = 1$, the concept of s-convex function reduces to convex function [11].
- $s \to 0$, the concept of s-convex function reduces reduces to P-functions [12].

In [13], Benaissa and Sarikaya established the required Lemma.

Lemma 1.2. *Let $\tau \in (0,1)$ and $s \in [0,1]$. The following inequality holds:*

$$\tau^s + (1-\tau)^s \leq 2^{1-s}, \tag{4}$$

Theorem 1.3. *(Hölder inequality). Let p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If Ψ and Φ are real functions defined on $[\lambda_1, \lambda_2]$ and if $|\Psi|^p$, $|\Phi|^q$ are integrable functions on $[\lambda_1, \lambda_2]$ then*

$$\int_{\lambda_1}^{\lambda_2} |\Psi(t)\Phi(t)|\, dt \leq \left(\int_{\lambda_1}^{\lambda_2} |\Psi(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |\Phi(t)|^q\, dt\right)^{\frac{1}{q}}.$$

The power-mean integral inequality, derived from the Hölder inequality, can be expressed as follows:

Theorem 1.4. *(Power mean integral inequality). Let $p \geq 1$ and W, Φ be two real functions defined on $[\lambda_1, \lambda_2]$. If $|W|$, $|W||\Phi|^q$ are integrable functions on $[a_1, a_2]$ then*

$$\int_{\lambda_1}^{\lambda_2} |W(t)\Phi(t)|\, dt \leq \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\, dt\right)^{1-\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\, |\Phi(t)|^p\, dt\right)^{\frac{1}{p}}.$$

For additional details and improvements of the power-mean integral inequality, consult references [15] and [14].
Depending on previous researches, we present a new form of Bullen inequality for third differentiable and s-convex functions utilizing a summation parameter $p \geq 1$ and the Riemann integral.

2. The basic identity

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable function on (a,b) such that $f''' \in L_1([a,b])$, then the following identity holds.*

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]$$
$$=\frac{(b-a)^3}{12}\left\{\int_0^{\frac{1}{2}} t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\left[f'''\left((1-t)a+tb\right)-f'''\left(ta+(1-t)b\right)\right]dt\right. \tag{5}$$
$$\left.+\int_{\frac{1}{2}}^1 (t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\left[f'''\left((1-t)a+tb\right)-f'''\left(ta+(1-t)b\right)\right]dt\right\}.$$

Proof. By applying integration by parts, we get

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{2}} t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\left[f'''\left((1-t)a+tb\right)-f'''\left(ta+(1-t)b\right)\right]dt \\
&= \left(\frac{1}{b-a}\right)t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\left[f''\left((1-t)a+tb\right)+f''\left(ta+(1-t)b\right)\right]\Big|_0^{\frac{1}{2}} \\
&\quad -\frac{1}{b-a}\int_0^{\frac{1}{2}}\left(3t^2-\frac{3}{2}t+\frac{1}{8}\right)\left[f''\left((1-t)a+tb\right)+f''\left(ta+(1-t)b\right)\right]dt \\
&= -\frac{1}{(b-a)^2}\left(3t^2-\frac{3}{2}t+\frac{1}{8}\right)\left[f'\left((1-t)a+tb\right)-f'\left(ta+(1-t)b\right)\right]\Big|_0^{\frac{1}{2}} \\
&\quad +\frac{1}{(b-a)^2}\int_0^{\frac{1}{2}}\left(6t-\frac{3}{2}\right)\left[f'\left((1-t)a+tb\right)-f'\left(ta+(1-t)b\right)\right]dt \\
&= -\frac{1}{(b-a)^2}\left(\frac{1}{8}\right)\left[f'(b)-f'(a)\right] \\
&\quad +\frac{1}{(b-a)^3}\left(6t-\frac{3}{2}\right)\left[f\left((1-t)a+tb\right)+f\left(ta+(1-t)b\right)\right]\Big|_0^{\frac{1}{2}} \\
&\quad -\frac{6}{(b-a)^3}\int_0^{\frac{1}{2}}\left[f\left((1-t)a+tb\right)+f\left(ta+(1-t)b\right)\right]dt.
\end{aligned}$$

Thus

$$\begin{aligned}
J_1 &= -\frac{1}{(b-a)^2}\left(\frac{1}{8}\right)\left[f'(b)-f'(a)\right]+\frac{1}{(b-a)^3}\left\{3f\left(\frac{a+b}{2}\right)+\frac{3}{2}\left[f(a)+f(b)\right]\right\} \\
&\quad -\frac{6}{(b-a)^3}\int_0^{\frac{1}{2}}\left[f\left((1-t)a+tb\right)+f\left(ta+(1-t)b\right)\right]dt.
\end{aligned}$$

Similarly

$$\begin{aligned}
J_2 &= \int_{\frac{1}{2}}^{1}(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\left[f'''\left((1-t)a+tb\right)-f'''\left(ta+(1-t)b\right)\right]dt \\
&= \left(\frac{1}{b-a}\right)(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\left[f''\left((1-t)a+tb\right)+f''\left(ta+(1-t)b\right)\right]\Big|_{\frac{1}{2}}^{1} \\
&\quad -\frac{1}{b-a}\int_0^{\frac{1}{2}}\left(3t^2-\frac{9}{2}t+\frac{13}{8}\right)\left[f''\left((1-t)a+tb\right)+f''\left(ta+(1-t)b\right)\right]dt \\
&= -\frac{1}{(b-a)^2}\left(3t^2-\frac{9}{2}t+\frac{13}{8}\right)\left[f'\left((1-t)a+tb\right)-f'\left(ta+(1-t)b\right)\right]\Big|_{\frac{1}{2}}^{1} \\
&\quad +\frac{1}{(b-a)^2}\int_{\frac{1}{2}}^{1}\left(6t-\frac{9}{2}\right)\left[f'\left((1-t)a+tb\right)-f'\left(ta+(1-t)b\right)\right]dt \\
&= -\frac{1}{(b-a)^2}\left(\frac{1}{8}\right)\left[f'(b)-f'(a)\right] \\
&\quad +\frac{1}{(b-a)^3}\left(6t-\frac{9}{2}\right)\left[f\left((1-t)a+tb\right)+f\left(ta+(1-t)b\right)\right]\Big|_{\frac{1}{2}}^{1} \\
&\quad -\frac{6}{(b-a)^3}\int_{\frac{1}{2}}^{1}\left[f\left((1-t)a+tb\right)+f\left(ta+(1-t)b\right)\right]dt,
\end{aligned}$$

then

$$\begin{aligned}
J_2 &= -\frac{1}{(b-a)^2}\left(\frac{1}{8}\right)\left[f'(b)-f'(a)\right]+\frac{1}{(b-a)^3}\left\{\frac{3}{2}\left[f(a)+f(b)\right]+3f\left(\frac{a+b}{2}\right)\right\} \\
&\quad -\frac{6}{(b-a)^3}\int_{\frac{1}{2}}^{1}\left[f\left((1-t)a+tb\right)+f\left(ta+(1-t)b\right)\right]dt.
\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}} f\left((1-t)a+tb\right)dt=\int_{\frac{1}{2}}^{1} f\left(ta+(1-t)b\right)dt=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(t)dt,$$

$$\int_0^{\frac{1}{2}} f\left(ta+(1-t)b\right)dt=\int_{\frac{1}{2}}^{1} f\left((1-t)a+tb\right)dt=\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f(t)dt,$$

we obtain

$$\begin{aligned}
J_1+J_2 &= \frac{1}{(b-a)^3}\left\{3\left[f(a)+f(b)\right]+6f\left(\frac{a+b}{2}\right)\right\}-\frac{12}{(b-a)^4}\int_a^b f(t)dt \\
&\quad -\frac{1}{(b-a)^2}\left(\frac{1}{4}\right)\left[f'(b)-f'(a)\right].
\end{aligned}$$

Multiplying the last equality by $\frac{(b-a)^3}{12}$ gives the equality (5). □

3. Fundamental Lemmas

These two lemmas are necessary for proving the important results. The next lemmas derive from the power mean integral inequality and the Hölder inequality.

Lemma 3.1. *Assuming $p \geq 1$ and $0 \leq \lambda_1 < \lambda_2 \leq 1$. Suppose the function f is three-time absolutely continuous such that $f''' \in L_1(a,b)$ and $v : (0,1) \to \mathbf{R}$ is an absolutely continuous function with $v \in L_p(\lambda_1, \lambda_2)$. If $\left|f'''\right|^p$ is a s-convex mapping on $[a,b]$.*

$$\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left|f'''\left((1-t)\,a + t\,b\right)\right| + \left|f'''\left(t\,a + (1-t)\,b\right)\right|\right] dt \leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|\ dt\right) 2\left(\frac{1}{2}\right)^{\frac{s}{p}} \left(\left|f'''(a)\right|^p + \left|f'''(b)\right|^p\right)^{\frac{1}{p}}. \quad (6)$$

Proof. We need the following inequality to prove the next results.
Let $A, B \geq 0$ and $\eta > 0$:

$$A^\eta + B^\eta \leq \max(1, 2^{1-\eta})(A+B)^\eta. \quad (7)$$

Let $p \geq 1$, $0 \leq \lambda_1 < \lambda_2 \leq 1$ and $v \in L_p(\lambda_1, \lambda_2)$, using power-mean integral inequality gives:

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left|f'''\left((1-t)\,a + t\,b\right)\right| + \left|f'''\left(t\,a + (1-t)\,b\right)\right|\right] dt\\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|\ dt\right)^{1-\frac{1}{p}} \left[\left(\int_{\lambda_1}^{\lambda_2} |v(t)| \left|f'''\left((1-t)\,a + t\,b\right)\right|^p\ dt\right)^{\frac{1}{p}} + \left(\int_{\lambda_1}^{\lambda_2} |v(t)| \left|f'''\left(t\,a + (1-t)\,b\right)\right|^p\ dt\right)^{\frac{1}{p}}\right].\end{aligned}$$

The inequality (7) yields $A^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, thus

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left|f'''\left((1-t)\,a + t\,b\right)\right| + \left|f'''\left(t\,a + (1-t)\,b\right)\right|\right] dt\\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|\ dt\right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[\int_{\lambda_1}^{\lambda_2} |v(t)| \left(\left|f'''\left((1-t)\,a + t\,b\right)\right|^p + \left|f'''\left(t\,a + (1-t)\,b\right)\right|^p\right) dt\right]^{\frac{1}{p}}.\end{aligned}$$

Given that $\left|f'''\right|^p$ is a s-convex function, we get

$$\left|f'''\left((1-t)\,a + t\,b\right)\right|^p \leq (1-t)^s \left|f'''(a)\right|^p + t^s \left|f'''(b)\right|^p.$$

Then, the inequality (4) provides us

$$\begin{aligned}\left|f'''\left((1-t)\,a + t\,b\right)\right|^p + \left|f'''\left(t\,a + (1-t)\,b\right)\right|^p &\leq \left[(1-t)^s + t^s\right]\left(\left|f'''(a)\right|^p + \left|f'''(b)\right|^p\right)\\ &\leq 2\left(\tfrac{1}{2}\right)^s \left(\left|f'''(a)\right|^p + \left|f'''(b)\right|^p\right).\end{aligned} \quad (8)$$

Hence

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left|f'''\left((1-t)\,a + t\,b\right)\right| + \left|f'''\left(t\,a + (1-t)\,b\right)\right|\right] dt\\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|\ dt\right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[\int_{\lambda_1}^{\lambda_2} |v(t)|\, 2\left(\frac{1}{2}\right)^s \left(\left|f'''(a)\right|^p + \left|f'''(b)\right|^p\right) dt\right]^{\frac{1}{p}}.\end{aligned}$$

For every value of $0 \leq \lambda_1 < \lambda_2 \leq 1$, we derive the next result:

$$\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left| f''' \left((1-t)\,a + t\,b \right) \right| + \left| f''' \left(ta + (1-t)\,b \right) \right| \right] dt \leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|\ dt \right) 2 \left(\frac{1}{2} \right)^{\frac{s}{p}} \left(\left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right)^{\frac{1}{p}}.$$

□

Lemma 3.2. *Assuming $p > 1$ and $0 \leq \lambda_1 < \lambda_2 \leq 1$. Suppose f is three-time absolutely continuous function such that $f''' \in L_1(a,b)$ and $v : (0,1) \to \mathbf{R}$ is an absolutely continuous function with $v \in L_p(\lambda_1, \lambda_2)$. If $\left| f''' \right|^p$ is a s-convex mapping on $[a,b]$.*

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left| f''' \left((1-t)\,a + t\,b \right) \right| + \left| f''' \left(t\,a + (1-t)\,b \right) \right| \right] dt \\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|^q\ dt \right)^{\frac{1}{q}} (\lambda_2 - \lambda_1)^{\frac{1}{p}}\ 2 \left(\frac{1}{2} \right)^{\frac{s}{p}} \left(\left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{9}$$

Proof. Let $p > 1$, $0 \leq \lambda_1 < \lambda_2 \leq 1$ and $w \in L_p(\lambda_1, \lambda_2)$, using Hölder inequality gives:

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left| f''' \left((1-t)\,a + t\,b \right) \right| + \left| f''' \left(t\,a + (1-t)\,b \right) \right| \right] dt \\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|^q\ dt \right)^{\frac{1}{q}} \left[\left(\int_{\lambda_1}^{\lambda_2} \left| f''' \left((1-t)\,a + t\,b \right) \right|^p\ dt \right)^{\frac{1}{p}} + \left(\int_{\lambda_1}^{\lambda_2} \left| f''' \left(t\,a + (1-t)\,b \right) \right|^p\ dt \right)^{\frac{1}{p}} \right]. \end{aligned}$$

Since $A^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}} (A+B)^{\frac{1}{p}}$, we get

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left| f''' \left((1-t)\,a + t\,b \right) \right| + \left| f''' \left(t\,a + (1-t)\,b \right) \right| \right] dt \\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|^q\ dt \right)^{\frac{1}{q}} 2^{1-\frac{1}{p}} \left[\int_{\lambda_1}^{\lambda_2} \left(\left| f''' \left((1-t)\,a + t\,b \right) \right|^p + \left| f''' \left(t\,a + (1-t)\,b \right) \right|^p \right) dt \right]^{\frac{1}{p}}. \end{aligned}$$

Given that $\left| f''' \right|^p$ is a s-convex function, the inequality (8) provides us

$$\left| f''' \left((1-t)\,a + t\,b \right) \right|^p + \left| f''' \left(t\,a + (1-t)\,b \right) \right|^p \leq 2 \left(\tfrac{1}{2} \right)^s \left(\left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right),$$

therefore

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left| f''' \left((1-t)\,a + t\,b \right) \right| + \left| f''' \left(t\,a + (1-t)\,b \right) \right| \right] dt \\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|^q\ dt \right)^{\frac{1}{q}} 2^{1-\frac{1}{p}} \left[\int_{\lambda_1}^{\lambda_2} 2 \left(\frac{1}{2} \right)^s \left(\left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right) dt \right]^{\frac{1}{p}}. \end{aligned}$$

For all values of $0 \leq \lambda_1 < \lambda_2 \leq 1$, we obtain the following result:

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} |v(t)| \left[\left| f''' \left((1-t)\,a + t\,b \right) \right| + \left| f''' \left(ta + (1-t)\,b \right) \right| \right] dt \\ &\leq \left(\int_{\lambda_1}^{\lambda_2} |v(t)|^q\ dt \right)^{\frac{1}{q}} (\lambda_2 - \lambda_1)^{\frac{1}{p}}\ 2 \left(\frac{1}{2} \right)^{\frac{s}{p}} \left(\left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right)^{\frac{1}{p}}. \end{aligned}$$

□

4. Bullen inequality with power mean inequality.

Now we provide the first Theorem.

Theorem 4.1. *Let $p \geq 1$, $s \in (0,1]$ and assume that f is defined as in Lemma 2.1. If $\left|f'''\right|^p$ is a s-convex mapping on $[a,b]$, then the following Bullen inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{1536}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right]^{\frac{1}{p}}, \tag{10}$$

where $s \in (0,1]$.

Proof. Using the modulus of identity (5) and applying the inequality (6), we deduce

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \\
&\leq \frac{(b-a)^3}{12}\left\{\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|\left[\left|f'''\left((1-t)a+\frac{t}{2}b\right)\right|+\left|f'''\left(\frac{t}{2}a+(1-t)b\right)\right|\right]dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|\left[\left|f'''\left((1-t)a+tb\right)\right|+\left|f'''\left(ta+(1-t)b\right)\right|\right]dt\right\} \\
&\leq \frac{(b-a)^3}{6}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left(\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right)^{\frac{1}{p}}\left\{\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|dt+\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|dt\right\}.
\end{aligned}$$

As

$$\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|dt=\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|dt=\frac{1}{512},$$

the desired inequality is achieved. □

Put $p = 1$ in the above Theorem 4.1, we get the following Corollary.

Corollary 4.2. *Assume that f is defined as in Lemma 2.1. If $\left|f'''\right|$ is a s-convex mapping on $[a,b]$, then the following Bullen inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{1536}\left(\frac{1}{2}\right)^{s}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right], \tag{11}$$

where $s \in (0,1]$.

Remark 4.3. *In the precedent Corollary 4.2, if we assume that $\left|f'''\right|$ is bounded i.e. $\sup_{x\in[a,b]}\left|f'''\right| = \left\|f'''\right\|_\infty$, then we obtain*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{768}\left(\frac{1}{2}\right)^{s}\left\|f'''\right\|_\infty. \tag{12}$$

The inequality (12) is a new one that uses s-convexity. Putting $s = 1$ yields

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{1536}\left\|f'''\right\|_\infty.$$

We present a various cases of Bullen inequality based on p and s.

4.1. Bullen inequality via convex function

If we choose $s = 1$ in the Theorem 4.1 and Corollary 4.2, we obtain the results bellow.

Corollary 4.4. *Let $p \geq 1$ and assume that f is defined as in Lemma 2.1. If $\left|f'''\right|^p$ is a convex mapping on $[a, b]$, then the following Bullen inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{1536}\left[\frac{\left|f'''(a)\right|^p+\left|f'''(b)\right|^p}{2}\right]^{\frac{1}{p}}. \tag{13}$$

For $p = 1$, we get

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{3072}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right]. \tag{14}$$

4.2. Bullen inequality via class P-functions

If we choose $s \to 0$ in the Theorem 4.1 and Corollary 4.2, we obtain the new results involving the class P-functions.

Corollary 4.5. *Let $p \geq 1$ and assume that f is defined as in Lemma 2.1. If $\left|f'''\right|^p$ is a P-functions mapping on $[a, b]$, then the following Bullen inequality holds*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{1536}\left[\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right]^{\frac{1}{p}}. \tag{15}$$

Set $p = 1$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{1536}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right]. \tag{16}$$

5. Bullen inequality via Hölder inequality

Now, we present the second Theorem.

Theorem 5.1. *Let $p > 1$ and assume that f is defined as in Lemma 2.1. If $\left|f'''\right|^p$ is a s-convex mapping on $[a, b]$, then the following Bullen inequality holds.*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right| \\ &\leq \frac{(b-a)^3}{6}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left(\frac{\left|f'''(a)\right|^p+\left|f'''(b)\right|^p}{2}\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|^q dt\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{17}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the modulus of identity (5) and applying the inequality (9), we deduce

$$\left|\frac{1}{24}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right|$$

$$\leq \frac{(b-a)^3}{12}\left\{\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|\left[\left|f'''\left((1-t)a+tb\right)\right|+\left|f'''\left(ta+(1-t)b\right)\right|\right]dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|\left[\left|f'''\left((1-t)a+tb\right)\right|+\left|f'''\left(ta+(1-t)b\right)\right|\right]dt\right\}$$

$$\leq \frac{(b-a)^3}{6}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left(\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{p}}\right\},$$

which finalizes the proof of Theorem 5.1. □

Remark 5.2. *In Theorem 5.1, setting $s=1$ and $s\to 0$ yields the Bullen inequality with convex and class P functions, respectively.*

6. Other results on the Bullen inequality.

Theorem 6.1. *Let $f:[a,b]\to\mathbb{R}$ be a three times differentiable function on (a,b) such that $f'''\in L_1([a,b])$. If there exist constants $-\infty<m<M<+\infty$ such that $m\leq f'''(x)\leq M$ for all $x\in[a,b]$, then the following inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right|\leq\frac{(M-m)(b-a)^3}{3072}.$$

Proof. Through the Lemma 2.1, we have

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]=\frac{(b-a)^3}{12}$$

$$\times\left\{\int_0^{\frac{1}{2}}t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\left[\left(f'''\left((1-t)a+tb\right)-\frac{M+m}{2}\right)-\left(f'''\left(ta+(1-t)b\right)-\frac{M+m}{2}\right)\right]dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\left[\left(f'''\left((1-t)a+tb\right)-\frac{M+m}{2}\right)-\left(f'''\left(ta+(1-t)b\right)-\frac{M+m}{2}\right)\right]dt\right\}.$$

Applying the absolute value to the previously equality, we obtain

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right|\leq\frac{(b-a)^3}{12}$$

$$\times\left\{\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|\left[\left|f'''\left((1-t)a+tb\right)-\frac{M+m}{2}\right|+\left|f'''\left(ta+(1-t)b\right)-\frac{M+m}{2}\right|\right]dt\right. \tag{18}$$

$$\left.+\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|\left[\left|f'''\left((1-t)a+tb\right)-\frac{M+m}{2}\right|+\left|f'''\left(ta+(1-t)b\right)-\frac{M+m}{2}\right|\right]dt\right\}.$$

Given that $m \leq f'''(x) \leq M$ for all $x \in [a,b]$,

$$\left| f'''\left((1-t)a+tb\right) - \frac{M+m}{2} \right| \leq \frac{M-m}{2}, \tag{19}$$

and

$$\left| f'''\left(ta+(1-t)b\right) - \frac{M+m}{2} \right| \leq \frac{M-m}{2}, \tag{20}$$

adding (19) and (20) to (18) yields

$$\left| \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt - \frac{b-a}{48}\left[f'(b)-f'(a)\right] \right|$$
$$\leq \frac{(M-m)(b-a)^3}{12}\left\{ \int_0^{\frac{1}{2}} \left| t\left(t^2 - \frac{3}{4}t + \frac{1}{8}\right)\right| dt + \int_{\frac{1}{2}}^1 \left|(t-1)\left(t^2 - \frac{5}{4}t + \frac{3}{8}\right)\right| dt \right\}$$
$$= \frac{(M-m)(b-a)^3}{3072}.$$

□

Theorem 6.2. *Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable function on (a,b) such that $f''' \in L_1([a,b])$. If f''' is an L-Lipschitzian function on $[a,b]$, then*

$$\left| \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt - \frac{b-a}{48}\left[f'(b)-f'(a)\right] \right| \leq \frac{L(b-a)^4}{3072}.$$

Proof. According to Lemma 2.1

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt - \frac{b-a}{48}\left[f'(b)-f'(a)\right]$$
$$= \frac{(b-a)^3}{12}\left\{ \int_0^{\frac{1}{2}} t\left(t^2 - \frac{3}{4}t + \frac{1}{8}\right)\left[\left(f'''\left((1-t)a+tb\right) - f'''(a)\right) - \left(f'''\left(ta+(1-t)b\right) - f'''(a)\right)\right] dt \right.$$
$$\left. + \int_{\frac{1}{2}}^1 (t-1)\left(t^2 - \frac{5}{4}t + \frac{3}{8}\right)\left[\left(f'''\left((1-t)a+tb\right) - f'''(b)\right) - \left(f'''\left(ta+(1-t)b\right) - f'''(b)\right)\right] dt \right\}.$$

Using the absolute value to the equality, we derive

$$\left| \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt - \frac{b-a}{48}\left[f'(b)-f'(a)\right] \right|$$
$$\leq \frac{(b-a)^3}{12}\left\{ \int_0^{\frac{1}{2}} \left| t\left(t^2 - \frac{3}{4}t + \frac{1}{8}\right)\right| \left[\left|f'''\left((1-t)a+tb\right) - f'''(a)\right| + \left|f'''\left(ta+(1-t)b\right) - f'''(a)\right|\right] dt \right.$$
$$\left. + \int_{\frac{1}{2}}^1 \left|(t-1)\left(t^2 - \frac{5}{4}t + \frac{3}{8}\right)\right| \left[\left|f'''\left((1-t)a+tb\right) - f'''(b)\right| + \left|f'''\left(ta+(1-t)b\right) - f'''(b)\right|\right] dt \right\}.$$

Given that f''' is a L-Lipschitzian function on $[a,b]$, so

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right|$$

$$\leq \frac{(b-a)^3}{12}L(b-a)\left\{\int_0^{\frac{1}{2}}\left|t\left(t^2-\frac{3}{4}t+\frac{1}{8}\right)\right|dt+\int_{\frac{1}{2}}^1\left|(t-1)\left(t^2-\frac{5}{4}t+\frac{3}{8}\right)\right|dt\right\}$$

$$=\frac{L\,(b-a)^4}{3072}.$$

□

7. Applications

For any positive values $\eta_1, \eta_2, a, b > 0$, we consider the following means:

- The weighted arithmetic mean:

$$W(\eta_1,\eta_2,a,b)=\frac{\eta_1 a+\eta_2 b}{\eta_1+\eta_2}.$$

- The arithmetic mean:

$$A(a,b)=\frac{a+b}{2}.$$

- The harmonic mean:

$$H(a,b)=\frac{2ab}{a+b}.$$

- The n-logarithmic mean:

$$L_n(a,b)=\left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)^{\frac{1}{n}}, \quad n\in\mathbf{R}-\{-1,0\},\ b>a.$$

- The logarithmic mean:

$$L(a,b)=\left(\frac{b-a}{\ln b-\ln a}\right), \quad n\in\mathbf{R}-\{-1,0\},\ b>a.$$

In [16], the following example is given: Let $s\in(0,1)$ and $d,k,c\in\mathbf{R}$. We define a function $\Phi:[0,+\infty)\to R$, as

$$\Phi(t)=\begin{cases} d & ,t=0\\ k\,t^s+c & ,t>0.\end{cases}$$

If $k\geq 0$ and $0\leq c\leq d$, then Φ is a s-convex function.

Example 7.1. *Let $t > 0$, $p \geq 1$, $0 < s < 1$ and consider the function $f(t) = t^{\left(\frac{s}{p}+3\right)}$, then*

$$f'''(t) = \left(\frac{s}{p}+3\right)\left(\frac{s}{p}+2\right)\left(\frac{s}{p}+1\right) t^{\frac{s}{p}}.$$

In reference to

$$\Phi(t) := \left|f'''(t)\right|^p = \left[\left(\frac{s}{p}+3\right)\left(\frac{s}{p}+2\right)\left(\frac{s}{p}+1\right)\right]^p t^s,$$

for $d = c = 0$, $k = \left[\left(\frac{s}{p}+3\right)\left(\frac{s}{p}+2\right)\left(\frac{s}{p}+1\right)\right]^p$, the function $\left|f'''\right|^p$ is a s-convex.

The following results are obtained by applying the preceding example to inequality (10).

Proposition 7.2. *Let $b > a > 0$, $p \geq 1$, $0 < s < 1$ and $n = \frac{s}{p} + 3$. Then the following inequality holds:*

$$\left|\frac{1}{2}A(a^n, b^n) + \frac{1}{2}A(a,b) - L_n^n(a,b) - \frac{n(n-1)(b-a)^2}{48} L_{n-2}^{n-2}(a,b)\right|$$

$$\leq \frac{(b-a)^3\, n(n-1)(n-2)}{1536} \left(\frac{1}{2}\right)^{\frac{s-1}{p}} A^{\frac{1}{p}}\left(a^s, b^s\right).$$

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