Filomat 39:13 (2025), 4285–4298
https://doi.org/10.2298/FIL2513285F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Abelian- and Tauberian-type results for the directional short-time fractional Fourier transform with fixed direction

Astrit Ferizi[a,*], Snježana Maksimović[b], Katerina Hadzi-Velkova Saneva[c]

[a]*Faculty of Natural Sciences and Mathematics, Ss. Cyril and Methodius University, Skopje, Republic of North Macedonia*
[b]*Faculty of Architecture, Civil Engineering and Geodesy, University of Banja Luka, Banja Luka, Bosnia and Herzegovina*
[c]*Faculty of Electrical Engineering and Information Technologies, Ss. Cyril and Methodius University, Skopje, Republic of North Macedonia*

Abstract. We introduce the short-time fractional Fourier transform (STFRFT) in the direction of $\mathbf{u}$. Then, following the duality approach, we develop a distributional framework for the STFRFT in the direction of $\mathbf{u}$ on the space of Schwartz tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. We provide several Abelian- and Tauberian-type results that characterize the quasiasymptotic behavior of tempered distributions at the origin in terms of the asymptotic behavior of their STFRFT in the direction of $\mathbf{u}$.

1. Introduction

The fractional Fourier transform (FRFT), a generalization of the Fourier transform (FT), was popularized in the ’80s by Namias, who used this new transform to solve certain classes of ordinary and partial differential equations [19]. McBride and Kerr refined Namias’s definition in [18], and showed that the FRFT is a homeomorphism of the Schwartz space $\mathcal{S}$ of test functions onto itself. Their theory was illustrated by solving a second-order ordinary differential equation, which the classical FT could not solve. A year later, Kerr, using the duality approach, extended the FRFT theory to the space $\mathcal{S}'$ of tempered distributions [13]. Zayed in [33] extended the FRFT to a space of generalized functions by using two different techniques, one analytic and the other algebraic. The FRFT was also applied for studying swept-frequency filters, which are used in frequency analyzers for high-frequency signals [1], in optics and signal processing [20], and in solving a generalized heat equation [21]. Toft et al. in [29] connected the FRFT with harmonic oscillator propagators on Pilipović spaces and modulation spaces. The n-dimensional FRFT, defined as n copies of the one-dimensional FRFT, was investigated by different authors [12, 20]. Naturally, several other fractional transforms were introduced and investigated, such as the short-time fractional Fourier transform (STFRFT), the fractional wavelet transform, and the fractional Stockwell transform.

The STFRFT was proposed in [28] with the purpose of locating the fractional Fourier domain of a signal, which is required in some applications. This paper also includes discussions on the estimations of the

2020 *Mathematics Subject Classification*. Primary 42C20, 41A27, 46F12; Secondary 40E05, 44A15, 46F10.
Keywords. Directional short-time fractional Fourier transform, Abelian and Tauberian theorems, quasiasymptotics of distributions, desingularization formula.
Received: 01 November 2024; Revised: 15 January 2025; Accepted: 13 March 2025
Communicated by Dragan S. Djordjević
* Corresponding author: Astrit Ferizi
Email addresses: ferizi.astrit@gmail.com (Astrit Ferizi), snjezana.maksimovic@aggf.unibl.org (Snježana Maksimović), saneva@feit.ukim.edu.mk (Katerina Hadzi-Velkova Saneva)
ORCID iDs: https://orcid.org/0009-0005-3649-0704 (Astrit Ferizi), https://orcid.org/0009-0002-0079-670X (Snježana Maksimović), https://orcid.org/0000-0001-8131-7920 (Katerina Hadzi-Velkova Saneva)

time-of-arrival and the pulsewidth, defined as the interval between the half points on the rising and falling edge of chirp signals, as well as on the STFRFD filtering. In a very natural way, Gao and Li introduced the multi-dimensional STFRFT as a tensor product of n-copies of the one-dimensional STFRFT [8]. Recently, the STFRFT was extended and studied in the space of tempered distributions in [2]. A different transform, called the novel short-time fractional Fourier transform, and its discrete form were presented in [27]. The authors of [27] also discussed the application of the novel STFRFT in autofocusing synthetic aperture radar and high resolution spectrograms as well as a generalization of the Stockwell transform.

Grafakos and Sansing in [10], and later Giv in [9], introduced a directional sensitive variant of the short-time Fourier transform. The authors of [23] developed the distributional framework for the directional short-time Fourier transform (DSTFT) defined by Giv. The extension of the DSTFT to the space $\mathcal{K}_1'$ of distributions of exponential type was discussed in [3]. Motivated by Giv's idea, the directional short-time fractional Fourier transform (DSTFRFT) and its synthesis operator were introduced in [7]. The authors also proved the Parseval identity and the reconstruction formula and extended the DSTRFRFT theory to the space of tempered distributions. Moreover, they provided an intrinsic connection between the DSTFRFT and the FRFT, along with a desingularization formula.

In general, distributions do not have point value, and Łojasiewicz was the first to give a satisfactory definition of the point value of distributions [17]. This idea was further extended by Zavialov, who introduced the concept of the quasiasymptotic behavior of distributions, which was developed in collaboration with Vladimirov and Drozinov [32]. Asymptotic analysis of distributions has proven to be a useful tool in diverse areas such as mathematical physics, number theory, and differential equations (see [5, 22, 32] and references therein). In recent years, characterizations of the asymptotic properties of distributions via various integral transforms [2, 4, 6, 14, 15, 31], wavelet coefficients [24], and Gabor frames [16] have been investigated.

The purpose of this paper is twofold. In Section 3, we study the DSTFRFT and its synthesis operator by fixing the direction $\mathbf{u} \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. This new transform, called the STFRFT in the direction of $\mathbf{u}$, is extended to the space of tempered distributions by following the duality approach. It is briefly mentioned that the Parseval identity and the reconstruction formula also hold. Most of our arguments in this article rely on the intrinsic connection between the FRFT and the STFRFT in the direction of $\mathbf{u}$ (see Prop. 3.2 below). Then, in Section 4, a desingularization formula and the characterization of the bounded sets in $\mathcal{S}'(\mathbb{R}^n)$ are provided.

The second aim of the paper is to present several Abelian- and Tauberian-type results, collected in Section 5, that fully characterize the asymptotic behavior of generalized functions at the origin via the asymptotic behavior of their STFRFT in the direction of $\mathbf{u}$. It is worth mentioning that similar assertions do not hold for asymptotics at infinity (see Remark 5.8 below).

2. Preliminaries

We use the following standard notations from multidimensional calculus: for $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ and the multi-index $\mathbf{m} = (m_1, m_2, ..., m_n) \in \mathbb{N}_0^n$, we write $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, $\partial_{\mathbf{x}}^{\mathbf{m}} = \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \cdots \partial_{x_n}^{m_n} = \frac{\partial^{|\mathbf{m}|}}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}}$, $|\mathbf{m}| = m_1 + m_2 + ... + m_n$, $|\mathbf{x}|$ denotes the Euclidean norm and $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots x_n y_n$ the scalar product of $\mathbf{x}$ and $\mathbf{y}$. The notation (f, φ) stands for the L^2-inner product of f and φ and $\langle f, \varphi \rangle$ stands for dual pairing between the distribution f and a test function φ; $(f, \varphi) = \langle f, \overline{\varphi} \rangle$. The FT of a function $f \in L^1(\mathbb{R}^n)$ is defined as $\mathcal{F} f(\xi) = \widehat{f(\xi)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\xi} d\mathbf{x}, \ \xi \in \mathbb{R}^n$, and it extends to $L^2(\mathbb{R}^n)$ in the usual way [11].

The space of rapidly decreasing smooth functions and its dual, the space of tempered distributions, are denoted by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. We refer to [25] for the well-known properties of these spaces. For the seminorms on $\mathcal{S}(\mathbb{R}^n)$, we make the following choice, $\rho_\nu(\varphi) = \sup_{\mathbf{x} \in \mathbb{R}^n, |\mathbf{m}| \le \nu} (1 + |\mathbf{x}|)^\nu |\partial_{\mathbf{x}}^{\mathbf{m}} \varphi(\mathbf{x})|$, $\nu \in \mathbb{N}_0$.

The natural range space for the STFRFT in the direction of $\mathbf{u}$ is the space $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ that consists of all smooth functions $\Phi \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfying the decay condition

$$\rho_{s,r}^{k,\mathbf{l}}(\Phi) = \sup_{(b,\mathbf{a}) \in \mathbb{R} \times \mathbb{R}^n} (1 + |b|^2)^{r/2} (1 + |\mathbf{a}|^2)^{s/2} |\partial_{\mathbf{a}}^{\mathbf{l}} \partial_b^k \Phi(b, \mathbf{a})| < \infty, \tag{1}$$

for all $s, r, k \in \mathbb{N}_0$ and $\mathbf{l} \in \mathbb{N}_0^n$. The family of seminorms (1) determines the topology of the space $\mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$, and its dual space is denoted by $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)$. We use $d\mu(b,\mathbf{a}) = dbd\mathbf{a}$ as a standard measure on $\mathbb{R}\times\mathbb{R}^n$, where db and $d\mathbf{a}$ are the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Furthermore, any locally integrable function F on $\mathbb{R}\times\mathbb{R}^n$ that satisfies

$$|F(b,\mathbf{a})| \le C\,(1+|\mathbf{a}|+|b|)^s\,, \quad (b,\mathbf{a})\in\mathbb{R}\times\mathbb{R}^n,$$

for some $C>0$ and $s\in\mathbb{N}_0$, will be identified with an element of $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)$, via the action

$$\langle F,\Phi\rangle := \int_{\mathbb{R}^n}\int_{\mathbb{R}} F(b,\mathbf{a})\Phi(b,\mathbf{a})dbd\mathbf{a},\quad \Phi\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^n). \tag{2}$$

We provide all dual spaces with the strong dual topology [30].

One can show that the nuclearity of the Schwartz spaces yields the following equality $\mathcal{S}(\mathbb{R}\times\mathbb{R}^n) = \mathcal{S}(\mathbb{R})\hat{\otimes}\mathcal{S}(\mathbb{R}^n)$ ([30], Thm. 51.6), where $X\hat{\otimes}Y$ is the topological tensor product space obtained as the completion of $X\otimes Y$ in the π-topology or, equivalently in the ε-topology. Then, we have the following isomorphisms $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)\cong\mathcal{S}'(\mathbb{R},\mathcal{S}'(\mathbb{R}^n))\cong\mathcal{S}'(\mathbb{R}^n,\mathcal{S}'(\mathbb{R}))$, which is being realized via the following identification

$$\langle F,\varphi\otimes\Psi\rangle = \langle\langle F,\Psi\rangle,\varphi\rangle = \langle\langle F,\varphi\rangle,\Psi\rangle,\quad \varphi\in\mathcal{S}(\mathbb{R}^n),\ \Psi\in\mathcal{S}(\mathbb{R}). \tag{3}$$

3. The STFRFT in the direction of u

In this section, we analyze the DSTFRFT by fixing the direction $\mathbf{u}$. First, we recall the definitions and some known facts concerning the FRFT, STFRFT and DSTFRFT.

Recall that the FRFT of order $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,...,\alpha_n)\in\mathbb{R}^n$ of a function $f\in L^1(\mathbb{R}^n)$, denoted as $\mathcal{F}_{\boldsymbol{\alpha}}f$, is defined as

$$\mathcal{F}_{\boldsymbol{\alpha}}f(\mathbf{a}) = \int_{\mathbb{R}^n} f(\mathbf{x})K_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{a})d\mathbf{x},\quad \mathbf{a}\in\mathbb{R}^n, \tag{4}$$

where

$$K_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{a}) = \prod_{k=1}^n K_{\alpha_k}(x_k,a_k), \tag{5}$$

$\mathbf{x}=(x_1,...,x_n),\ \mathbf{a}=(a_1,...,a_n)\in\mathbb{R}^n$ and $K_{\alpha_k}(x_k,a_k)$ are defined by

$$K_{\alpha_k}(x_k,a_k) = \begin{cases} C_{\alpha_k} e^{i\left(\frac{x_k^2+a_k^2}{2}c_1^{(k)} - x_k a_k c_2^{(k)}\right)} & if\ \alpha_k\notin\pi\mathbb{Z} \\ \delta(x_k-a_k) & if\ \alpha_k\in 2\pi\mathbb{Z} \\ \delta(x_k+a_k) & if\ \alpha_k\in 2\pi\mathbb{Z}+\pi \end{cases},$$

$c_1^{(k)}=\cot(\alpha_k)$, $c_2^{(k)}=\csc(\alpha_k)$, $C_{\alpha_k}=\sqrt{\frac{1-ic_1^{(k)}}{2\pi}}$ [1, 12, 20, 29, 33]. Throughout the paper, we use the following convenient notations, $c_1=(c_1^{(1)},...,c_1^{(n)})$ and $c_2=(c_2^{(1)},...,c_2^{(n)})$. If we put $\boldsymbol{\alpha}=(\pi/2,\pi/2,...,\pi/2)$ in (4), we obtain the definition of the FT. One can also notice that

$$\mathcal{F}_{\boldsymbol{\alpha}}(e^{-i\frac{c_1\cdot(a_1^2,...,a_n^2)}{2}}f(\mathbf{a}))(\mathbf{x}) = \prod_{k=1}^n\sqrt{1-ic_1^{(k)}}\,e^{i\frac{c_1\cdot(x_1^2,...,x_n^2)}{2}}\mathcal{F}f(c_2^{(1)}x_1,\dots,c_2^{(n)}x_n). \tag{6}$$

Following [18, 21], one can easily prove that $\mathcal{F}_{\boldsymbol{\alpha}}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is a continuous map, and can be extended to the space of tempered distributions by duality

$$\langle\mathcal{F}_{\boldsymbol{\alpha}}f,\varphi\rangle = \langle f,\mathcal{F}_{\boldsymbol{\alpha}}\varphi\rangle,\ f\in\mathcal{S}'(\mathbb{R}^n),\ \varphi\in\mathcal{S}(\mathbb{R}^n).$$

Recall that the STFRFT of order $\boldsymbol{\alpha} \in \mathbb{R}^n$ of an integrable function $f \in L^1(\mathbb{R}^n)$ with respect to a window function $\psi \in \mathcal{S}(\mathbb{R}^n)$, denoted as $S^{\boldsymbol{\alpha}}_{\psi} f$, is defined as

$$S^{\boldsymbol{\alpha}}_{\psi} f(\mathbf{b}, \mathbf{a}) = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi}(\mathbf{x} - \mathbf{b}) K_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{a}) d\mathbf{x}, \tag{7}$$

where $(\mathbf{b}, \mathbf{a}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $K_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{a})$ is given by (5) ([8], Def. 2). For $\boldsymbol{\alpha} = (\pi/2, ..., \pi/2)$ in (7), we obtain the definition of the STFT, namely,

$$S_{\psi} f(\mathbf{b}, \mathbf{a}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi}(\mathbf{x} - \mathbf{b}) e^{-i\mathbf{x}\cdot\mathbf{a}} d\mathbf{x}.$$

We refer to the definition of the DSTFRFT as provided in [7]. The DSTFRFT of order $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$ of an integrable function $f \in L^1(\mathbb{R}^n)$ with respect to $\psi \in \mathcal{S}(\mathbb{R})$, denoted as $DS^{\boldsymbol{\alpha}}_{\psi} f$, is defined as

$$DS^{\boldsymbol{\alpha}}_{\psi} f(\mathbf{u}, b, \mathbf{a}) = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi}(\mathbf{u} \cdot \mathbf{x} - b) K_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{a}) d\mathbf{x}, \tag{8}$$

where $(\mathbf{u}, b, \mathbf{a}) \in \mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$ ($\mathbb{S}^{n-1}$ stands for the unit sphere in $\mathbb{R}^n$).

If we fix the direction $\mathbf{u}$ in the definition (8), we use the notation $DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f(b, \mathbf{a})$ and call this new transform the STFRFT in the direction of $\mathbf{u}$. For $\boldsymbol{\alpha} = (\pi/2, ..., \pi/2)$, we obtain the STFT in the direction of $\mathbf{u}$, defined in [3], namely,

$$DS_{\psi,\mathbf{u}} f(b, \mathbf{a}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi}(\mathbf{u} \cdot \mathbf{x} - b) e^{-i\mathbf{x}\cdot\mathbf{a}} d\mathbf{x}.$$

Since $K_{\alpha_k}(x_k, a_k)$ is a 2π-periodic function with respect to α_k, we may suppose that $\alpha_k \in (-\pi, \pi) \setminus \{0\}$ for $k \in \{1, 2, ..., n\}$.

One can show, similarly as for the DSTFRFT ([7], Prop. 2), that for a non-trivial window $\psi \in \mathcal{S}(\mathbb{R})$ with synthesis window $\eta \in \mathcal{S}(\mathbb{R})$, namely $(\eta, \psi) \neq 0$, the following Parseval formula holds,

$$\int_{\mathbb{R}^n} f(\mathbf{x})\overline{g(\mathbf{x})} d\mathbf{x} = \frac{1}{(\eta, \psi)} \int_{\mathbb{R}^n} \int_{\mathbb{R}} DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f(b, \mathbf{a}) \overline{DS^{\boldsymbol{\alpha}}_{\eta,\mathbf{u}} g(b, \mathbf{a})} db d\mathbf{a},$$

for $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Furthermore, if $f \in L^1(\mathbb{R}^n)$ such that $\mathcal{F}_{\boldsymbol{\alpha}} f \in L^1(\mathbb{R}^n)$, then the following reconstruction formula holds pointwisely,

$$f(\mathbf{x}) = \frac{1}{(\eta, \psi)} \int_{\mathbb{R}^n} \int_{\mathbb{R}} DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f(b, \mathbf{a}) \eta(\mathbf{x} \cdot \mathbf{u} - b) K_{-\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{a}) db d\mathbf{a}, \quad \textit{a.e.} \ \mathbf{x} \in \mathbb{R}^n, \tag{9}$$

where $\eta \in \mathcal{S}(\mathbb{R})$ is a synthesis window for $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$.

The reconstruction formula (9) allows us to define the STFRFT synthesis operator in the direction of $\mathbf{u}$, which maps the function on $\mathbb{R} \times \mathbb{R}^n$ to a function on $\mathbb{R}^n$. For the given $\psi \in \mathcal{S}(\mathbb{R}^n)$, we define this synthesis operator as

$$(DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}})^* \Phi(\mathbf{x}) := \int_{\mathbb{R}^n} \int_{\mathbb{R}} \Phi(b, \mathbf{a}) \psi(\mathbf{x} \cdot \mathbf{u} - b) K_{-\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{a}) db d\mathbf{a}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{10}$$

The last integral is absolutely convergent if $\Phi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$. Now, by using the reconstruction formula (9), we have

$$(DS^{\boldsymbol{\alpha}}_{\eta,\mathbf{u}})^* (DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f)(\mathbf{x}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f(b, \mathbf{a}) \eta(\mathbf{x} \cdot \mathbf{u} - b) K_{-\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{a}) db d\mathbf{a} = (\eta, \psi) f(\mathbf{x}),$$

for a.e. $\mathbf{x} \in \mathbb{R}^n$.

By using the Fubini's theorem, one can show that the short-time fractional Fourier synthesis operator in the direction of $\mathbf{u}$ is in fact the transpose of the STFRFT in the direction of $\mathbf{u}$ in the following sense: If $\psi \in \mathcal{S}(\mathbb{R})$, $f \in L^1(\mathbb{R}^n)$ and $\Phi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(\mathbf{x})\overline{(DS^{\alpha}_{\psi,\mathbf{u}})^*(\overline{\Phi})}(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^n}\int_{\mathbb{R}} DS^{\alpha}_{\psi,\mathbf{u}} f(b,\mathbf{a})\Phi(b,\mathbf{a})dbd\mathbf{a}.$$

Under the standard identification (2), the last relation may be written as $\langle f, \overline{(DS^{\alpha}_{\psi,\mathbf{u}})^*(\overline{\Phi})}\rangle = \langle DS^{\alpha}_{\psi,\mathbf{u}} f, \Phi\rangle$. As in [7], the last relation motivates our definition of the distributional STFRFT in the direction of $\mathbf{u}$.

One can readily note that the continuity of the DSTFRFT and its synthesis operator remains valid if we fix the direction $\mathbf{u}$, namely, the bilinear mapping $DS^{\alpha}_{\mathbf{u}} : \mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$ defined as $(f,\psi) \to DS^{\alpha}_{\psi,\mathbf{u}} f$, and $(DS^{\alpha}_{\mathbf{u}})^* : \mathcal{S}(\mathbb{R}\times\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^n)$ defined as $(\Phi,\psi) \to (DS^{\alpha}_{\psi,\mathbf{u}})^*\Phi$ are continuous.

Similarly to the case of the DSTFRFT ([7], Prop. 7), it can be proven that if $\eta \in \mathcal{S}(\mathbb{R})$ is a synthesis window for $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$, the following reconstruction formula holds

$$\frac{1}{(\eta,\psi)}(DS^{\alpha}_{\eta,\mathbf{u}})^* \circ DS^{\alpha}_{\psi,\mathbf{u}} = Id_{\mathcal{S}(\mathbb{R}^n)}. \tag{11}$$

The continuity results allow us to define the STFRFT in the direction of $\mathbf{u}$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to ψ as the element $DS^{\alpha}_{\psi,\mathbf{u}} f \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)$ whose action on the test function $\Phi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$ is given by

$$\langle DS^{\alpha}_{\psi,\mathbf{u}} f, \Phi\rangle := \langle f, \overline{(DS^{\alpha}_{\psi,\mathbf{u}})^*(\overline{\Phi})}\rangle, \tag{12}$$

as well as the synthesis operator $(DS^{\alpha}_{\psi,\mathbf{u}})^* : \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle (DS^{\alpha}_{\psi,\mathbf{u}})^* F, \varphi\rangle := \langle F, \overline{DS^{\alpha}_{\psi,\mathbf{u}}(\overline{\varphi})}\rangle, \quad F \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n), \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{13}$$

By taking the transposes, we obtain the following immediate result:

Proposition 3.1. *Let $\psi \in \mathcal{S}(\mathbb{R})$. The STFRFT in the direction of $\mathbf{u}$, $DS^{\alpha}_{\psi,\mathbf{u}} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)$, and the short-time fractional Fourier synthesis operator in the direction of $\mathbf{u}$, $(DS^{\alpha}_{\psi,\mathbf{u}})^* : \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, are continuous linear maps.*

One can show a generalization of the reconstruction formula (11) for the space of tempered distributions, namely,

$$\frac{1}{(\eta,\psi)}(DS^{\alpha}_{\eta,\mathbf{u}})^* \circ DS^{\alpha}_{\psi,\mathbf{u}} = Id_{\mathcal{S}'(\mathbb{R}^n)}, \tag{14}$$

where $\eta \in \mathcal{S}(\mathbb{R})$ is a synthesis window for a non-trivial window $\psi \in \mathcal{S}(\mathbb{R})$.

The next proposition provides an intrinsic relation between the FRFT and the STFRFT in the direction of $\mathbf{u}$ on $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 3.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R})$. Then*

$$\langle DS^{\alpha}_{\psi,\mathbf{u}} f, \Phi\rangle = \int_{\mathbb{R}} \langle \mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u} - b)\right)(\mathbf{a}), \Phi(b,\mathbf{a})\rangle_{\mathbf{a}} db, \quad \Phi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^n). \tag{15}$$

Furthermore, $DS^{\alpha}_{\psi,u} f \in \mathbb{C}^{\infty}(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))$ and it is of slow growth on $\mathbb{R}$.

Proof. Using (12) and (10), as well as the Fubini's theorem, for $\Phi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$, we obtain

$$\langle DS^{\alpha}_{\psi,\mathbf{u}}f, \Phi\rangle = \Big\langle f(\mathbf{x}), \overline{(DS^{\alpha}_{\psi,\mathbf{u}})^{*}(\overline{\Phi})(\mathbf{x})}\Big\rangle = \Big\langle f(\mathbf{x}), \int_{\mathbb{R}^n}\int_{\mathbb{R}} \Phi(b,\mathbf{a})\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)K_{\alpha}(\mathbf{x},\mathbf{a})db\,d\mathbf{a}\Big\rangle$$
$$= \langle f(\mathbf{x}), \int_{\mathbb{R}} \overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})db\rangle. \tag{16}$$

Since $f \in \mathcal{S}'(\mathbb{R}^n)$, it follows that $f = \partial^{\mathbf{m}}h$, for some continuous function h of at most polynomial growth on $\mathbb{R}^n$, and some $\mathbf{m} \in \mathbb{N}_0^n$ ([25], Thm. VI, page 239). Then, by the Lebesgue dominated convergence theorem, (16) takes the form

$$\langle DS^{\alpha}_{\psi,\mathbf{u}}f, \Phi\rangle = (-1)^{|\mathbf{m}|}\int_{\mathbb{R}^n} h(\mathbf{x})\partial^{\mathbf{m}}_{\mathbf{x}}\left(\int_{\mathbb{R}} \overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})db\right)d\mathbf{x}$$
$$=(-1)^{|\mathbf{m}|}\int_{\mathbb{R}^n} h(\mathbf{x})d\mathbf{x}\int_{\mathbb{R}} \partial^{\mathbf{m}}_{\mathbf{x}}\left(\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})\right)db.$$

One can easily show that for given $\mathbf{s}, \mathbf{m} \in \mathbb{N}_0^n$, and $\nu \in \mathbb{N}_0$, it is true

$$\left|\mathbf{x}^{\mathbf{s}}\partial^{\mathbf{m}}_{\mathbf{x}}\left(\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})\right)\right| = \left|\mathbf{x}^{\mathbf{s}}\sum_{\mathbf{k}+\mathbf{l}=\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k},\mathbf{l}}\mathbf{u}^{\mathbf{k}}\,\overline{\psi^{(|\mathbf{k}|)}}(\mathbf{x}\cdot\mathbf{u}-b)\partial^{\mathbf{l}}_{\mathbf{x}}\left(\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})\right)\right|$$
$$= \Big|\sum_{\mathbf{k}+\mathbf{l}=\mathbf{m}}\sum_{|\mathbf{d}|,|\mathbf{r}|\le|\mathbf{l}|}\binom{\mathbf{m}}{\mathbf{k},\mathbf{l}}\mathbf{u}^{\mathbf{k}}C^{\mathbf{l}}_{\alpha,\mathbf{d},\mathbf{r}}\overline{\psi^{(|\mathbf{k}|)}}(\mathbf{x}\cdot\mathbf{u}-b)\mathbf{x}^{\mathbf{d}+\mathbf{s}}\int_{\mathbb{R}^n}\mathbf{a}^{\mathbf{r}}\Phi(b,\mathbf{a})K_{\alpha}(\mathbf{x},\mathbf{a})d\mathbf{a}\Big|$$
$$\le C'\sum_{\mathbf{k}+\mathbf{l}=\mathbf{m}}\sum_{|\mathbf{d}|,|\mathbf{r}|\le|\mathbf{l}|}\Big|\int_{\mathbb{R}^n}\mathbf{a}^{\mathbf{r}}\Phi(b,\mathbf{a})e^{i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}}\partial^{\mathbf{d}+\mathbf{s}}_{\mathbf{a}}(e^{-i\mathbf{x}\cdot(a_1c_2^{(1)},\dots,a_nc_2^{(n)})})d\mathbf{a}\Big|$$
$$= \sum_{\mathbf{k}+\mathbf{l}=\mathbf{m}}\sum_{|\mathbf{d}|,|\mathbf{r}|\le|\mathbf{l}|}\Big|\int_{\mathbb{R}^n}\partial^{\mathbf{d}+\mathbf{s}}_{\mathbf{a}}\left(\mathbf{a}^{\mathbf{r}}\Phi(b,\mathbf{a})e^{i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}}\right)e^{-i\mathbf{x}\cdot(a_1c_2^{(1)},\dots,a_nc_2^{(n)})}d\mathbf{a}\Big|$$
$$\le C(1+|b|)^{-\nu},$$

for some $C, C' > 0$. Then, there exists $D > 0$ such that

$$\left|h(\mathbf{x})\partial^{\mathbf{m}}_{\mathbf{x}}\left(\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})\right)\right| \le D\frac{1}{(1+|\mathbf{x}|)^{n+1}(1+|b|)^2}$$

and, we obtain

$$\int_{\mathbb{R}^n} d\mathbf{x}\int_{\mathbb{R}}\left|h(\mathbf{x})\partial^{\mathbf{m}}_{\mathbf{x}}\left(\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})\right)\right|db < \infty.$$

Finally by the Fubini's theorem, we have

$$\langle DS^{\alpha}_{\psi,\mathbf{u}}f, \Phi\rangle = (-1)^{|\mathbf{m}|}\int_{\mathbb{R}} db\int_{\mathbb{R}^n} h(\mathbf{x})\partial^{\mathbf{m}}_{\mathbf{x}}\left(\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})\right)d\mathbf{x}$$
$$= \int_{\mathbb{R}}\langle f(\mathbf{x}), \overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(\Phi(b,\cdot)\right)(\mathbf{x})\rangle_{\mathbf{x}} = \int_{\mathbb{R}}\langle \mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}), \Phi(b,\mathbf{a})\rangle_{\mathbf{a}}db,$$

which proves the relation (15).

One can show that for fixed $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the function

$$b \to \langle \mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}), \varphi(\mathbf{a})\rangle_{\mathbf{a}}, \quad b \in \mathbb{R},$$

is smooth of at most polynomial growth on $\mathbb{R}$, then it may be identified with an element of $\mathcal{S}'(\mathbb{R})$ via the following action on $\Psi \in \mathcal{S}(\mathbb{R})$,

$$\langle\langle \mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}), \varphi(\mathbf{a})\rangle_{\mathbf{a}}, \Psi(b)\rangle_b = \int_{\mathbb{R}}\langle \mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}), \varphi(\mathbf{a})\rangle_{\mathbf{a}}\Psi(b)db.$$

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\Psi \in \mathcal{S}(\mathbb{R})$, under the identification (3), we obtain

$$\langle\langle DS^{\alpha}_{\psi,\mathbf{u}}f(b,\mathbf{a}),\varphi(\mathbf{a})\rangle_{\mathbf{a}},\Psi(b)\rangle_b = \langle DS^{\alpha}_{\psi,\mathbf{u}}f,\varphi\Psi\rangle = \int_{\mathbb{R}}\langle\mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}),\varphi(\mathbf{a})\rangle_{\mathbf{a}}\Psi(b)db$$
$$= \langle\langle\mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}),\varphi(\mathbf{a})\rangle_{\mathbf{a}},\Psi(b)\rangle_b.$$

Then,

$$\langle DS^{\alpha}_{\psi,\mathbf{u}}f(b,\mathbf{a}),\varphi(\mathbf{a})\rangle_{\mathbf{a}} = \langle\mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}),\varphi(\mathbf{a})\rangle_{\mathbf{a}} = \langle f(\mathbf{x}),\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\varphi(\mathbf{x})\rangle_{\mathbf{x}}. \tag{17}$$

So, we conclude that $DS^{\alpha}_{\psi,\mathbf{u}}f \in \mathbb{C}^{\infty}(\mathbb{R},\mathcal{S}'(\mathbb{R}^n))$ and it is of slow growth on $\mathbb{R}$. □

4. Desingularization formula and characterization of bounded subsets of $\mathcal{S}'(\mathbb{R}^n)$

As a corollary of Prop. 3.2, we have the following desingularization formula.

Corollary 4.1. (Desingularization formula) *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R})$ be a non-trivial window. If $\eta \in \mathcal{S}(\mathbb{R})$ is a synthesis window for ψ, then*

$$\langle f,\varphi\rangle = \frac{1}{(\eta,\psi)}\int_{\mathbb{R}}\left\langle\mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}),\overline{DS^{\alpha}_{\eta,u}(\overline{\varphi})}(b,\mathbf{a})\right\rangle_{\mathbf{a}}db,\ \ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proof. By relations (14), (13), and Prop. 3.2, we have

$$\langle f,\varphi\rangle = \frac{1}{(\eta,\psi)}\left\langle(DS^{\alpha}_{\eta,\mathbf{u}})^*(DS^{\alpha}_{\psi,\mathbf{u}}f),\varphi\right\rangle = \frac{1}{(\eta,\psi)}\left\langle DS^{\alpha}_{\psi,\mathbf{u}}f,\overline{DS^{\alpha}_{\eta,\mathbf{u}}(\overline{\varphi})}\right\rangle$$
$$= \frac{1}{(\eta,\psi)}\int_{\mathbb{R}}\left\langle\mathcal{F}_{\alpha}\left(f(\cdot)\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\right)(\mathbf{a}),\overline{DS^{\alpha}_{\eta,\mathbf{u}}(\overline{\varphi})}(b,\mathbf{a})\right\rangle_{\mathbf{a}}db.$$

□

Now, we characterize the bounded subsets in $\mathcal{S}'(\mathbb{R}^n)$ via the STFRFT in the direction of $\mathbf{u}$. Note that weak boundedness is equivalent to strong boundedness, due to the Banach–Steinhaus theorem [30].

Proposition 4.2. *Let $\psi \in \mathcal{S}(\mathbb{R})\setminus\{0\}$. A subset $\mathcal{B} \subset \mathcal{S}'(\mathbb{R}^n)$ is weakly (strongly) bounded in $\mathcal{S}'(\mathbb{R}^n)$ if and only if there exists $l = l_{\mathcal{B}} \in \mathbb{N}_0$ such that for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ one can find $C = C_{\varphi,\mathcal{B}} > 0$ with*

$$\left|\left\langle DS^{\alpha}_{\psi,\mathbf{u}}f(b,\mathbf{a}),\varphi(\mathbf{a})\right\rangle_{\mathbf{a}}\right| \le C(1+|b|)^l, \tag{18}$$

for all $f \in \mathcal{B}$ and $b \in \mathbb{R}$.

Proof. Suppose that $\mathcal{B}$ is weakly bounded in $\mathcal{S}'(\mathbb{R}^n)$ then, by the Banach-Steinhaus theorem, it is equicontinuous ([30], Thm. 33.2), i.e., there exists $C' = C'_{\mathcal{B}} > 0$ and $N = N_{\mathcal{B}} \in \mathbb{N}_0$ such that

$$|\langle f,\varphi\rangle| \le C'\rho_N(\varphi), \tag{19}$$

for all $f \in \mathcal{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Using relations (17) and (19), for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we obtain

$$\left|\left\langle DS^{\alpha}_{\psi,\mathbf{u}}f(b,\mathbf{a}),\varphi(\mathbf{a})\right\rangle_{\mathbf{a}}\right| = \left|\langle f(\mathbf{x}),\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\varphi(\mathbf{x})\rangle_{\mathbf{x}}\right| \le C'\rho_N\left(\overline{\psi}((\cdot)\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\varphi(\cdot)\right)$$
$$= C'\sup_{\mathbf{x}\in\mathbb{R}^n,\,|\mathbf{m}|\le N}(1+|\mathbf{x}|)^N\left|\sum_{\mathbf{k}+\mathbf{l}=\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k},\mathbf{l}}\partial^{\mathbf{k}}_{\mathbf{x}}\left(\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\right)\partial^{\mathbf{l}}_{\mathbf{x}}\left(\mathcal{F}_{\alpha}\varphi(\mathbf{x})\right)\right|$$

$$\leq C' \rho_N(\mathcal{F}_{\boldsymbol{\alpha}}\varphi) \sum_{\mathbf{k}+\mathbf{l}=\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k},\mathbf{l}} \rho_{|\mathbf{k}|}(\psi) \leq C_{\varphi,\mathcal{B}}$$

for all $f \in \mathcal{B}$ and $b \in \mathbb{R}$.

The converse, using relations (15), (17) and (18), for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\Psi \in \mathcal{S}(\mathbb{R})$, we obtain

$$\left|\left\langle DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f, \varphi\Psi\right\rangle\right| = \left|\int_{\mathbb{R}} \langle DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f(b,\mathbf{a}), \varphi(\mathbf{a})\rangle_{\mathbf{a}} \Psi(b) db\right| \leq C \int_{\mathbb{R}} (1+|b|)^l |\Psi(b)| db,$$

for all $f \in \mathcal{B}$. Now, since $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n) \cong \mathcal{S}'(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))$, we may conclude that $\{DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f : f \in \mathcal{B}\}$ is weakly bounded in $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)$, and by the inversion formula (14), it follows that $\mathcal{B}$ is weakly (strongly) bounded in $\mathcal{S}'(\mathbb{R}^n)$. □

5. Abelian- and Tauberian-type results

In this section, we prove several Abelian- and Tauberian-type results that characterize the quasiasymptotic behavior at the origin of tempered distributions via the asymptotic behavior of their STFRFT in the direction of $\mathbf{u}$. First, we briefly explain the notion of quasiasymptotics of distributions. For a complete treatment of this theory, we refer the reader to [5, 22, 26, 32].

Throughout this paper, L stands for Karamata's slowly varying function at the origin, namely, a positive measurable function on an interval $(0, A]$, for some $A > 0$, such that

$$\lim_{\varepsilon\to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1, \quad \forall a > 0.$$

It is said that the distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ has quasiasymptotic behavior of degree $\beta \in \mathbb{R}$ at the origin with respect to L if there exists $g \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon\to 0^+} \left\langle \frac{f(\varepsilon\mathbf{x})}{\varepsilon^\beta L(\varepsilon)}, \varphi(\mathbf{x})\right\rangle = \langle g(\mathbf{x}), \varphi(\mathbf{x})\rangle \tag{20}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We use the following notation

$$f(\varepsilon\mathbf{x}) \sim \varepsilon^\beta L(\varepsilon) g(\mathbf{x}) \text{ as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}^n),$$

which always should be interpreted in the weak topology of $\mathcal{S}'(\mathbb{R}^n)$, i.e. in the sense of (20). One can prove that the limit distribution g is a homogeneous distribution of degree β, namely, $g(a\mathbf{x}) = a^\beta g(\mathbf{x})$ for each $a > 0$ [5, 22, 32]. The quasiasymptotic behavior of distributions at infinity is defined in a similar way.

We start with a useful lemma that connects the quasiasymptotics at a point and the quasiasymptotics with oscillation at the same point.

Lemma 5.1. ([2], Lemma 3.1) *Let $f \in \mathcal{S}'(\mathbb{R})$. If*

$$\langle f(\varepsilon x)/(\varepsilon^\beta L(\varepsilon)), \varphi(x)\rangle \text{ converges as } \varepsilon \to 0^+, \ \forall \varphi \in \mathcal{S}(\mathbb{R}), \tag{21}$$

then

$$\langle e^{ic(\varepsilon x)^2/2} f(\varepsilon x)/(\varepsilon^\beta L(\varepsilon)), \varphi(x)\rangle \text{ converges as } \varepsilon \to 0^+, \forall \varphi \in \mathcal{S}(\mathbb{R}), \tag{22}$$

where c is a real constant. Conversely, if (22) holds and for some $\varepsilon_0 \in (0,1)$, the family $\{f(\varepsilon x)/(\varepsilon^\beta L(\varepsilon)) : \varepsilon \in (0, \varepsilon_0)\}$ is bounded in $\mathcal{S}'(\mathbb{R})$, then (21) holds.

Remark 5.2. *One can easily show that Lemma 5.1 also holds for the multidimensional case, that is for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$.*

The direct parts of the next two propositions are Abelian-type results, while their converses may be regarded as Tauberian theorems.

Proposition 5.3. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R})$ so that $\psi(0) = 1$. If f satisfies*

$$f(\varepsilon \mathbf{x}) \sim \varepsilon^{\beta} L(\varepsilon) g(\mathbf{x}) \quad as \quad \varepsilon \to 0^+ \quad in \quad \mathcal{S}'(\mathbb{R}^n), \tag{23}$$

then, for every fixed $b \in \mathbb{R}$,

$$e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right) \sim \varepsilon^{n+\beta} L(\varepsilon) \prod_{k=1}^{n} \sqrt{1 - ic_1^{(k)}}\, \mathcal{F} g(c_2^{(1)} a_1, \dots, c_2^{(n)} a_n), \tag{24}$$

as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. Conversely, if (24) holds and

$$\left\{ \frac{f(\varepsilon \mathbf{x})}{\varepsilon^{\beta} L(\varepsilon)} \right\}_{0<\varepsilon<1} \text{ is bounded in } \mathcal{S}'(\mathbb{R}^n), \tag{25}$$

then (23) holds.

Proof. By relations (17) and (6), for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^{n+\beta} L(\varepsilon)} \left\langle e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right), \varphi(\mathbf{a}) \right\rangle_{\mathbf{a}} = \frac{1}{\varepsilon^{\beta} L(\varepsilon)} \left\langle f(\mathbf{x}), \overline{\psi}(\mathbf{x}\cdot\mathbf{u} - \varepsilon b) \mathcal{F}_{\boldsymbol{\alpha}}(e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}} \varphi(\varepsilon \mathbf{a}))(\mathbf{x}) \right\rangle_{\mathbf{x}} \\
&= \frac{\prod_{k=1}^n \sqrt{1 - ic_1^{(k)}}}{\varepsilon^{\beta} L(\varepsilon)} \left\langle e^{i\frac{\varepsilon^2}{2} c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}), \overline{\psi}(\varepsilon(\mathbf{u}\cdot\mathbf{x} - b)) \mathcal{F}\varphi(c_2^{(1)} x_1, \dots, c_2^{(n)} x_n) \right\rangle_{\mathbf{x}} \\
&= \frac{\prod_{k=1}^n \sqrt{1 - ic_1^{(k)}}}{\varepsilon^{\beta} L(\varepsilon)} \left(\left\langle h_\varepsilon(\mathbf{x}), \mathcal{F}\varphi(c_2^{(1)} x_1, \dots, c_2^{(n)} x_n) \right\rangle_{\mathbf{x}} + \left\langle h_\varepsilon(\mathbf{x}), e_{\varepsilon,b,\mathbf{u}}(\mathbf{x}) \right\rangle_{\mathbf{x}} \right),
\end{aligned} \tag{26}$$

where $h_\varepsilon(\mathbf{x}) = e^{i\frac{\varepsilon^2}{2} c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x})$, and

$$e_{\varepsilon,b,\mathbf{u}}(\mathbf{x}) = \overline{\psi}(\varepsilon(\mathbf{u}\cdot\mathbf{x} - b)) \mathcal{F}\varphi(c_2^{(1)} x_1, \dots, c_2^{(n)} x_n) - \mathcal{F}\varphi(c_2^{(1)} x_1, \dots, c_2^{(n)} x_n).$$

Since $\left\{ \frac{h_\varepsilon(\mathbf{x})}{\varepsilon^{\beta} L(\varepsilon)} \right\}_{0<\varepsilon<1}$ is weakly bounded in $\mathcal{S}'(\mathbb{R}^n)$, the net $e_{\varepsilon,b,\mathbf{u}}(\mathbf{x})$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and converges to 0 as $\varepsilon \to 0^+$ in $\mathcal{S}(\mathbb{R}^n)$, by relations (26), (23), and Lemma 5.1 (see Remark 5.2) we conclude that

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{n+\beta} L(\varepsilon)} \left\langle e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi,\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right), \varphi(\mathbf{a}) \right\rangle_{\mathbf{a}} &= \prod_{k=1}^{n} \sqrt{1 - ic_1^{(k)}} \left\langle g(\mathbf{x}), \mathcal{F}\varphi(c_2^{(1)} x_1, \dots, c_2^{(n)} x_n) \right\rangle_{\mathbf{x}} \\
&= \prod_{k=1}^{n} \sqrt{1 - ic_1^{(k)}} \left\langle \mathcal{F} g(c_2^{(1)} a_1, \dots, c_2^{(n)} a_n), \varphi(\mathbf{a}) \right\rangle_{\mathbf{a}}.
\end{aligned}$$

To prove that (24) and (25) imply (23), we reverse the procedure described above. Specifically, we start from (26) and use the assumption (25). This assumption indicates that the second term on the right-hand side of (26) tends to zero. By applying Lemma 5.1 and Remark 5.2, we can see that (24) implies (23). This completes the proof. □

In the rest of the paper, we use the notation $\psi_{1/\varepsilon}(x) = \psi(x/\varepsilon)$, for $\psi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$.

Proposition 5.4. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. Suppose that $f(\varepsilon \mathbf{x}) \sim \varepsilon^\beta L(\varepsilon) g(\mathbf{x})$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. Then,*

$$e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right) \sim \varepsilon^{n+\beta} L(\varepsilon) \prod_{k=1}^{n} \sqrt{1 - ic_1^{(k)}} DS_{\psi,\mathbf{u}} g(b, (a_1 c_2^{(1)}, \dots, a_n c_2^{(n)})) \tag{27}$$

as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^n)$. The converse, if (27) is true, then f has quasiasymptotic behavior of degree β at the origin with respect to L.

Proof. Let $\Phi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$. By the change of variables, (12) and (10), we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^{n+\beta}L(\varepsilon)} \langle e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right), \Phi(b,\mathbf{a})\rangle_{b,\mathbf{a}} = \frac{1}{\varepsilon^{\beta+1}L(\varepsilon)} \left\langle DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f(b,\mathbf{a}), e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}} \Phi\left(\frac{b}{\varepsilon}, \varepsilon \mathbf{a}\right)\right\rangle_{b,\mathbf{a}} \\
&= \frac{1}{\varepsilon^{\beta+1}L(\varepsilon)} \left\langle f(\mathbf{x}), \overline{(DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}})^*(e^{i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}} \overline{\Phi}\left(\frac{b}{\varepsilon}, \varepsilon\mathbf{a}\right))(\mathbf{x})}\right\rangle_{\mathbf{x}} \\
&= \frac{1}{\varepsilon^{\beta+1}L(\varepsilon)} \left\langle f(\mathbf{x}), \int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}} \Phi\left(\frac{b}{\varepsilon}, \varepsilon\mathbf{a}\right) \overline{\psi_{1/\varepsilon}}(\mathbf{x}\cdot\mathbf{u} - b) K_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{a}) db d\mathbf{a} \right\rangle_{\mathbf{x}}.
\end{aligned}$$

By the change of variables $\mathbf{x} \to \varepsilon \mathbf{x}$, $b \to \varepsilon b$, $\mathbf{a} \to \frac{\mathbf{a}}{\varepsilon}$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^{n+\beta}L(\varepsilon)} \langle e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right), \Phi(b,\mathbf{a})\rangle_{b,\mathbf{a}} \\
&= \frac{\prod_{k=1}^n \sqrt{1 - ic_1^{(k)}}}{(2\pi)^{n/2}\varepsilon^\beta L(\varepsilon)} \left\langle e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon\mathbf{x}), \int_{\mathbb{R}^n}\int_{\mathbb{R}} \Phi(b,\mathbf{a})\,\overline{\psi}(\mathbf{x}\cdot\mathbf{u} - b) e^{-i\mathbf{x}\cdot(a_1c_2^{(1)},\dots,a_nc_2^{(n)})} db d\mathbf{a} \right\rangle_{\mathbf{x}} \\
&= \frac{\prod_{k=1}^n \sqrt{1 - ic_1^{(k)}}}{(2\pi)^{n/2}\varepsilon^\beta L(\varepsilon)} \left\langle f(\varepsilon\mathbf{x}), \left(e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} - 1\right) \int_{\mathbb{R}^n}\int_{\mathbb{R}} \Phi(b,\mathbf{a})\,\overline{\psi}(\mathbf{x}\cdot\mathbf{u} - b) e^{-i\mathbf{x}\cdot(a_1c_2^{(1)},\dots,a_nc_2^{(n)})} db d\mathbf{a} \right\rangle_{\mathbf{x}} \\
&\quad + \frac{\prod_{k=1}^n \sqrt{1 - ic_1^{(k)}}}{(2\pi)^{n/2}\varepsilon^\beta L(\varepsilon)} \left\langle f(\varepsilon\mathbf{x}), \int_{\mathbb{R}^n}\int_{\mathbb{R}} \Phi(b,\mathbf{a})\,\overline{\psi}(\mathbf{x}\cdot\mathbf{u} - b) e^{-i\mathbf{x}\cdot(a_1c_2^{(1)},\dots,a_nc_2^{(n)})} db d\mathbf{a} \right\rangle_{\mathbf{x}}.
\end{aligned}$$

Now, using the same idea as in the proof of Prop. 5.3, i.e. using the fact that $\{\frac{f(\varepsilon\mathbf{x})}{\varepsilon^\beta L(\varepsilon)}\}_{0<\varepsilon<1}$ is weakly bounded in $\mathcal{S}'(\mathbb{R}^n)$, the net

$$\left\{ \left(e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} - 1\right) \int_{\mathbb{R}^n}\int_{\mathbb{R}} \Phi(b,\mathbf{a})\,\overline{\psi}(\mathbf{x}\cdot\mathbf{u} - b) e^{-i\mathbf{x}\cdot(a_1c_2^{(1)},\dots,a_nc_2^{(n)})} db d\mathbf{a} \right\}_{0<\varepsilon<1}$$

belongs to $\mathcal{S}(\mathbb{R}^n)$ and converges to 0 as $\varepsilon \to 0^+$ in $\mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{n+\beta}L(\varepsilon)} \langle e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right), \Phi(b,\mathbf{a})\rangle_{b,\mathbf{a}} \\
&= \prod_{k=1}^n \sqrt{1 - ic_1^{(k)}}\,(2\pi)^{-n/2} \left\langle g(\mathbf{x}), \int_{\mathbb{R}^n}\int_{\mathbb{R}} \Phi(b,\mathbf{a})\,\overline{\psi}(\mathbf{x}\cdot\mathbf{u} - b) e^{-i\mathbf{x}\cdot(a_1c_2^{(1)},\dots,a_nc_2^{(n)})} db d\mathbf{a} \right\rangle_{\mathbf{x}} \\
&= \prod_{k=1}^n \sqrt{1 - ic_1^{(k)}} \left\langle DS_{\psi,\mathbf{u}} g(b, (a_1c_2^{(1)},\dots,a_nc_2^{(n)})), \Phi(b,a) \right\rangle_{b,\mathbf{a}}.
\end{aligned}$$

For the converse, doing similar steps as above, for $\Phi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ we obtain

$$\frac{1}{\varepsilon^\beta L(\varepsilon)} \left\langle DS_{\psi,\mathbf{u}} \left(e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon\mathbf{x})\right), \Phi \right\rangle$$

$$= \prod_{k=1}^{n} \frac{|c_2^{(k)}|}{\sqrt{1 - ic_1^{(k)}}} \frac{1}{\varepsilon^{n+\beta} L(\varepsilon)} \Big\langle e^{-i\frac{c_1 \cdot (a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon b, \frac{\mathbf{a}}{\varepsilon}\right), \Phi\big(b, (a_1 c_2^{(1)}, \dots, a_n c_2^{(n)})\big)\Big\rangle_{b,\mathbf{a}}$$

$$\to \prod_{k=1}^{n} |c_2^{(k)}| \Big\langle DS_{\psi,\mathbf{u}} g\big(b, (a_1 c_2^{(1)}, \dots, a_n c_2^{(n)})\big), \Phi\big(b, (a_1 c_2^{(1)}, \dots, a_n c_2^{(n)})\big)\Big\rangle_{b,\mathbf{a}}$$

$$= \Big\langle DS_{\psi,\mathbf{u}} g(b, \mathbf{a}), \Phi(b, \mathbf{a})\Big\rangle$$

as $\varepsilon \to 0^+$. We may conclude that $\left\{\frac{1}{\varepsilon^\beta L(\varepsilon)} DS_{\psi,\mathbf{u}}\left(e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x})\right)\right\}_{0<\varepsilon<1}$ converges as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^n)$.

Let $\eta \in \mathcal{S}(\mathbb{R})$ be a synthesis window for ψ, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, by relations (14) and (13) for $\boldsymbol{\alpha} = (\pi/2, \dots, \pi/2)$, we obtain

$$\frac{1}{\varepsilon^\beta L(\varepsilon)} \Big\langle e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}), \varphi(\mathbf{x})\Big\rangle = \frac{1}{\varepsilon^\beta L(\varepsilon)(\eta, \psi)} \Big\langle DS_{\psi,\mathbf{u}}\left(e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x})\right), \overline{DS_{\eta,\mathbf{u}}(\overline{\varphi})}\Big\rangle.$$

Since $\overline{DS_{\eta,\mathbf{u}}(\overline{\varphi})} \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$, it follows that $\{\frac{1}{\varepsilon^\beta L(\varepsilon)} e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x})\}_{0<\varepsilon<1}$ converges as $\varepsilon \to 0^+$, and then $\{\frac{1}{\varepsilon^\beta L(\varepsilon)} e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}))\}_{0<\varepsilon<1}$ is also weakly bounded in $\mathcal{S}'(\mathbb{R}^n)$.

Now, we aim to show that $\{\frac{1}{\varepsilon^\beta L(\varepsilon)} f(\varepsilon \mathbf{x})\}_{0<\varepsilon<1}$ converges as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. For fixed $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\frac{1}{\varepsilon^\beta L(\varepsilon)} \langle f(\varepsilon \mathbf{x}), \varphi(\mathbf{x})\rangle = \frac{1}{\varepsilon^\beta L(\varepsilon)} \langle e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}), e^{-i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} \varphi(\mathbf{x})\rangle$$
$$= \frac{1}{\varepsilon^\beta L(\varepsilon)} \langle e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}), \varphi(\mathbf{x})\rangle + \frac{1}{\varepsilon^\beta L(\varepsilon)} \langle e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}), e^{-i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} \varphi(\mathbf{x}) - \varphi(\mathbf{x})\rangle. \tag{28}$$

Since $\{\frac{1}{\varepsilon^\beta L(\varepsilon)} e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}))\}_{0<\varepsilon<1}$ is weakly bounded in $\mathcal{S}'(\mathbb{R}^n)$, the net $\{e^{-i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} \varphi(\mathbf{x}) - \varphi(\mathbf{x})\}_{0<\varepsilon<1}$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and converges to 0 as $\varepsilon \to 0^+$ in $\mathcal{S}(\mathbb{R}^n)$, then

$$\frac{1}{\varepsilon^\beta L(\varepsilon)} \langle e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x}), e^{-i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} \varphi(\mathbf{x}) - \varphi(\mathbf{x})\rangle \to 0, \quad \text{as} \quad \varepsilon \to 0^+. \tag{29}$$

So by (28) and (29), and the fact that $\{\frac{1}{\varepsilon^\beta L(\varepsilon)} e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon \mathbf{x})\}_{0<\varepsilon<1}$ converges as $\varepsilon \to 0^+$, we conclude that $\{\frac{1}{\varepsilon^\beta L(\varepsilon)} f(\varepsilon \mathbf{x})\}_{0<\varepsilon<1}$ converges as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. Then by the Banach-Steinhaus theorem ([30], p. 348), there exists a homogeneous distribution $h \in \mathcal{S}'(\mathbb{R}^n)$ such that $f(\varepsilon \mathbf{x}) \sim \varepsilon^\beta L(\varepsilon) h(\mathbf{x})$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. □

We provide here another Abelian-type result.

Proposition 5.5. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R})$. Suppose that $f(\varepsilon \mathbf{x}) \sim \varepsilon^\beta L(\varepsilon) g(\mathbf{x})$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. Then for any given $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $b \in \mathbb{R}$, is true*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{n+\beta} L(\varepsilon)} \Big\langle e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon^2 b, \frac{\mathbf{a}}{\varepsilon}\right), \varphi(\mathbf{a})\Big\rangle_{\mathbf{a}} = \prod_{k=1}^{n} \sqrt{1 - ic_1^{(k)}} \Big\langle DS_{\psi,\mathbf{u}} g(0, (a_1 c_2^{(1)}, \dots, a_n c_2^{(n)})), \varphi(\mathbf{a})\Big\rangle_{\mathbf{a}}. \tag{30}$$

Proof. Let $b \in \mathbb{R}$. By the change of variables and relations (17) and (6), we obtain

$$\frac{1}{\varepsilon^{n+\beta} L(\varepsilon)} \Big\langle e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2\varepsilon^2}} DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon^2 b, \frac{\mathbf{a}}{\varepsilon}\right), \varphi(\mathbf{a})\Big\rangle_{\mathbf{a}} = \frac{1}{\varepsilon^\beta L(\varepsilon)} \Big\langle DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}} f\left(\varepsilon^2 b, \mathbf{a}\right), e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}} \varphi(\varepsilon \mathbf{a})\Big\rangle_{\mathbf{a}}$$
$$= \frac{1}{\varepsilon^\beta L(\varepsilon)} \Big\langle f(\mathbf{x}), \overline{\psi}\left(\frac{\mathbf{x}}{\varepsilon} \cdot \mathbf{u} - \varepsilon b\right) \mathcal{F}_{\boldsymbol{\alpha}}\big(e^{-i\frac{c_1\cdot(a_1^2,\dots,a_n^2)}{2}} \varphi(\varepsilon \mathbf{a})\big)(\mathbf{x})\Big\rangle_{\mathbf{x}}$$

$$=\frac{\prod_{k=1}^{n}\sqrt{1-ic_1^{(k)}}}{\varepsilon^{\beta}L(\varepsilon)}\left\langle f(\varepsilon\mathbf{x}),\overline{\psi}\,(\mathbf{x}\cdot\mathbf{u}-\varepsilon b)\,e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\ldots,x_n^2)}\mathcal{F}\varphi(x_1c_2^{(1)},\ldots,x_nc_2^{(n)})\right\rangle_{\mathbf{x}}$$
$$\to\prod_{k=1}^{n}\sqrt{1-ic_1^{(k)}}\left\langle g(\mathbf{x}),\overline{\psi}(\mathbf{x}\cdot\mathbf{u})\mathcal{F}\varphi(x_1c_2^{(1)},\ldots,x_nc_2^{(n)})\right\rangle_{\mathbf{x}}$$
$$=\prod_{k=1}^{n}\sqrt{1-ic_1^{(k)}}\left\langle DS_{\psi,\mathbf{u}}g(0,(a_1c_2^{(1)},\ldots,a_nc_2^{(n)})),\varphi(\mathbf{a})\right\rangle_{\mathbf{a}},$$

as $\varepsilon\to 0^+$. □

Remark 5.6. *The limit (30) holds uniformly for b in bounded subsets of $\mathbb{R}$.*

The following proposition is a Tauberian counterpart of Prop. 5.5.

Proposition 5.7. *Let $f\in\mathcal{S}'(\mathbb{R}^n)$ and $\psi\in\mathcal{S}(\mathbb{R})\setminus\{0\}$. Suppose that the following two conditions hold:*

(i) The limit

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{n+\beta-1}L(\varepsilon)}\left\langle e^{-i\frac{c_1\cdot(a_1^2,\ldots,a_n^2)}{2\varepsilon^2}}DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}}f\left(\varepsilon^2 b,\frac{\mathbf{a}}{\varepsilon}\right),\varphi(\mathbf{a})\right\rangle_{\mathbf{a}}=M_b(\varphi)$$

exists and is finite for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $b\in\mathbb{R}$;

(ii) There exists $l>1$ such that for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$,

$$\frac{1}{\varepsilon^{n+\beta-1}L(\varepsilon)}\left|\left\langle e^{-i\frac{c_1\cdot(a_1^2,\ldots,a_n^2)}{2\varepsilon^2}}DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}}f\left(\varepsilon^2 b,\frac{\mathbf{a}}{\varepsilon}\right),\varphi(\mathbf{a})\right\rangle_{\mathbf{a}}\right|\le C_{\varphi}(1+|b|)^{-l}$$

for all $b\in\mathbb{R}$ and $0<\varepsilon<1$.

Then, there exists a homogeneous distribution $g\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$f(\varepsilon\mathbf{x})\sim\varepsilon^{\beta}L(\varepsilon)g(\mathbf{x})\ \text{ as }\ \varepsilon\to 0^+\ \text{in}\ \mathcal{S}'(\mathbb{R}^n).$$

Proof. Using (15) for $\boldsymbol{\alpha}=(\pi/2,\ldots,\pi/2)$, $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $\Psi\in\mathcal{S}(\mathbb{R})$, (6), and the change of variables, we get

$$\frac{1}{\varepsilon^{\beta}L(\varepsilon)}\left\langle DS_{\psi,\mathbf{u}}\left(e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\ldots,x_n^2)}f(\varepsilon\mathbf{x})\right),\varphi\Psi\right\rangle=\frac{1}{\varepsilon^{\beta}L(\varepsilon)}\int_{\mathbb{R}}\langle e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\ldots,x_n^2)}f(\varepsilon\mathbf{x}),\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}\varphi(\mathbf{x})\rangle_{\mathbf{x}}\Psi(b)db$$
$$=\prod_{k=1}^{n}\frac{|c_2^{(k)}|}{\sqrt{1-ic_1^{(k)}}}\frac{1}{\varepsilon^{\beta-n}L(\varepsilon)}\int_{\mathbb{R}}\left\langle f(\varepsilon\mathbf{x}),\overline{\psi}(\mathbf{x}\cdot\mathbf{u}-b)\mathcal{F}_{\alpha}\left(e^{-i\frac{c_1\cdot(a_1^2,\ldots,a_n^2)}{2}}\varphi(\varepsilon a_1c_2^{(1)},\ldots,\varepsilon a_nc_2^{(n)})\right)(\varepsilon\mathbf{x})\right\rangle_{\mathbf{x}}\Psi(b)db$$
$$=\prod_{k=1}^{n}\frac{|c_2^{(k)}|}{\sqrt{1-ic_1^{(k)}}}\frac{1}{\varepsilon^{\beta}L(\varepsilon)}\int_{\mathbb{R}}\left\langle f(\mathbf{x}),\overline{\psi}_{1/\varepsilon}(\mathbf{x}\cdot\mathbf{u}-\varepsilon b)\mathcal{F}_{\alpha}\left(e^{-i\frac{c_1\cdot(a_1^2,\ldots,a_n^2)}{2}}\varphi(\varepsilon a_1c_2^{(1)},\ldots,\varepsilon a_nc_2^{(n)})\right)(\mathbf{x})\right\rangle_{\mathbf{x}}\Psi(b)db$$
$$=\prod_{k=1}^{n}\frac{|c_2^{(k)}|}{\sqrt{1-ic_1^{(k)}}}\frac{1}{\varepsilon^{\beta}L(\varepsilon)}\int_{\mathbb{R}}\left\langle DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}}f(\varepsilon b,\mathbf{a}),e^{-i\frac{c_1\cdot(a_1^2,\ldots,a_n^2)}{2}}\varphi(\varepsilon a_1c_2^{(1)},\ldots,\varepsilon a_nc_2^{(n)})\right\rangle_{\mathbf{a}}\Psi(b)db$$
$$=\prod_{k=1}^{n}\frac{|c_2^{(k)}|}{\sqrt{1-ic_1^{(k)}}}\frac{1}{\varepsilon^{n+\beta}L(\varepsilon)}\int_{\mathbb{R}}\left\langle e^{-i\frac{c_1\cdot(a_1^2,\ldots,a_n^2)}{2\varepsilon^2}}DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}}f\left(\varepsilon b,\frac{\mathbf{a}}{\varepsilon}\right),\varphi(a_1c_2^{(1)},\ldots,a_nc_2^{(n)})\right\rangle_{\mathbf{a}}\Psi(b)db$$
$$=\prod_{k=1}^{n}\frac{|c_2^{(k)}|}{\sqrt{1-ic_1^{(k)}}}\frac{1}{\varepsilon^{n+\beta-1}L(\varepsilon)}\int_{\mathbb{R}}\left\langle e^{-i\frac{c_1\cdot(a_1^2,\ldots,a_n^2)}{2\varepsilon^2}}DS^{\boldsymbol{\alpha}}_{\psi_{1/\varepsilon},\mathbf{u}}f\left(\varepsilon^2 b,\frac{\mathbf{a}}{\varepsilon}\right),\varphi(a_1c_2^{(1)},\ldots,a_nc_2^{(n)})\right\rangle_{\mathbf{a}}\Psi(\varepsilon b)db$$

$$\to \prod_{k=1}^{n} \frac{|c_2^{(k)}|}{\sqrt{1 - ic_1^{(k)}}} \int_{\mathbb{R}} M_b(\varphi)\Psi(0)db, \ \ as \ \ \varepsilon \to 0^+,$$

where the convergence follows from the Lebesgue convergence theorem and the assumptions (i) and (ii). Now since $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n) = \mathcal{S}(\mathbb{R})\hat{\otimes}\mathcal{S}(\mathbb{R}^n)$, we may conclude that $\left\{\frac{1}{\varepsilon^\beta L(\varepsilon)} DS_{\psi,\mathbf{u}}\left(e^{i\frac{\varepsilon^2}{2}c_1\cdot(x_1^2,\dots,x_n^2)} f(\varepsilon\mathbf{x})\right)\right\}_{0<\varepsilon<1}$ converges as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)$. Then, similar as in Prop. 5.4, the inversion formula (14) implies that $\{\frac{1}{\varepsilon^\beta L(\varepsilon)} f(\varepsilon\mathbf{x})\}_{0<\varepsilon<1}$ converges as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. By the Banach-Steinhaus theorem ([30], p. 348), there exists a homogeneous distribution $g \in \mathcal{S}'(\mathbb{R}^n)$ such that $f(\varepsilon\mathbf{x}) \sim \varepsilon^\beta L(\varepsilon) g(\mathbf{x})$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$. □

Remark 5.8. *In general, similar results* ***do not*** *hold for asymptotics at infinity (see Remark 3.2 in [2]).*

Disclosure statement

The authors report there are no competing interests to declare.

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