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Compactness of the generalized Volterra type integral operator between the Bergman spaces with logarithmic weights

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Abstract. By using the Carleson measures, we characterize the compactness of the generalized Volterra type integral operator between the Bergman spaces with logarithmic weights

1. Introduction

Let $0 < p < \infty$, $-1 < \gamma < \infty$ and $\delta \leq 0$. We define the weighted Bergman space with logarithmic weight by $A^p_{\omega_{\gamma,\delta}}$, consisting of analytic functions f on the unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ for which

$$\|f\|^p_{\omega_{\gamma,\delta}} = \int_{\mathbb{D}} |f(z)|^p \omega_{\gamma,\delta}(z) dm(z) < \infty,$$

where the weight $\omega_{\gamma,\delta}$ is defined by

$$\omega_{\gamma,\delta}(z) = \Big(\log \frac{1}{|z|}\Big)^{\gamma} \Big[\log\Big(1 - \log \frac{1}{|z|}\Big)\Big]^{\delta}$$

and dm is the Lebesgue measure on $\mathbb{D}$ normalized to be $m(\mathbb{D}) = 1$. When $\delta = 0$, this space will be the weighted Bergman space A^p_γ and for $\gamma = 0$ and $\delta = 0$, it is the Bergman space A^p. We refer the interested reader to [19] for the details on the Bergman spaces.

Definition 1.1. *(See, for example [7]) A continuous function ω on $\mathbb{D}$ is called normal weight if*

(i) ω is a radial weight, that is $\omega(z) = \omega(|z|)$ for every z;
(ii) there exist $t > s > 0$ such that

$$\frac{\omega(r)}{(1-r)^s} \searrow 0, \quad \frac{\omega(r)}{(1-r)^t} \nearrow \infty,$$

as $r \to 1^-$.

We say ω is admissible weight if it non-increasing and $\omega(r)(1-r)^{1+\alpha}$ is non-decreasing for some $\alpha > 0$.

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The notation $U(z) \lesssim V(z)$ (or respectively $U(z) \gtrsim V(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ (or respectively $CU(z) \geq V(z)$) holds for all z in the set in question. We write $U(z) \approx V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \gtrsim U(z)$ hold.

Let $0 < p < \infty$ and ω be a normal weight function on $\mathbb{D}$. Then the space $\mathcal{A}(p, \omega)$ is defined as follows:

$$\mathcal{A}(p,\omega) = \Big\{ f \in \mathcal{H}(\mathbb{D}) : \quad \|f\|^p_{\mathcal{A}(p,\omega)} = \int_{\mathbb{D}} |f(z)|^p \frac{\omega^p|z|}{1-|z|} dA(z) < \infty \Big\},$$

where $dA(z)$ is the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. For $1 \leq p < \infty$, $\mathcal{A}(p,\omega)$ is a Banach space equipped with the norm $\|.\|_{\mathcal{A}(p,\omega)}$. When $0 < p < 1$, $\|.\|_{\mathcal{A}(p,\omega)}$ is a quasinorm on $\mathcal{A}(p,\omega)$ and $\mathcal{A}(p,\omega)$ is a Frechet space, but not a Banach space. Moreover, the following asymptotic relation holds

$$\|f\|_{\mathcal{A}(p,\omega)} \approx \sum_{j=0}^{n-1} |f^{(j)}(0)| + \Big(\int_{\mathbb{D}} \Big| f^{(n)}(z) \Big|^p (1-|z|^2)^{pn} \frac{\omega^p|z|}{1-|z|} dA(z) \Big)^{\frac{1}{p}}. \tag{1}$$

This relation is well known and can be found for standard power weights in [6].

For $r \in (0,1)$ and $a \in \mathbb{D}$ the pseudohyperbolic metric ρ on $\mathbb{D}$ is defined as $\rho(z,a) = |\phi_a(z)|$, where $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$. Moreover, the pseudohyperbolic disc is defined as

$$E(a,r) = \{z \in \mathbb{D} : \rho(z,a) < r\}.$$

For every $z \in \mathbb{D}$ we have

$$m(E(a,r)) \approx (1-|a|^2)^2 \approx (1-|z|^2)^2 \approx |1-\bar{a}z|^2 \approx m(E(z,r)).$$

Carleson measures were first introduced by Carleson [3], who studied positive Borel measures μ on the unit disk that satisfy for any function f in the Hardy space $H^p(\mathbb{D})$ the condition

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \int_0^{2\pi} |f(e^{i\theta})|^p dt,$$

where C is positive constant. Following similar notation, we define (vanishing) $\omega_{\gamma,\delta}$-Carleson measures on the weighted Hilbert spaces.

Definition 1.2. *Let μ be a positive Borel measure. We say μ is a $\omega_{\gamma,\delta}$-Carleson measure if there exists a constant $C > 0$ such that for all $f \in A^p_{\omega_{\gamma,\delta}}$,*

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C\|f\|^2_{A^p_{\omega_{\gamma,\delta}}}.$$

Moreover, we say μ is a vanishing $\omega_{\gamma,\delta}$-Carleson measure if

$$\lim_{k\to\infty} \int_{\mathbb{D}} |f_k(z)|^2 d\mu(z) = 0,$$

for any bounded sequence $\{f_k\} \in A^p_{\omega_{\gamma,\delta}}$ that converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$.

Definition 1.3. *Let $f, g \in H(\mathbb{D})$. If $g(z) = O(f(z))$, $|z| \to 1$ and $f(z) = O(g(z))$, $|z| \to 1$ simultaneous, then we denote this concept by $O(f(z)) = O(g(z))$, $|z| \to 1$. Namely, there exists $r_0 \in [0,1)$ such that $g(z) \approx f(z)$ for $r_0 \leq |z| < 1$.*

Let X and Y be Banach spaces of analytic functions on a domain Ω in $\mathbb{C}$, u an analytic function on Ω and φ be an analytic function mapping Ω into itself. The weighted composition operator with symbols u and φ from X to Y is the operator uC_φ with range in Y defined by

$$uC_\varphi f = M_u C_\varphi f = u(f \circ \varphi), \quad f \in X,$$

where M_u is the multiplication operator with symbol u and C_φ is the composition operator with symbol φ. We refer the interested reader to [5] and [15] for the theory of composition operators.
There exists some generalizations of the above operator as an integral type operator, by many researchers, for example see [1, 9–12, 16–18].

Let X and Y be two Banach spaces. The essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : \text{K is compact}\}.$$

The operator is compact if and only if $\|T\|_{e,X\to Y} = 0$.
The essential norm of the composition operator on $A^2_\alpha(\mathbb{D})$ in terms of the generalized Nevanlinna counting function was studied by Shapiro in [14]. Also, Kwon and Lee in [8] have studied the similar argument for the composition operators on Bergman spaces of logarithmic weights in terms of the modified Nevanlinna counting function. Pérez-González, Rättyä and Vukotić, in [13] gave several quantities for the essential norm of the composition operators acting between Hardy and weighted Bergman spaces.

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. The generalized Volterra type integral operator induced by the function $g \in \mathcal{H}(\mathbb{D})$ and the self-map φ of $\mathbb{D}$, is defined as follows:

$$J_g^\varphi : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D}), \quad f(z) \to \int_0^z f(\varphi(\xi))g'(\xi)d\xi, \qquad z \in \mathbb{D}.$$

Voltra-type operators on Zygmond spaces are investigated by Li and Stevic in [9]. In this article, we characterize compactness of the above generalized Volterra type integral operator between the Bergman spaces with logarithmic weights, by using the Carleson measures.

2. Preliminaries

Now, we quote several lemmas which will be used in the proofs of the main results in this paper.

Lemma 2.1. [4, Lemma 3.1]

$$\log\Big(1 - \frac{1}{\log x}\Big) \approx \log\frac{1}{1-x} \quad 1/2 \le x < 1. \tag{2}$$

Lemma 2.2. [5, Lemma 3.2] *For a fixed $r_0 \in [0, 1)$,*

$$\|f\|^p_{A^p_{\omega_{\gamma,\delta}}} \approx \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \omega_{\gamma,\delta}(z)dA(z). \tag{3}$$

Lemma 2.3. [5, Lemma 3.3] *Let $0 < p < \infty$, $-1 < \gamma < \infty$ and $\delta \le 0$. If $f \in A^p_{\omega_{\gamma,\delta}}$, then*

$$|f(z)| \le \Big[\Big(\log\frac{1}{|z|}\Big)^2 \omega_{\gamma,\delta}(z)\Big]^{\frac{-1}{p}} \|f\|_{\omega_{\gamma,\delta}}, \tag{4}$$

for $z \in \mathbb{D}$ with $|z| \ge \frac{1}{2}$.

In order to prove the Lemma 2.6, we need the following well-known estimate (see [19]).

Lemma 2.4. *If $d > 0$ and $c > -1$, then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^c dm(z)}{|1-\overline{\lambda}z|^{2+c+d}} \approx \frac{1}{(1-|z|^2)^d}.$$

Lemma 2.5. *For $-1 \le \gamma < \infty$ and $\delta \le 0$, the weight $\omega_{\gamma,\delta}$ is an admissible weight.*

Proof. The proof can be done by straight calculations, using the Lemma 2.1. □

Lemma 2.6. *Let $0 < p < \infty$, $-1 < \gamma < \infty$ and $\delta \leq 0$. For an analytic function $f \in H(\mathbb{D})$,*

$$\|f\|^p_{A^p_{\omega_{\gamma,\delta}}} \approx \int_{\mathbb{D}} |f'(z)|^p \omega_{(\gamma+1)/p+p-1,\delta/p}(z)dm(z). \tag{5}$$

Proof. We put

$$\theta_{\gamma,\delta}(z) = \big(1-|z|\big)^{\gamma} \log\Big(\frac{1}{1-|z|}\Big)^{\delta}. \tag{6}$$

Since we know that $1-|\lambda|$ and $\log\frac{1}{|\lambda|}$ are comparable for $\frac{1}{2} \leq |z| < 1$, so,

$$\theta_{\gamma,\delta}(z) \approx \omega_{\gamma,\delta}(z), \qquad 1/2 \leq |z| < 1. \tag{7}$$

For a fixed $r_0 \in [\frac{1}{2}, 1)$ we can see

$$\int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \theta_{\gamma,\delta}(z)dA(z) \leq \int_{\mathbb{D}} |f(z)|^p \theta_{\gamma,\delta}(z)dA(z) \tag{8}$$

and from (7),

$$\int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \omega_{\gamma,\delta}(z)dA(z) \approx \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \theta_{\gamma,\delta}(z)dA(z). \tag{9}$$

Now, by Lemma 2.2 and (9), we get

$$\begin{aligned}\|f\|^p_{A^p_{\omega_{\gamma,\delta}}} &\approx \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \omega_{\gamma,\delta}(z)dA(z)\\ &\approx \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \theta_{\gamma,\delta}(z)dA(z)\\ &\leq \int_{\mathbb{D}} |f(z)|^p \theta_{\gamma,\delta}(z)dA(z).\end{aligned} \tag{10}$$

On the other hand we have,

$$\|f\|^p_{\mathcal{A}_{(p,\omega)}} = \int_{\mathbb{D}} |f(z)|^p \frac{\omega^p(|z|)}{1-|z|} dA(z) \tag{11}$$

for all $f \in \mathcal{A}_{(p,\omega)}$. If we put $\omega(z) = (\theta_{\gamma+1,\delta(z)})^{\frac{1}{p}}$, by the relations (10) and (11) we get,

$$\begin{aligned}\|f\|^p_{\mathcal{A}_{(p,\omega)}} &= \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \frac{(1-|z|)^{\gamma+1}(\log\frac{1}{1-|z|})^{\delta}}{1-|z|} dA(z)\\ &= \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^p \theta_{\gamma,\delta}(z)dA(z)\\ &\approx \|f\|^p_{A^p_{\omega_{\gamma,\delta}}}.\end{aligned} \tag{12}$$

If we put $n = 1$ in the relation (1) and since $f(0) = 0$,

$$\begin{aligned}\|f\|^p_{A^p_{\omega_{\gamma,\delta}}} &\approx \Big(\int_{\mathbb{D}\setminus r_0\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \frac{\omega|z|)}{1-|z|} dA(z)\Big)\\ &\approx \int_{\mathbb{D}} |f'(z)|^p \omega_{(\gamma+1)/p+p-1,\delta/p}(z)dm(z).\end{aligned} \tag{13}$$

The theorem is proved. □

Lemma 2.7. *If g is non-negative measurable function on $\mathbb{D}$, then*

$$\int_{\mathbb{D}}(g\circ\varphi)(z)|\varphi'(z)|^2\omega_{\gamma,\delta}(z)dA(z)=\int_{\mathbb{D}}g(u)N_{\varphi,\gamma,\delta}(u)dA(u).$$

3. Compactness of the generalized Volterra type integral operators

In this section, we characterize compactness of the generalized Volterra type integral operator between the Bergman spaces with logarithmic weights by using the $\omega_{\beta,\sigma}$-Carleson measures, where $\beta=\frac{\gamma+1}{p}+p-1$ and $\sigma=\delta/p$. Let λ be sufficiently close to the boundary of $\mathbb{D}$. So, $|\lambda|>\frac{1}{2}$. We consider the test function

$$k_\lambda(z):=\frac{(1-|\lambda|^2)^{\frac{\alpha}{p}}}{\sqrt[p]{\omega_{\beta,\sigma}(\lambda)}(1-\overline{\lambda}z)^{\frac{\alpha+2}{p}}}. \tag{14}$$

By Lemma 2.4 we get $k_\lambda(z)\in A^p_{\omega_{\beta,\sigma}}$. We have

$$\begin{aligned}\frac{\mu(E(\lambda,r))}{(\log\frac{1}{|\lambda|})^2\omega_{\beta,\sigma}(\lambda)}&\approx\frac{\mu(E(\lambda,r))}{\omega_{\beta,\sigma}(\lambda)\left(1-|\lambda|^2\right)^2}\\&=\int_{E(\lambda,r)}\frac{(1-|\lambda|^2)^\alpha}{\omega_{\beta,\sigma}(\lambda)(1-|\lambda|^2)^{\alpha+2}}d\mu(z)\\&\le\int_{E(\lambda,r)}\frac{(1-|\lambda|^2)^\alpha}{\omega_{\beta,\sigma(\lambda)}|1-\overline{\lambda}z|^{\alpha+2}}d\mu(z)\\&\le\int_{\mathbb{D}}\frac{(1-|\lambda|^2)^\alpha}{\omega_{\beta,\sigma(\lambda)}|1-\overline{\lambda}z|^{\alpha+2}}d\mu(z)\\&=\int_{\mathbb{D}}|k_\lambda(z)|^pd\mu(z).\end{aligned} \tag{15}$$

For convenience, we will use the notation

$$\tilde{\mu}_{\omega,r}(\lambda)=\frac{\mu(E(\lambda,r))}{\left(\log\frac{1}{|\lambda|}\right)^2\omega_{\beta,\sigma}(\lambda)}.$$

The following theorem characterizes the vanishing $\omega_{\beta,\sigma}$-Carleson measure on Bergman spaces with logarithmic weights. The proof of this theorem is similar to [2], Theorem 6, and so we have eliminated its proof.

Theorem 3.1. *Let $r\in(0,1)$ and μ be a positive Borel measure on $\mathbb{D}$. Then the followings are equivalent.*

1. The measure μ is a vanishing $\omega_{\beta,\sigma}$-Carleson measure.

2. For any $a\in\mathbb{D}$,

$$\lim_{|\lambda|\to1}\int_{\mathbb{D}}|k_\lambda(z)|^pd\mu(z)=0.$$

3. For any $a\in\mathbb{D}$,

$$\lim_{|\lambda|\to1}\tilde{\mu}_{\omega,r}(\lambda)=0.$$

We will use the modified Nevanlinna counting function defined in [8] as follows:
For an analytic self map φ on $\mathbb{D}$, $0 \le r < 1, 0 \le \gamma < \infty, \delta \le 0$ and $a \in \mathbb{D} \setminus \{\varphi(0)\}$,

$$N_{\varphi,\gamma,\delta}(r,a) = \sum_{z_j(a)\in\varphi^{-1}(a)} \omega_{\gamma,\delta}\Big(\frac{z_j(a)}{r}\Big)$$

with $|z_j(a)| < r$, counting multiplicities, and

$$N_{\varphi,\gamma,\delta}(a) = N_{\varphi,\gamma,\delta}(1,a) = \sum_{z_j(a)\in\varphi^{-1}(a)} \omega_{\gamma,\delta}(z_j(a)).$$

$N_{\varphi,\gamma,\delta}(r,a) = 0$ if a is not in $\varphi(r\mathbb{D})$ where $r\mathbb{D} = \{z \in r\mathbb{D} : |z| < r\}$. When $\delta = 0$ we denote, as introduced by Shapiro ([14]),

$$N_{\varphi,\gamma}(r,a) = \sum_{z\in\varphi^{-1}(a),|z|<r} \Big(\log\frac{r}{|z|}\Big)^{\gamma}$$

and

$$N_{\varphi,\gamma}(a) = N_{\varphi,\gamma}(1,a) = \sum_{z\in\varphi^{-1}(a)} \Big(\log\frac{r}{|z|}\Big)^{\gamma}.$$

Let $d\mu_N(z) = N_{\varphi,\beta,\sigma}(z)dm(z)$, where $\beta = \frac{\gamma+1}{p} + p - 1$ and $\sigma = \frac{\delta}{p}$.

Theorem 3.2. *Let $0 \le p < \infty$, $-1 < \beta < \infty$ and $\sigma \le 0$. Also, Let $g \in \mathcal{H}(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ such that $O(|\varphi'(z)|^2) = O(|g'(z)|^p), |z| \to 1^-$. Then the followings are equivalent:*

1. *The operator $J_g^{\varphi} f : A^p_{\omega_{\beta,\sigma}} \to A^p_{\omega_{\beta,\sigma}}$ is compact.*
2. *The measure μ_N is vanishing $\omega_{\omega_{\beta,\sigma}}$-Carleson measure.*
3. *For any $\lambda \in \mathbb{D}$, $\quad \limsup_{|\lambda|\to 1} \|J_g^{\varphi} k_\lambda\|^p_{A^p_{\omega_{\beta,\sigma}}} = 0$.*

Proof. We show (2) implies (1). Suppose that μ_N is vanishing $\omega_{\beta,\sigma}$-Carleson measure. Then, by Theorem 3.1 , we have

$$\lim_{|\lambda|\to 1} \frac{\mu_N(E(\lambda,r))}{\Big(\log\frac{1}{|\lambda|}\Big)^2 \omega_{\beta,\sigma}(\lambda)} = 0. \tag{16}$$

Let $\{f_k\}$ be a bounded sequence in $A^p_{\omega_{\beta,\sigma}}$ which convergence to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Then, there exists a constant $M > 0$ such hat $\|f_k\|^p_{\omega_{\beta,\sigma}} \le M$. Now by lemma 2.3, there exists a constant C such that for any $z \in \mathbb{D}$,

$$|f_k(z)|^p \le \frac{C}{\Big(\log\frac{1}{|z|}\Big)^2 \omega_{\beta,\sigma}(z)} \int_{E(z,r)} |f_k(\lambda)|^p \omega_{\beta,\sigma}(\lambda) dm(\lambda). \tag{17}$$

By using Lemma 2.6, Lemma 2.7, Fubini's Theorem and $O(|\varphi'(z)|^2) = O(|g'(z)|^p), |z| \to 1^-$, there is $r_0 \in (0,1)$

such that,

$$\begin{aligned}
\|J_g^{\varphi} f_k\|^p_{A^p_{\omega_{\beta,\sigma}}} &= \int_{\mathbb{D}} |(J_g^{\varphi} f_k)'|^p \omega_{\beta,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |f_k \circ \varphi(z)|^p |g'(z)|^p \omega_{\beta,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |f_k \circ \varphi(z)|^p |\varphi'(z)|^2 \omega_{\beta,\sigma}(z) dm(z) \qquad \underbrace{O(|\varphi'(z)|^2) = O(|g'(z)|^p)} \\
&= \int_{\mathbb{D}} |f_k(z)|^p N_{\varphi,\beta,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |f_k(z)|^p d\mu_N(z) \\
&\leq C \int_{\mathbb{D}} \frac{1}{\left(\log \frac{1}{|z|}\right)^2 \omega_{\beta,\sigma}(z)} \int_{E(z,r)} |f_k(\lambda)|^p \omega_{\beta,\sigma}(\lambda) dm(\lambda) d\mu_N(z) \\
&\leq C \int_{\mathbb{D}} |f_k(\lambda)|^p \omega_{\beta,\sigma}(\lambda) \Big(\int_{E(z,r)} \frac{1}{\left(\log \frac{1}{|z|}\right)^2 \omega_{\beta,\sigma}(z)} d\mu_N(z) \Big) dm(\lambda).
\end{aligned} \tag{18}$$

Since $\chi_{E(\lambda,r)} = \chi_{E(z,r)}$, for all $z \in E(\lambda, r)$,we get

$$\|J_g^{\varphi} f_k\|^p_{A^p_{\omega_{\beta,\sigma}}} \leq C \int_{\mathbb{D}} |f_k(\lambda)|^p \omega_{\beta,\sigma}(\lambda) \frac{\mu_N(E(\lambda, r))}{\left(\log \frac{1}{|\lambda|}\right)^2 \omega_{\beta,\sigma}(\lambda)} dm(\lambda). \tag{19}$$

Equation (16) implies that, for a given $\epsilon > 0$ there exists $r \in (0, 1)$ such that

$$\begin{aligned}
&\int_{|\lambda|>r} |f_k(\lambda)|^p \omega_{\beta,\sigma}(\lambda) \frac{\mu_N(E(\lambda, r))}{\left(\log \frac{1}{|\lambda|}\right)^2 \omega_{\beta,\sigma}(\lambda)} dm(\lambda) \\
&\leq \epsilon \int_{\mathbb{D}} |f_k(\lambda)|^p \omega_{\beta,\sigma}(\lambda) dm(\lambda) \\
&= \epsilon \|f_k\|^p_{A^p_{\omega_{\beta,\sigma}}} \\
&\leq \epsilon M^2.
\end{aligned} \tag{20}$$

On the other hand, since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, for some constant $C_1 > 0$ we obtain

$$\begin{aligned}
&\int_{|\lambda|\leq r} |f_k(\lambda)|^p \omega_{\beta,\sigma}(\lambda) \frac{\mu_N(E(\lambda, r))}{\left(\log \frac{1}{|\lambda|}\right)^2 \omega_{\beta,\sigma}(\lambda)} dm(\lambda) \\
&\leq \frac{\epsilon}{(1-r)^2} \int_{\mathbb{D}} \mu_N(E(\lambda, r)) dm(\lambda) \\
&\leq \frac{\epsilon}{(1-r)^2} \mu_N(E(\lambda, r)) \\
&\leq \epsilon C_1.
\end{aligned} \tag{21}$$

So, by using the inequalities (19), (20) and (21), we have

$$\|J_g^{\varphi} f_k\|^p_{A^p_{\omega_{\beta,\sigma}}} \leq C(\epsilon M^2 + \epsilon C_1).$$

Since, ϵ is arbitrary, so, $\lim_{k\to\infty} \|J_g^{\varphi} f_k\|^p_{A^p_{\omega_{\beta,\sigma}}} = 0$. Therefore, J_g^{φ} is compact operator.

Now, we show (2) is equivalent to (3). For any $\lambda \in \mathbb{D}$, we have

$$\begin{aligned}
\|J_g^{\varphi} k_\lambda\|^p_{A^p_{\omega_{\gamma,\delta}}} &= \int_{\mathbb{D}} |(J_g^{\varphi} k_\lambda)'|^p \omega_{\beta,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |k_\lambda \circ \varphi(z)|^p |g'(z)|^p \omega_{\beta,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |k_\lambda \circ \varphi(z)|^p |\varphi'(z)|^2 \omega_{\beta,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |k_\lambda(z)|^p N_{\varphi,\beta,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |k_\lambda(z)|^p d\mu_N(z).
\end{aligned} \tag{22}$$

By Theorem 3.1, we get that (2) is equivalent to (3).

Finally, we show (1) implies (3). Since k_λ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|\lambda| \to 1$, so, for a fixed compact operator I on $A^p_{\omega_{\beta,\sigma}}$ we have $\|Ik_\lambda\|_{A^p_{\omega_{\beta,\sigma}}} \to 0, |\lambda| \to 1$. Hence, there exists a constant $C > 0$ such that

$$\begin{aligned}
C\|J_g^{\varphi} - I\|_{A^p_{\omega_{\beta,\sigma}}} &\geq \limsup_{|\lambda|\to 1} \|(J_g^{\varphi} - I)k_\lambda\|_{A^p_{\omega_{\beta,\sigma}}} \\
&\geq \limsup_{|\lambda|\to 1} \|J_g^{\varphi} k_\lambda\|_{A^p_{\omega_{\beta,\sigma}}} - \|Ik_\lambda\|_{A^p_{\omega_{\beta,\sigma}}} \\
&= \limsup_{|\lambda|\to 1} \|J_g^{\varphi} k_\lambda\|_{A^p_{\omega_{\beta,\sigma}}}.
\end{aligned}$$

Taking infimum over all compact operators I, we get

$$C\|J_g^{\varphi}\|_{e,A^p_{\omega_{\beta,\sigma}}} \geq \limsup_{|\lambda|\to 1} \|J_g^{\varphi} k_\lambda\|_{A^p_{\omega_{\beta,\sigma}}}. \tag{23}$$

Now, if we suppose that J_g^{φ} is compact on $A^p_{\omega_{\beta,\sigma}}$, Then $\|J_g^{\varphi}\|_{e,A^p_{\omega_{\beta,\sigma}}} = 0$. Hence, using relation (23), we get the condition (3). The theorem is proved. □

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References

[1] F. Alighadr Ardebili, H. Vaezi, M. Hassanlou, *Generalized integration operator between the Bloch-type space and weighted Dirichlet-type spaces*, Kyungpook Math. J. **60** (2020), 519-534.

[2] W. Al-Rawashdeh, *Generalized Composition Operators on Weighted Hilbert Spaces of Analytic Functions*, International Journal of Advanced Research in Mathematics **10** (2017), 1-13.

[3] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. Math. **76** (1962), 547-559.

[4] F. Colonna, S. Li, *Weighted composition operators from the Lipschitz space into the Zygmund space*, Math. Inequal. Appl. **17**(3) (2014), 963-975.

[5] C. C. Cowen, B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Stud. Adv. Math. CRC Press, Boca Raton, Fla, USA, 1995.

[6] T. M. Flett, *The daul of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. **38** (1972), 746-765.

[7] Z. J. Hu, *Extended cesaro operators on mixed normed spaces*, Proc. Amer. Math. Soc. **131** (2003), 2171-2179.

[8] E. G. Kwon, J. Lee, *Composition operator between Bergman spaces of logarithmic weights*, Internat. J. Math. **26**(9) (2015), Article 1550068.

[9] S. Li, S. Stevic, *Volterra-type operators on Zygmund spaces*, J. Inequal. Appl. **2007** (2007), Article ID 32124.

[10] S. Li, S. Stevic, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl. **338** (2008), 1282-1295.
[11] S. Li, S. Stevic, *Products of integral-type operators and composition operators between Bloch-type spaces*, J. Math. Anal. Appl. **349** (2009), 596-610.
[12] A. Manavia, M. Hassanlou, H. Vaezi, *Essential norm of generalized integral type operator from $QK(p,q)$ to Zygmund Spaces*, Filomat **37**(16) (2023), 5273-5282.
[13] F. Pérez-González, J. Rättyä, D. Vukotić, *On composition operators acting between Hardy and weighted Bergman spaces*, Expo. Math. **25**(4) (2007), 309-323.
[14] J. Shapiro, *The essential norm of a compositin operator*, Ann. Math. **127**(2) (1987), 375-404.
[15] J. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.
[16] S. Stevic, *Generalized composition operators between mixed norm space and some weighted spaces*, Numer. Funct. Anal. Optim. **29** (2008), 959-978.
[17] S. Stevic, *Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces*, Util. Math. **77** (2008), 167-172.
[18] S. Stevic, *Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces*, Sib. Math. J. **50** (2009), 726-736.
[19] K. Zhu, *Operator Theory in Function Spaces*, Pure and Applied Mathematics, Marcel Dekker, New York, 1990.