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The impact of monotone functions on dynamic inequalities of Hardy type with a negative parameter on time scales

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Abstract. In this manuscript, we establish various novel Hardy-type inequalities involving a single negative parameter and monotone functions on time scales. In the continuous case, our results reduce to the integral inequalities proved by Benaissa, Sarikaya [6] and Azzouz et al. [5], while in the discrete or quantum case, the inequalities obtained are fundamentally new.

1. Introduction

In 1920, Hardy [9] established the well-known discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\varsigma(i)\right)^{ł} \le \left(\frac{ł}{ł-1}\right)^{ł}\sum_{n=1}^{\infty}\varsigma^{ł}(n), \tag{1}$$

where $ł > 1$ and $\{\varsigma(n)\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers.

In 1925, Hardy [10, Theorem A] gave a continuous analogue of (1) and proved that if $ł > 1$ and $\Theta \ge 0$ is a $ł-$integrable function on $(0,\infty)$, then Θ is integrable over any finite interval $(0,\varphi)$ for each $\varphi \in (0,\infty)$ and

$$\int_0^{\infty}\left(\frac{1}{\varphi}\int_0^{\varphi}\Theta(\vartheta)d\vartheta\right)^{ł} d\varphi \le \left(\frac{ł}{ł-1}\right)^{ł}\int_0^{\infty}\Theta^{ł}(\varphi)d\varphi. \tag{2}$$

The constant $(ł/(ł-1))^{ł}$ in both (1) and (2) is sharp, i.e., it cannot be lowered to a smaller one without affecting the validity of (1) and (2) for all possible sequences and functions.

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In [13] Leindler generalized the discrete Hardy inequality (1) and proved that if $\text{ł} > 1$, $\lambda(n), \varpi(n) > 0$, then

$$\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{s=1}^{n} \varpi(s) \right)^{\text{ł}} \leq \text{ł}^{\text{ł}} \sum_{n=1}^{\infty} \lambda^{1-\text{ł}}(n) \left(\sum_{s=n}^{\infty} \lambda(s) \right)^{\text{ł}} \varpi^{\text{ł}}(n). \tag{3}$$

In 2007, Bicheng [7] established some inequalities of Hardy type with a negative parameter ł and proved that if $\text{ł} < 0$, $\iota < 1$, $\Theta > 0$ and $0 < \int_0^{\infty} \varphi^{-\iota} \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \mathrm{d}\varphi < \infty$, then

$$\int_0^{\infty} \varphi^{-\iota} \left(\int_0^{\varphi} \Theta(\vartheta) \mathrm{d}\vartheta \right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{\iota - 1} \right)^{\text{ł}} \int_0^{\infty} \varphi^{-\iota} \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \mathrm{d}\varphi, \tag{4}$$

and if $\text{ł} < 0$, $\iota > 1$, $\Theta > 0$ and $0 < \int_0^{\infty} \varphi^{-\iota} \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \mathrm{d}\varphi < \infty$, then

$$\int_0^{\infty} \varphi^{-\iota} \left(\int_{\varphi}^{\infty} \Theta(\vartheta) \mathrm{d}\vartheta \right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{1 - \iota} \right)^{\text{ł}} \int_0^{\infty} \varphi^{-\iota} \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \mathrm{d}\varphi, \tag{5}$$

where the constants $\left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}$ and $\left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}}$, respectively, are best possible.

In 2020, Benaissa and Sarikaya [6] generalized (5) and proved that, for $\text{ł} < 0$ and any $\Theta, \varpi > 0$ on $(0, \infty)$ with $0 < \int_0^{\infty} \varpi^{-\iota}(\varphi) \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \mathrm{d}\varphi < \infty$,

$$\int_0^{\infty} \varpi^{-\iota}(\varphi) \left(\int_{\varphi}^{\infty} \Theta(\vartheta) \mathrm{d}\vartheta \right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{1 - \iota} \right)^{\text{ł}} \int_0^{\infty} \varpi^{-\iota}(\varphi) \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \mathrm{d}\varphi \tag{6}$$

provided $\iota > 1$ and $\varphi/\varpi(\varphi)$ is nondecreasing on $(0, \infty)$, see [6, Theorem 1], and

$$\int_0^{\infty} \varpi^{-\iota}(\varphi) \left(\int_0^{\varphi} \Theta(\vartheta) \mathrm{d}\vartheta \right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{\iota - 1} \right)^{\text{ł}} \int_0^{\infty} \varpi^{-\iota}(\varphi) \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \mathrm{d}\varphi \tag{7}$$

provided either $0 \leq \iota < 1$ and $\varphi/\varpi(\varphi)$ is nonincreasing on $[0, \infty)$, see [6, Theorem 2], or $\iota < 0$ and $\varphi/\varpi(\varphi)$ is nondecreasing on $[0, \infty)$, see [6, Theorem 3].

In 2023, Azzouz et al. [5] generalized (6) and (7) and showed that for $0 \leq \varsigma < \varrho \leq \infty$, $\text{ł} < 0$ and $\Theta, \varpi > 0$ on (ς, ϱ),

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left(\int_{\varphi}^{\varrho} \Theta(\vartheta) \mathrm{d}\vartheta \right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{1 - \iota} \right)^{\text{ł}} \int_{\varsigma}^{\varrho} \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \varpi^{-\iota}(\varphi) \left(1 - \left(\frac{\varphi}{\varrho} \right)^{\frac{1-\iota}{\text{ł}}} \right)^{\text{ł}-1} \mathrm{d}\varphi \tag{8}$$

provided $\iota > 1$ and $\varphi/\varpi(\varphi)$ is nondecreasing on (ς, ϱ), see [5, Theorem 3], and

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left(\int_{\varsigma}^{\varphi} \Theta(\vartheta) \mathrm{d}\vartheta \right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{\iota - 1} \right)^{\text{ł}} \int_{\varsigma}^{\varrho} \left(\varphi \Theta(\varphi)\right)^{\text{ł}} \varpi^{-\iota}(\varphi) \left(1 - \left(\frac{\varsigma}{\varphi} \right)^{\frac{\iota-1}{\text{ł}}} \right)^{\text{ł}-1} \mathrm{d}\varphi \tag{9}$$

provided either $0 \leq \iota < 1$ and $\varphi/\varpi(\varphi)$ is nonincreasing on (ς, ϱ) or $\iota < 0$ and $\varphi/\varpi(\varphi)$ is nondecreasing on (ς, ϱ), see [5, Theorem 4].

As an auxiliary result, the authors in [5] also proved that for $0 \leq \varsigma < \varrho < \infty$, $\text{ł} < 0$, $\eta > 0$, $\iota \in \mathbb{R}$ and $\Theta, \varpi > 0$ on (ς, ϱ)

$$\left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \mathrm{d}\varphi \right)^{\frac{\eta-\text{ł}}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left[(\varphi - \varsigma) \Theta(\varphi) \right]^{\eta} \mathrm{d}\varphi \right)^{\frac{\text{ł}}{\eta}} \leq \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left(\int_{\varsigma}^{\varphi} \Theta(\vartheta) \mathrm{d}\vartheta \right)^{\text{ł}} \mathrm{d}\varphi, \tag{10}$$

provided Θ is nondecreasing on (ς, ϱ), and

$$\left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\mathrm{d}\varphi\right)^{\frac{\eta-\text{ł}}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho-\varphi)\,\Theta(\varphi)\right]^{\eta}\mathrm{d}\varphi\right)^{\frac{\text{ł}}{\eta}} \leq \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\int_{\varphi}^{\varrho}\Theta(\vartheta)\mathrm{d}\vartheta\right)^{\text{ł}}\mathrm{d}\varphi. \tag{11}$$

provided Θ is nonincreasing on (ς, ϱ), see [5, Theorem 5].

Finally, combining (9) with (10) and (8) with (11), they obtained their main result which states that for $0 \leq \varsigma < \varrho < \infty$, $\text{ł} < 0$, $\eta > 0$ and $\Theta, \varpi > 0$ on (ς, ϱ), the following inequalities hold:
(a) If Θ is nondecreasing and either $0 \leq \iota < 1$ and $\varphi/\varpi(\varphi)$ is nonincreasing or $\iota < 0$, and $\varphi/\varpi(\varphi)$ is nondecreasing, then

$$\begin{aligned}&\left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\mathrm{d}\varphi\right)^{\frac{\eta-\text{ł}}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varphi-\varsigma)\,\Theta(\varphi)\right]^{\eta}\mathrm{d}\varphi\right)^{\frac{\text{ł}}{\eta}}\\ &\leq\left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}\left(\varphi\Theta(\varphi)\right)^{\text{ł}}\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right)^{\text{ł}-1}\mathrm{d}\varphi,\end{aligned} \tag{12}$$

see [5, Corollary 2].
(b) If $\iota > 1$, Θ is nonincreasing and $\varphi/\varpi(\varphi)$ is nondecreasing, then

$$\begin{aligned}&\left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\mathrm{d}\varphi\right)^{\frac{\eta-\text{ł}}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho-\varphi)\,\Theta(\varphi)\right]^{\eta}\mathrm{d}\varphi\right)^{\frac{\text{ł}}{\eta}}\\ &\leq\left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}\left(\varphi\Theta(\varphi)\right)^{\text{ł}}\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\mathrm{d}\varphi,\end{aligned} \tag{13}$$

see [5, Corollary 3].

In view of all the above, the interesting question arises: can we state a continuous (integral) analogue of Leindler's result (3) as well as the corresponding formula in, e.g., quantum calculus - also known as a calculus without limits? A positive answer will certainly include the use of time scale calculus - a tool, introduced by Hilger in his seminal work [12] to unify continuous and discrete analysis (i.e., the theories of differential equations and difference equations) which also opens the possibility of formulating additional discrete analogues of the result in diffferent less standard calculi.

By a time scale $\mathbb{T}$, we understand an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ (with the subspace topology inherited from the standard topology on $\mathbb{R}$). Notably, considerable attention has been directed towards Hardy-type inequalities on time scales, see the book by Agarwal et al. [1] and the papers [3, 4, 14–19]. In particular, Saker [16] established the time scale version of Leindler's inequality (3) and proved that for $\varsigma \in \mathbb{T}$, $\text{ł} > 1$, $\lambda, \varpi \in C_{\mathrm{rd}}([\varsigma, \infty)_{\mathbb{T}}], \mathbb{R}^{+})$ such that

$$\Lambda(\vartheta) = \int_{\vartheta}^{\infty} \lambda(s)\Delta s \quad \text{and} \quad \Phi(\vartheta) = \int_{\varsigma}^{\vartheta} \varpi(s)\Delta s \quad \text{for any} \quad \vartheta \in [\varsigma, \infty)_{\mathbb{T}},$$

then

$$\int_{\varsigma}^{\infty} \lambda(\vartheta)(\Phi^{\sigma}(\vartheta))^{\text{ł}}\Delta\vartheta \leq \text{ł}^{\text{ł}}\int_{\varsigma}^{\infty}\lambda^{1-\text{ł}}(\vartheta)\varpi^{\text{ł}}(\vartheta)\Lambda^{\text{ł}}(\vartheta)\Delta\vartheta. \tag{14}$$

Clearly, in this case, the one-line time scale inequality (14) takes on different forms depending on the time scale. Specifically, when $\mathbb{T} = \mathbb{N}$, it yields the discrete inequality (3). For $\mathbb{T} = \mathbb{R}$, it provides the continuous analogue of (3). Lastly, when $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, the formula applicable in quantum calculus is obtained.

However, the derivation of dynamic inequalities on time scales can often result in cases that need to be treated separately. As a matter of fact, the intricacies of time-scale calculus can lead to final results that are not straightforward (1:1 extensions of their continuous/discrete counterparts) but rather need to be analyzed within distinct scenarios. This case-by-case analysis is crucial for ensuring the accuracy and applicability of the inequalities across different types of time scales.

Building on this trend, the aim of this paper is to generalize contiunous inequalities (12) and (13) by Azzouz et al. [5] to their dynamic counterparts. By carefully employing of classical time-scale calculus techniques, we will obtain novel forms of the Hardy-type dynamic inequalities involving negative parameters on time scales. Notably, it turns out that the monotonicity of the functions involved in the considered inequalities has a significant role in changing their resulting forms.

The paper is organized as follows. Section 2 introduces basic lemmas on time scales, including the chain rule, integration by parts, and the reversed Hölder inequality. Section 3 states our main results, preceded by a series of auxiliary Lemmas, generalizing, step by step, existing results known in the continuous case. Several comments and remarks are included to emphasize the distinctions resulting from the choice of time scale.

2. Auxiliary lemmas

Instead of repeating the basic facts of time scales and time scale notation, we refer the reader to the monograph by Bohner and Peterson [8] summarizes and organizes much of the theory. Herein, we only recall three fundamental time scale results, we will need to establish our main result: a chain rule, the integration rules on time scales and the reversed Hölder dynamic inequality, respectively.

Lemma 2.1 (Chain Rule, see [8, Theorem 1.87]). *Assume $v : \mathbb{R} \to \mathbb{R}$ is continuous, $v : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$, and $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[\vartheta, \sigma(\vartheta)]$ with*

$$(u \circ v)^{\Delta}(\vartheta) = u'(v(c))v^{\Delta}(\vartheta). \tag{15}$$

Theorem 2.2 (See [8, Theorem 1.77]). *If $\varsigma, \varrho, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $u, v \in C_{\mathrm{rd}}([\varsigma, \varrho]_{\mathbb{T}}, \mathbb{R})$, then*

1. $\int_{\varsigma}^{\varrho} [\alpha u(\vartheta) + \beta v(\vartheta)] \Delta\vartheta = \alpha \int_{\varsigma}^{\varrho} u(\vartheta)\Delta\vartheta + \beta \int_{\varsigma}^{\varrho} v(\vartheta)\Delta\vartheta$;
2. $\int_{\varsigma}^{\varrho} u(\vartheta)\Delta\vartheta = -\int_{\varrho}^{\varsigma} u(\vartheta)\Delta\vartheta$;
3. $\int_{\varsigma}^{\varrho} u(\vartheta)\Delta\vartheta = \int_{\varsigma}^{c} u(\vartheta)\Delta\vartheta + \int_{c}^{\varrho} u(\vartheta)\Delta\vartheta$;
4. $\int_{\varsigma}^{\varsigma} u(\vartheta)\Delta\vartheta = 0$;
5. $\left|\int_{\varsigma}^{\varrho} u(\vartheta)\Delta\vartheta\right| \leq \int_{\varsigma}^{\varrho} |u(\vartheta)| \Delta\vartheta$;
6. *$u(\vartheta) \geq 0$ for all $\vartheta \in [\varsigma, \varrho]_{\mathbb{T}}$, implies $\int_{\varsigma}^{\varrho} u(\vartheta)\Delta\vartheta \geq 0$;*
7. *the integration by parts rule*

$$\int_{\varsigma}^{\varrho} u(\vartheta)v^{\Delta}(\vartheta)\Delta\vartheta = [u(\vartheta)v(\vartheta)]_{\varsigma}^{\varrho} - \int_{\varsigma}^{\varrho} u^{\Delta}(\vartheta)v^{\sigma}(\vartheta)\Delta\vartheta \tag{16}$$

holds.

Lemma 2.3 (Reversed Hölder inequality, see [2, Theorem 6.2]). *If $\varsigma, \varrho \in \mathbb{T}$ and $\Theta, \varpi \in C_{rd}([\varsigma, \varrho]_{\mathbb{T}}, \mathbb{R}^{+})$, then*

$$\int_{\varsigma}^{\varrho} \Theta(\vartheta)\varpi(\vartheta)\Delta\vartheta \geq \left[\int_{\varsigma}^{\varrho} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right]^{\frac{1}{\text{ł}}} \left[\int_{\varsigma}^{\varrho} \varpi^{\text{ł}^*}(\vartheta)\Delta\vartheta\right]^{\frac{1}{\text{ł}^*}}, \tag{17}$$

where $\text{ł} < 0$ and $\text{ł}^* = \text{ł}/(\text{ł} - 1)$.

In the proofs of our results, we will make use of the following two elementary algebraic inequalities

$$\begin{aligned} &\lambda\varsigma^{\lambda-1}(\varrho - \varsigma) \leq \varrho^{\lambda} - \varsigma^{\lambda} \leq \lambda\varrho^{\lambda-1}(\varrho - \varsigma) \quad \text{for } \lambda \geq 1 \text{ or } \lambda < 0, \\ &\lambda\varrho^{\lambda-1}(\varrho - \varsigma) \leq \varrho^{\lambda} - \varsigma^{\lambda} \leq \lambda\varsigma^{\lambda-1}(\varrho - \varsigma) \quad \text{for } 0 < \lambda < 1, \end{aligned} \tag{18}$$

which are valid for $\varrho \geq \varsigma > 0$, see [11, Theorem 41]. For the sake of convenience, we also state a basic assumption that all the integrals considered are well defined.

3. Main Results

To start with, we establish the time scale version of inequalities (10) and (11) (see below Lemma 3.2). As a needed preliminary, we state the following auxiliary result.

Lemma 3.1. *Assume that* $\varsigma, \varrho \in \mathbb{T}$, $\varsigma < \varrho$, $ł < 0$, $\eta > 0$ *and* $\phi,\ \psi \in C_{\mathrm{rd}}\left([\varsigma,\varrho]_{\mathbb{T}}, \mathbb{R}^{+}\right)$. *Then*

$$\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{ł}(\varphi)\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\Delta\varphi\right)^{\frac{\eta-ł}{\eta}} \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{\eta}(\varphi)\Delta\varphi\right)^{\frac{ł}{\eta}} \tag{19}$$

and

$$\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{\eta}(\varphi)\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\Delta\varphi\right)^{\frac{ł-\eta}{ł}} \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{ł}(\varphi)\Delta\varphi\right)^{\frac{\eta}{ł}}. \tag{20}$$

Proof. Applying (17) on R.H.S. of (19) with $\eta/ł < 0$ and $\eta/\left(\eta - ł\right)$, we see that

$$\begin{aligned}\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{ł}(\varphi)\Delta\varphi &= \int_{\varsigma}^{\varrho} \phi^{\frac{\eta-ł}{\eta}}(\varphi)\phi^{\frac{ł}{\eta}}(\varphi)\psi^{ł}(\varphi)\Delta\varphi \\ &\geq \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\Delta\varphi\right)^{\frac{\eta-ł}{\eta}} \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{\eta}(\varphi)\Delta\varphi\right)^{\frac{ł}{\eta}},\end{aligned}$$

which is (19). Again by applying (17) on R.H.S. of (20) with $ł/\eta < 0$ and $ł/\left(ł - \eta\right)$, we get

$$\begin{aligned}\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{\eta}(\varphi)\Delta\varphi &= \int_{\varsigma}^{\varrho} \phi^{\frac{ł-\eta}{ł}}(\varphi)\phi^{\frac{\eta}{ł}}(\varphi)\psi^{\eta}(\varphi)\Delta\varphi \\ &\geq \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\Delta\varphi\right)^{\frac{ł-\eta}{ł}} \left(\int_{\varsigma}^{\varrho} \phi(\varphi)\psi^{ł}(\varphi)\Delta\varphi\right)^{\frac{\eta}{ł}},\end{aligned}$$

which is (20). □

Lemma 3.2. *Assume* $\varsigma, \varrho \in \mathbb{T}$, $0 \leq \varsigma < \varrho < \infty$, $ł < 0$, $\eta > 0$, $\iota \in \mathbb{R}$ *and* $\Theta,\ \varpi \in C_{\mathrm{rd}}\left([\varsigma,\varrho]_{\mathbb{T}}, \mathbb{R}^{+}\right)$, *then*
(a) If Θ *is nondecreasing on* (ς, ϱ), *then*

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)F^{\eta}(\varphi)\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{ł-\eta}{ł}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho-\varphi)\Theta(\varphi)\right]^{ł}\Delta\varphi\right)^{\frac{\eta}{ł}}, \tag{21}$$

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\Omega^{\sigma}(\varphi)\right)^{ł}\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-ł}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi)-\varsigma)\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{ł}{\eta}}. \tag{22}$$

(b) If Θ *is nonincreasing on* (ς, ϱ), *then*

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)F^{ł}(\varphi)\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-ł}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho-\varphi)\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{ł}{\eta}}, \tag{23}$$

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\Omega^{\sigma}(\varphi)\right)^{\eta}\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{ł-\eta}{ł}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi)-\varsigma)\Theta(\varphi)\right]^{ł}\Delta\varphi\right)^{\frac{\eta}{ł}}, \tag{24}$$

where $F(\varphi) = \int_{\varphi}^{\varrho} \Theta(\vartheta)\Delta\vartheta$ *and* $\Omega(\varphi) = \int_{\varsigma}^{\varphi} \Theta(\vartheta)\Delta\vartheta$.

Proof. Let us prove part (a) first. Since Θ is nondecreasing on (ς, ϱ), then

$$F(\varphi) = \int_{\varphi}^{\varrho} \Theta(\vartheta)\Delta\vartheta \geq (\varrho - \varphi)\,\Theta(\varphi),$$

and then we have for $\eta > 0$ that

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) F^{\eta}(\varphi)\,\Delta\varphi \geq \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho - \varphi)\,\Theta(\varphi)\right]^{\eta}\Delta\varphi. \tag{25}$$

Applying (20) with $\phi(\varphi) = \varpi^{-\iota}(\varphi)$ and $\psi(\varphi) = (\varrho - \varphi)\,\Theta(\varphi)$ on R.H.S. of (25), we get

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) F^{\eta}(\varphi)\,\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\text{ł}-\eta}{\text{ł}}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho - \varphi)\,\Theta(\varphi)\right]^{\text{ł}}\Delta\varphi\right)^{\frac{\eta}{\text{ł}}},$$

which satisfies (21).

Note that

$$\Omega^{\sigma}(\varphi) = \int_{\varsigma}^{\sigma(\varphi)} \Theta(\vartheta)\Delta\vartheta = \int_{\varsigma}^{\varphi} \Theta(\vartheta)\Delta\vartheta + \int_{\varphi}^{\sigma(\varphi)} \Theta(\vartheta)\Delta\vartheta = \int_{\varsigma}^{\varphi} \Theta(\vartheta)\Delta\vartheta + \mu(\varphi)\,\Theta(\varphi). \tag{26}$$

Since Θ is nondecreasing on (ς, ϱ), we have from (26) that

$$\Omega^{\sigma}(\varphi) \leq (\varphi - \varsigma)\,\Theta(\varphi) + \mu(\varphi)\,\Theta(\varphi) = (\sigma(\varphi) - \varsigma)\,\Theta(\varphi),$$

and then we have for $\text{ł} < 0$, that

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\Omega^{\sigma}(\varphi)\right)^{\text{ł}}\Delta\varphi \geq \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi) - \varsigma)\,\Theta(\varphi)\right]^{\text{ł}}\Delta\varphi. \tag{27}$$

Applying (19) on R.H.S. of (27) with $\phi(\varphi) = \varpi^{-\iota}(\varphi)$ and $\psi(\varphi) = (\sigma(\varphi) - \varsigma)\,\Theta(\varphi)$, we get

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\Omega^{\sigma}(\varphi)\right)^{\text{ł}}\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-\text{ł}}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi) - \varsigma)\,\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{\text{ł}}{\eta}},$$

which is (22). Now, we are prepared to prove the part (b).
Since Θ is nonincreasing on (ς, ϱ), then

$$F(\varphi) = \int_{\varphi}^{\varrho} \Theta(\vartheta)\Delta\vartheta \leq (\varrho - \varphi)\,\Theta(\varphi),$$

and then we have for $\text{ł} < 0$ that

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) F^{\text{ł}}(\varphi)\,\Delta\varphi \geq \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho - \varphi)\,\Theta(\varphi)\right]^{\text{ł}}\Delta\varphi. \tag{28}$$

Applying (19) on R.H.S. of (28) with $\phi(\varphi) = \varpi^{-\iota}(\varphi)$ and $\psi(\varphi) = (\varrho - \varphi)\,\Theta(\varphi)$, we obtain

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) F^{\text{ł}}(\varphi)\,\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-\text{ł}}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\varrho - \varphi)\,\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{\text{ł}}{\eta}},$$

which is (23). To complete the proof, we prove (24).

Since Θ is nonincreasing on (ς, ϱ) and $\eta > 0$, we have from (26) that

$$\Omega^{\sigma}(\varphi) = \int_{\varsigma}^{\varphi} \Theta(\vartheta)\Delta\vartheta + \mu(\varphi)\Theta(\varphi) \geq (\varphi - \varsigma)\Theta(\varphi) + \mu(\varphi)\Theta(\varphi) = (\sigma(\varphi) - \varsigma)\Theta(\varphi),$$

and then

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\Omega^{\sigma}(\varphi)\right)^{\eta}\Delta\varphi \geq \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi) - \varsigma)\Theta(\varphi)\right]^{\eta}\Delta\varphi. \tag{29}$$

Applying (20) on R.H.S. of (29) with $\phi(\varphi) = \varpi^{-\iota}(\varphi)$ and $\psi(\varphi) = (\sigma(\varphi) - \varsigma)\Theta(\varphi)$, we have

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\Omega^{\sigma}(\varphi)\right)^{\eta}\Delta\varphi \geq \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\text{ł}-\eta}{\text{ł}}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi) - \varsigma)\Theta(\varphi)\right]^{\text{ł}}\Delta\varphi\right)^{\frac{\eta}{\text{ł}}},$$

which is (24). □

Corollary 3.3. *If* $\mathbb{T} = \mathbb{R}$, *then the inequalities* (22) *and* (23) *reduce to* (10) *and* (11), *respectively.*

Remark 3.4. *It is clear that the monotonicity of the function* Θ *in Lemma 3.2 has a clear role in changing the shape of inequalities and creating new and different inequalities.*

The following result can be seen as a time scale generalization of the integral inequality (6).

Lemma 3.5. *Assume* $\varsigma, \varrho \in \mathbb{T}$, $0 \leq \varsigma < \varrho < \infty$, $\text{ł} < 0$, $\iota > 1$ *and* $\Theta, \varpi \in C_{\mathrm{rd}}\left([\varsigma, \varrho]_{\mathbb{T}}, \mathbb{R}^{+}\right)$ *such that* $\varphi/\varpi(\varphi)$ *is nondecreasing on* (ς, ϱ). *Then*

$$\begin{aligned} &\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)F^{\text{ł}}(\varphi)\Delta\varphi \\ &\leq \left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}-1} \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(1 - \left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1} \left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1-\iota}{\text{ł}}-1}\Delta\vartheta\right)\Delta\varphi, \end{aligned} \tag{30}$$

where $\Theta(\varphi) = \int_{\varphi}^{\varrho} \Theta(\vartheta)\Delta\vartheta$.

Proof. Note that

$$\Theta(\varphi) = \int_{\varphi}^{\varrho} \Theta(\vartheta)\Delta\vartheta \qquad = \int_{\varphi}^{\varrho} \vartheta^{-\frac{1+\text{ł}-\iota}{\text{łł}^*}}\left[\vartheta^{\frac{1+\text{ł}-\iota}{\text{łł}^*}}\Theta(\vartheta)\right]\Delta\vartheta, \tag{31}$$

where $\text{ł}^* = \text{ł}/(\text{ł}-1)$. Applying Lemma (2.3) on R.H.S. of (31), we get

$$\int_{\varphi}^{\varrho} \vartheta^{-\frac{1+\text{ł}-\iota}{\text{łł}^*}}\left[\vartheta^{\frac{1+\text{ł}-\iota}{\text{łł}^*}}\Theta(\vartheta)\right]\Delta\vartheta \geq \left(\int_{\varphi}^{\varrho} \vartheta^{\frac{\iota-1-\text{ł}}{\text{ł}}}\Delta\vartheta\right)^{\frac{1}{\text{ł}^*}} \left(\int_{\varphi}^{\varrho} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\text{ł}}}. \tag{32}$$

Applying (15) with $v(\vartheta) = \vartheta$ and $u(\varphi) = \varphi^{\frac{\iota-1}{\text{ł}}}$, we see that

$$(u \circ v)^{\Delta}(\vartheta) = u'(v(c))v^{\Delta}(\vartheta) = \frac{\iota-1}{\text{ł}}c^{\frac{\iota-1-\text{ł}}{\text{ł}}} \quad \text{for some } c \in [\vartheta, \sigma(\vartheta)]. \tag{33}$$

Since $(\iota - 1 - \text{ł})/\text{ł} < 0$, then $c^{\frac{\iota-1-\text{ł}}{\text{ł}}} \leq \vartheta^{\frac{\iota-1-\text{ł}}{\text{ł}}}$ and (33) becomes

$$(u \circ v)^{\Delta}(\vartheta) \geq \frac{\iota-1}{\text{ł}}\vartheta^{\frac{\iota-1-\text{ł}}{\text{ł}}}. \tag{34}$$

Integrating (34) over ϑ from φ to ϱ, (note $\iota > 1$ and $\text{ł} < 0$), we observe that

$$\int_{\varphi}^{\varrho} \vartheta^{\frac{\iota-1-\text{ł}}{\text{ł}}} \Delta\vartheta \geq \frac{\text{ł}}{\iota-1} \int_{\varphi}^{\varrho} (u \circ v)^{\Delta}(\vartheta)\Delta\vartheta = \frac{\text{ł}}{\iota-1} \int_{\varphi}^{\varrho} \left[(u \circ v)(\varrho) - (u \circ v)(\varphi)\right] = \frac{\text{ł}}{1-\iota}\left(\varphi^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right).$$

Since $\text{ł}^{*} > 0$, then we have that

$$\left(\int_{\varphi}^{\varrho} \vartheta^{\frac{\iota-1-\text{ł}}{\text{ł}}} \Delta\vartheta\right)^{\frac{1}{\text{ł}^{*}}} \geq \left(\frac{\text{ł}}{1-\iota}\right)^{\frac{1}{\text{ł}^{*}}} \left(\varphi^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\frac{1}{\text{ł}^{*}}}. \tag{35}$$

Take in account (32) and (35), we see that

$$\int_{\varphi}^{\varrho} \vartheta^{-\frac{1+\text{ł}-\iota}{\text{łł}^{*}}} \left[\vartheta^{\frac{1+\text{ł}-\iota}{\text{łł}^{*}}} \Theta(\vartheta)\right] \Delta\vartheta \geq \left(\frac{\text{ł}}{1-\iota}\right)^{\frac{1}{\text{ł}^{*}}} \left(\varphi^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\frac{1}{\text{ł}^{*}}} \left(\int_{\varphi}^{\varrho} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\text{ł}}}. \tag{36}$$

From (31) and (36), we have for $\text{ł} < 0$ that

$$F^{\text{ł}}(\varphi) \leq \left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}-1} \left(\varphi^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\text{ł}-1} \int_{\varphi}^{\varrho} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta. \tag{37}$$

Multiplying (37) by $\varpi^{-\iota}(\varphi)$ and then integrating over φ from ς to ϱ, we get

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) F^{\text{ł}}(\varphi)\Delta\varphi \leq \left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}-1} \int_{\varsigma}^{\varrho} \left(\varphi^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\text{ł}-1} \varpi^{-\iota}(\varphi) \left(\int_{\varphi}^{\varrho} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi. \tag{38}$$

Applying (16) on R.H.S. of (38), with

$$u(\varphi) = \int_{\varphi}^{\varrho} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta \quad \text{and} \quad v(\varphi) = \int_{\varsigma}^{\varphi} \left(\vartheta^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\text{ł}-1} \varpi^{-\iota}(\vartheta)\Delta\vartheta,$$

we have that

$$\begin{aligned}
&\int_{\varsigma}^{\varrho} \left(\varphi^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\text{ł}-1} \varpi^{-\iota}(\varphi) \left(\int_{\varphi}^{\varrho} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
&= \int_{\varsigma}^{\varrho} u(\vartheta) v^{\Delta}(\vartheta)\Delta\vartheta = u(\varrho)v(\varrho) - u(\varsigma)v(\varsigma) - \int_{\varsigma}^{\varrho} u^{\Delta}(\vartheta) v^{\sigma}(\vartheta)\Delta\vartheta \\
&= -\int_{\varsigma}^{\varrho} u^{\Delta}(\vartheta) v^{\sigma}(\vartheta)\Delta\vartheta = \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\varphi) v^{\sigma}(\varphi)\Delta\varphi \\
&= \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\varphi) \left(\int_{\varsigma}^{\sigma(\varphi)} \left(\vartheta^{\frac{\iota-1}{\text{ł}}} - \varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\text{ł}-1} \varpi^{-\iota}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
&= \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^{*}}} \Theta^{\text{ł}}(\varphi) \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota} \left[1 - \left(\frac{\vartheta}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1} \left[\frac{\vartheta}{\varpi(\vartheta)}\right]^{\iota} \Delta\vartheta\right)\Delta\varphi.
\end{aligned} \tag{39}$$

Note that

$$\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota} \left[1 - \left(\frac{\vartheta}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1} \left[\frac{\vartheta}{\varpi(\vartheta)}\right]^{\iota} \Delta\vartheta \tag{40}$$

$$\begin{aligned}
= \quad & \int_{\varsigma}^{\varphi} \vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota} \left[1 - \left(\frac{\vartheta}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1} \left[\frac{\vartheta}{\varpi(\vartheta)}\right]^{\iota} \Delta + \int_{\varphi}^{\sigma(\varphi)} \vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota} \left[1 - \left(\frac{\vartheta}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1} \left[\frac{\vartheta}{\varpi(\vartheta)}\right]^{\iota} \Delta\vartheta \\
= \quad & \int_{\varsigma}^{\varphi} \vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota} \left[1 - \left(\frac{\vartheta}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1} \left[\frac{\vartheta}{\varpi(\vartheta)}\right]^{\iota} \Delta\vartheta + \mu(\varphi) \left[1 - \left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1} \left[\frac{\varphi}{\varpi(\varphi)}\right]^{\iota} \varphi^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}.
\end{aligned}$$

Since $1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}$ is nonincreasing on (ς,ϱ) and $\text{ł}<0$, $\vartheta/\varpi(\vartheta)$ is nondecreasing on (ς,ϱ) and $\iota>1$, we have

$$\begin{aligned}&\int_{\varsigma}^{\varphi}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\left[1-\left(\frac{\vartheta}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1}\left[\frac{\vartheta}{\varpi(\vartheta)}\right]^{\iota}\Delta\vartheta\\ \leq\;&\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\int_{\varsigma}^{\varphi}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta.\end{aligned}\tag{41}$$

Substituting (41) into (40), we obtain

$$\begin{aligned}&\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\left[1-\left(\frac{\vartheta}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right]^{\text{ł}-1}\left[\frac{\vartheta}{\varpi(\vartheta)}\right]^{\iota}\Delta\vartheta\\ \leq\;&\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\int_{\varsigma}^{\varphi}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta\\ &+\mu(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\varphi^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\\ =\;&\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(\int_{\varsigma}^{\varphi}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta+\varphi^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\mu(\varphi)\right)\\ =\;&\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(\int_{\varsigma}^{\varphi}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta+\int_{\varphi}^{\sigma(\varphi)}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta\right)\\ =\;&\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta.\end{aligned}\tag{42}$$

Substituting (42) into (39), we observe that

$$\begin{aligned}&\int_{\varsigma}^{\varrho}\left(\varphi^{\frac{\iota-1}{\text{ł}}}-\varrho^{\frac{\iota-1}{\text{ł}}}\right)^{\text{ł}-1}\varpi^{-\iota}(\varphi)\left(\int_{\varphi}^{\varrho}\vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi\\ \leq\;&\int_{\varsigma}^{\varrho}\varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta\right)\Delta\varphi.\end{aligned}\tag{43}$$

Take in account (38) and (43), we get

$$\begin{aligned}&\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)F^{\text{ł}}(\varphi)\Delta\varphi\\ &\leq\left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}-1}\int_{\varsigma}^{\varrho}\varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{(\iota-1)(\text{ł}-1)}{\text{ł}}-\iota}\Delta\vartheta\right)\Delta\varphi\\ &=\left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}-1}\int_{\varsigma}^{\varrho}\varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{1-\iota}{\text{ł}}-1}\Delta\vartheta\right)\Delta\varphi,\end{aligned}$$

which is (30). □

Corollary 3.6. *If $\mathbb{T} = \mathbb{R}$, $\varsigma, \varrho \in \mathbb{T}$, $0 \leq \varsigma < \varrho < \infty$, $\text{ł} < 0$, $\iota > 1$ and Θ, ϖ are positive continuous functions such that $\varphi/\varpi(\varphi)$ is nondecreasing on (ς, ϱ), then*

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\int_{\varphi}^{\varrho} \Theta(\vartheta)\mathrm{d}\vartheta\right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}} \int_{\varsigma}^{\varrho} \varphi^{\text{ł}}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1} \mathrm{d}\varphi. \tag{44}$$

Proof. For the continuous case of (30), we see that

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\int_{\varphi}^{\varrho} \Theta(\vartheta)\mathrm{d}\vartheta\right)^{\text{ł}} \mathrm{d}\varphi$$
$$\leq \left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}-1} \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1} \left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\left(\int_{\varsigma}^{x} \vartheta^{\frac{1-\iota}{\text{ł}}-1}\mathrm{d}\vartheta\right)\mathrm{d}\varphi. \tag{45}$$

Since

$$\int_{\varsigma}^{x} \vartheta^{\frac{1-\iota}{\text{ł}}-1}\mathrm{d}\vartheta = \frac{\text{ł}}{1-\iota}\left[\varphi^{\frac{1-\iota}{\text{ł}}} - \varsigma^{\frac{1-\iota}{\text{ł}}}\right] \leq \frac{\text{ł}}{1-\iota}\varphi^{\frac{1-\iota}{\text{ł}}},$$

the inequality (45) becomes

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left(\int_{\varphi}^{\varrho} \Theta(\vartheta)\mathrm{d}\vartheta\right)^{\text{ł}} \mathrm{d}\varphi \leq \left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}} \int_{\varsigma}^{\varrho} \varphi^{\text{ł}}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1} \mathrm{d}\varphi,$$

which is (44). □

By similar steps than before, one can easily obtain the following.

Remark 3.7. *If $\mathbb{T} = \mathbb{R}$, $\varsigma = 0$ and $\varrho = \infty$, then the inequality* (44) *reduces to* (6).

In the discrete case, the estimation of the dynamic integral $\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1-\iota}{\text{ł}}-1}\Delta\vartheta$ always requires consideration of two cases: $1-\iota \leq p$ and $1-\iota \geq p$.

Remark 3.8. *Combining Corollaries 3.3 and 3.6, we obtain the classical integral inequality* (13) *proved by Azzouz et al. [5].*

Corollary 3.9. *If $\mathbb{T} = \mathbb{N}$, $\varsigma, \varrho \in \mathbb{N}$, $1 \leq \varsigma < \varrho < \infty$, $\text{ł} < 0$, $\iota > 1$ and Θ, ϖ are positive sequences such that $\varphi/\varpi(\varphi)$ is nondecreasing on (ς, ϱ), then*
(a) For $1-\iota \leq p$,

$$\sum_{\varphi=\varsigma}^{\varrho-1} \varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho-1} \Theta(\vartheta)\right)^{\text{ł}} \leq \left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}}\left(\frac{\varsigma+1}{\varsigma}\right)^{\frac{1-\iota}{\text{ł}}} \sum_{\varphi=\varsigma}^{\varrho-1} \varphi^{\text{ł}}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}. \tag{46}$$

(b) For $1-\iota \geq p$,

$$\sum_{\varphi=\varsigma}^{\varrho-1} \varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho-1} \Theta(\vartheta)\right)^{\text{ł}} \leq \frac{\varsigma+1}{\varsigma}\left(\frac{\text{ł}}{1-\iota}\right)^{\text{ł}} \sum_{\varphi=\varsigma}^{\varrho-1} \varphi^{\text{ł}}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\text{ł}}}\right)^{\text{ł}-1}. \tag{47}$$

Proof. For the discrete case of (30), we observe that

$$\sum_{\varphi=\varsigma}^{\varrho-1}\varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho-1}\Theta(\vartheta)\right)^{\l}$$
$$\leq\left(\frac{\l}{1-\iota}\right)^{\l-1}\sum_{\varphi=\varsigma}^{\varrho-1}\varphi^{\frac{1+\l-\iota}{\l^{*}}}\Theta^{\l}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\l}}\right)^{\l-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\sum_{\vartheta=\varsigma}^{\varphi}\vartheta^{\frac{1-\iota}{\l}-1}. \tag{48}$$

Case (a): $1-\iota\leq p$. Using (18), since $(1-\iota)/\l\geq 1$, we see that

$$\Delta\vartheta^{\frac{1-\iota}{\l}}=(\vartheta+1)^{\frac{1-\iota}{\l}}-\vartheta^{\frac{1-\iota}{\l}}\geq\frac{1-\iota}{\l}\vartheta^{\frac{1-\iota}{\l}-1},$$

so,

$$\sum_{\vartheta=\varsigma}^{\varphi}\vartheta^{\frac{1-\iota}{\l}-1}\leq\frac{\l}{1-\iota}\sum_{\vartheta=\varsigma}^{\varphi}\Delta\vartheta^{\frac{1-\iota}{\l}}=\frac{\l}{1-\iota}\left[(\varphi+1)^{\frac{1-\iota}{\l}}-\varsigma^{\frac{1-\iota}{\l}}\right]\leq\frac{\l}{1-\iota}(\varphi+1)^{\frac{1-\iota}{\l}}. \tag{49}$$

Substituting (49) into (48), we get

$$\sum_{\varphi=\varsigma}^{\varrho-1}\varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho-1}\Theta(\vartheta)\right)^{\l}\leq\left(\frac{\l}{1-\iota}\right)^{\l}\sum_{\varphi=\varsigma}^{\varrho-1}\varphi^{\l}\Theta^{\l}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\l}}\right)^{\l-1}\left(\frac{\varphi+1}{\varphi}\right)^{\frac{1-\iota}{\l}}. \tag{50}$$

Since $\varphi\geq\varsigma$, $\iota>1$ and $\l<0$, we observe that $(\varphi+1)/\varphi\leq(\varsigma+1)/\varsigma$, thus (50) becomes

$$\sum_{\varphi=\varsigma}^{\varrho-1}\varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho-1}\Theta(\vartheta)\right)^{\l}\leq\left(\frac{\l}{1-\iota}\right)^{\l}\left(\frac{\varsigma+1}{\varsigma}\right)^{\frac{1-\iota}{\l}}\sum_{\varphi=\varsigma}^{\varrho-1}\varphi^{\l}\Theta^{\l}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{\l}}\right)^{\l-1},$$

which is (46).
Case (b): $1-\iota\geq p$. Using (18), since $0<(1-\iota)/\l\leq 1$, we have that

$$\Delta\vartheta^{\frac{1-\iota}{\l}}=(\vartheta+1)^{\frac{1-\iota}{\l}}-\vartheta^{\frac{1-\iota}{\l}}\geq\frac{1-\iota}{\l}(\vartheta+1)^{\frac{1-\iota}{\l}-1},$$

and then, since $\vartheta+1\leq\frac{\varsigma+1}{\varsigma}\vartheta$, we get

$$\Delta\vartheta^{\frac{1-\iota}{\l}}\geq\frac{1-\iota}{\l}\left(\frac{\varsigma+1}{\varsigma}\right)^{\frac{1-\iota}{\l}-1}\vartheta^{\frac{1-\iota}{\l}-1}.$$

Therefore

$$\sum_{\vartheta=\varsigma}^{\varphi}\vartheta^{\frac{1-\iota}{\l}-1}\leq\frac{\l}{1-\iota}\left(\frac{\varsigma}{\varsigma+1}\right)^{\frac{1-\iota}{\l}-1}\sum_{\vartheta=\varsigma}^{\varphi}\Delta\vartheta^{\frac{1-\iota}{\l}}$$
$$=\frac{\l}{1-\iota}\left(\frac{\varsigma}{\varsigma+1}\right)^{\frac{1-\iota}{\l}-1}\left[(\varphi+1)^{\frac{1-\iota}{\l}}-\varsigma^{\frac{1-\iota}{\l}}\right]$$
$$\leq\frac{\l}{1-\iota}\left(\frac{\varsigma}{\varsigma+1}\right)^{\frac{1-\iota}{\l}-1}(\varphi+1)^{\frac{1-\iota}{\l}}. \tag{51}$$

Substituting (51) into (48), we obtain

$$
\begin{aligned}
&\sum_{\varphi=\varsigma}^{\varrho-1} \varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho-1}\Theta(\vartheta)\right)^{ł} \\
&\le \left(\frac{\varsigma}{\varsigma+1}\right)^{\frac{1-\iota}{ł}-1}\left(\frac{ł}{1-\iota}\right)^{ł}\sum_{\varphi=\varsigma}^{\varrho-1}\varphi^{ł}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{ł}}\right)^{ł-1}\left(\frac{\varphi+1}{\varphi}\right)^{\frac{1-\iota}{ł}} \\
&\le \frac{\varsigma+1}{\varsigma}\left(\frac{ł}{1-\iota}\right)^{ł}\sum_{\varphi=\varsigma}^{\varrho-1}\varphi^{ł}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{ł}}\right)^{ł-1},
\end{aligned}
$$

which is (47). □

In contrast with the previous result, in the following corollary given for quantum case, we will see that there is no need to divide the proof into two cases while applying (18).

Corollary 3.10. *If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, $\varsigma, \varrho \in \mathbb{T}$, $\varsigma < \varrho < \infty$, $ł < 0$, $\iota > 1$ and Θ, ϖ are positive sequences such that $\varphi/\varpi(\varphi)$ is nondecreasing on (ς, ϱ), then*

$$
\begin{aligned}
&\sum_{\varphi=\varsigma}^{\varrho/q}\varphi\varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho/q}(q-1)\vartheta\Theta(\vartheta)\right)^{ł}(\varphi)\Delta\varphi \\
&\le q^{\frac{1-\iota}{ł}+1}\left(\frac{ł}{1-\iota}\right)^{ł-1}\frac{ł(q-1)}{1-\iota+ł}\sum_{\varphi=\varsigma}^{\varrho/q}\varphi^{ł+1}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{ł}}\right)^{ł-1}. \qquad (52)
\end{aligned}
$$

Proof. For the discrete case of (30), we have

$$
\begin{aligned}
&\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)\varphi\varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho/q}(q-1)\vartheta\Theta(\vartheta)\right)^{ł}\Delta\varphi \\
&\le \left(\frac{ł}{1-\iota}\right)^{ł-1}\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)^2\varphi^{\frac{1+ł-\iota}{ł*}+1}\Theta^{ł}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{ł}}\right)^{ł-1}\left(\frac{\varphi}{\varpi(\varphi)}\right)^{\iota}\sum_{\vartheta=\varsigma}^{\varphi}\vartheta^{\frac{1-\iota}{ł}}. \qquad (53)
\end{aligned}
$$

Since $ł < 0$ and $\iota > 1$, we observe that $(1-\iota)/ł > 0$ and by applying (18) with $\lambda = (1-\iota)/ł + 1 \ge 1$, we see that

$$
\Delta_q\vartheta^{\frac{1-\iota}{ł}+1} = \frac{(qt)^{\frac{1-\iota}{ł}+1} - \vartheta^{\frac{1-\iota}{ł}+1}}{(q-1)t} \ge \left(\frac{1-\iota}{ł}+1\right)\vartheta^{\frac{1-\iota}{ł}},
$$

and then

$$
\begin{aligned}
&\sum_{\vartheta=\varsigma}^{\varphi}\vartheta^{\frac{1-\iota}{ł}} \le \frac{ł}{1-\iota+ł}\sum_{\vartheta=\varsigma}^{\varphi}\Delta_q\vartheta^{\frac{1-\iota}{ł}+1} \\
&= \frac{ł}{1-\iota+ł}\left[(q\varphi)^{\frac{1-\iota}{ł}+1} - \varsigma^{\frac{1-\iota}{ł}+1}\right] \le \frac{ł}{1-\iota+ł}(q\varphi)^{\frac{1-\iota}{ł}+1}. \qquad (54)
\end{aligned}
$$

Substituting (54) into (53), we have

$$
\begin{aligned}
&\sum_{\varphi=\varsigma}^{\varrho/q}\varphi\varpi^{-\iota}(\varphi)\left(\sum_{\vartheta=\varphi}^{\varrho/q}(q-1)\vartheta\Theta(\vartheta)\right)^{ł}(\varphi)\Delta\varphi \\
&\le q^{\frac{1-\iota}{ł}+1}\left(\frac{ł}{1-\iota}\right)^{ł-1}\frac{ł(q-1)}{1-\iota+ł}\sum_{\varphi=\varsigma}^{\varrho/q}\varphi^{ł+1}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left(1-\left(\frac{\varphi}{\varrho}\right)^{\frac{1-\iota}{ł}}\right)^{ł-1},
\end{aligned}
$$

which is (52). □

In the following, we can prove the time scale version of (7). To complete that by using the chain rule for $0 \le \iota < 1$, it is necessary to split it into two cases: $(\iota - 1)/ł \ge 1$ and $0 < (\iota - 1)/ł \le 1$.

Lemma 3.11. *Assume that $\varsigma, \varrho \in \mathbb{T}$, $0 \le \varsigma < \varrho < \infty$, $ł < 0$, $0 \le \iota < 1$ and $\Theta, \varpi \in C_{\mathrm{rd}}([\varsigma, \varrho)_{\mathbb{T}}, \mathbb{R}^+)$ such that $\sigma(\varphi)/\varpi(\varphi)$ is nonincreasing on (ς, ϱ).*
(a) If $(\iota - 1)/ł \ge 1$, then

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{ł} \Delta\varphi \le \left(\frac{ł}{\iota - 1}\right)^{ł} \int_{\varsigma}^{\varrho} \left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{ł}} (\sigma(\varphi))^{ł}\, \Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1 - \left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1} \Delta\varphi. \tag{55}$$

(b) If $0 < (\iota - 1)/ł \le 1$, then

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{ł} \Delta\varphi \le \left(\frac{ł}{\iota - 1}\right)^{ł} \int_{\varsigma}^{\varrho} \left(\frac{\sigma(\varphi)}{\varphi}\right)^{\iota} (\varphi\Theta(\varphi))^{ł}\, \varpi^{-\iota}(\varphi)\left[1 - \left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1} \Delta\varphi, \tag{56}$$

where $\Omega(\varphi) = \int_{\varsigma}^{\varphi} \Theta(\vartheta)\Delta\vartheta$.

Proof. To prove this theorem, we have two cases:
Case (a): $(\iota - 1)/ł \ge 1$. Note that

$$\Omega^{\sigma}(\varphi) = \int_{\varsigma}^{\sigma(\varphi)} \Theta(\vartheta)\Delta\vartheta = \int_{\varsigma}^{\sigma(\varphi)} \left[(\sigma(\vartheta))^{\frac{\iota-ł-1}{łł^*}}\right]\left[(\sigma(\vartheta))^{\frac{1+ł-\iota}{łł^*}}\, \Theta(\vartheta)\right] \Delta\vartheta, \tag{57}$$

where $ł^* = ł/(ł - 1)$. Applying Lemma 2.3 on R.H.S. of (57), we get

$$\int_{\varsigma}^{\sigma(\varphi)} \left[(\sigma(\vartheta))^{\frac{\iota-ł-1}{łł^*}}\right]\left[(\sigma(\vartheta))^{\frac{1+ł-\iota}{łł^*}}\, \Theta(\vartheta)\right] \Delta\vartheta \ge \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-ł-1}{ł}}\, \Delta\vartheta\right)^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}}\, \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}}. \tag{58}$$

From (57) and (58), we see that

$$\Omega^{\sigma}(\varphi) \ge \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-ł-1}{ł}}\, \Delta\vartheta\right)^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}}\, \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}}. \tag{59}$$

Applying (15) with $v(\vartheta) = \vartheta$ and $u(\varphi) = \varphi^{\frac{\iota-1}{ł}}$, we see that

$$(u \circ v)^{\Delta}(\vartheta) = u'(v(c))v^{\Delta}(\vartheta) = \frac{\iota - 1}{ł} c^{\frac{\iota-1-ł}{ł}} \quad \text{for some } c \in [\vartheta, \sigma(\vartheta)]. \tag{60}$$

Since $(\iota - 1)/ł \ge 1$ then $c^{\frac{\iota-ł-1}{ł}} \le (\sigma(\vartheta))^{\frac{\iota-ł-1}{ł}}$,and (60) becomes

$$(u \circ v)^{\Delta}(\vartheta) \le \frac{\iota - 1}{ł}(\sigma(\vartheta))^{\frac{\iota-1-ł}{ł}}. \tag{61}$$

By integrating (61) over ϑ from ς to $\sigma(\varphi)$, we get

$$\begin{aligned} \int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-ł-1}{ł}}\, \Delta\vartheta &\ge \frac{ł}{\iota - 1} \int_{\varsigma}^{\sigma(\varphi)} (u \circ v)^{\Delta}(\vartheta)\Delta\vartheta \\ &= \frac{ł}{\iota - 1}\left[(u \circ v)(\sigma(\varphi)) - (u \circ v)(\varsigma)\right] = \frac{ł}{\iota - 1}\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]. \end{aligned} \tag{62}$$

Substituting (62) into (59), since $ł^* > 0$, we observe that

$$\Omega^{\sigma}(\varphi) \geq \left(\frac{ł}{\iota-1}\right)^{\frac{1}{ł^*}} \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}},$$

and then we have for $ł < 0$, that

$$[\Omega^{\sigma}(\varphi)]^{ł} \leq \left(\frac{ł}{\iota-1}\right)^{ł-1} \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta. \tag{63}$$

Multiplying (63) by $\varpi^{-\iota}(\varphi)$ and then integrating over φ from ς to ϱ, we see that

$$\begin{aligned}&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\,[\Omega^{\sigma}(\varphi)]^{ł}\,\Delta\varphi \\ \leq\ & \left(\frac{ł}{\iota-1}\right)^{ł-1} \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi. \end{aligned}\tag{64}$$

Applying (16) on R.H.S. of (64) with

$$u(\varphi) = -\int_{\varphi}^{\varrho} \varpi^{-\iota}(\vartheta)\left[(\sigma(\vartheta))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta \quad \text{and} \quad v(\varphi) = \int_{\varsigma}^{\varphi} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta,$$

we get

$$\begin{aligned}&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\ &= \int_{\varsigma}^{\varrho} u^{\Delta}(\vartheta)v^{\sigma}(\vartheta)\Delta\vartheta = u(\varrho)v(\varrho) - u(\varsigma)v(\varsigma) - \int_{\varsigma}^{\varrho} u(\vartheta)v^{\Delta}(\vartheta)\Delta\vartheta \\ &= -\int_{\varsigma}^{\varrho} u(\varphi)\,(\sigma(\varphi))^{\frac{1+ł-\iota}{ł^*}}\,\Theta^{ł}(\varphi)\Delta\varphi \\ &= \int_{\varsigma}^{\varrho} (\sigma(\varphi))^{\frac{1+ł-\iota}{ł^*}}\,\Theta^{ł}(\varphi)\left(\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{(\iota-1)(ł-1)}{ł}-\iota}\,\varpi^{-\iota}(\vartheta)\left[1-\left(\frac{\varsigma}{\sigma(\vartheta)}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}(\sigma(\vartheta))^{\iota}\,\Delta\vartheta\right)\Delta\varphi \\ &\leq \int_{\varsigma}^{\varrho} (\sigma(\varphi))^{\frac{1+ł-\iota}{ł^*}}\,\Theta^{ł}(\varphi)\left(\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{(\iota-1)(ł-1)}{ł}-\iota}\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\frac{\sigma(\vartheta)}{\varpi(\vartheta)}\right)^{\iota}\Delta\vartheta\right)\Delta\varphi. \end{aligned}\tag{65}$$

Since $\sigma(\vartheta)/\varpi(\vartheta)$ and $\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}$ are nonincreasing on (ς,ϱ) (note $0 \leq \iota < 1$), we have for $\vartheta \geq \varphi$, that $(\sigma(\vartheta)/\varpi(\vartheta))^{\iota} \leq (\sigma(\varphi)/\varpi(\varphi))^{\iota}$ and then (65) becomes

$$\begin{aligned}&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\ &\leq \int_{\varsigma}^{\varrho} (\sigma(\varphi))^{\frac{1+ł-\iota}{ł^*}+\iota}\,\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{1-\iota}{ł}-1}\,\Delta\vartheta\right)\Delta\varphi. \end{aligned}\tag{66}$$

By using (15) with $v(\vartheta) = \vartheta$ and $u(\vartheta) = \vartheta^{\frac{1-\iota}{ł}}$, we have

$$(u \circ v)^{\Delta}(\vartheta) = u'(v(c))v^{\Delta}(\vartheta) = \frac{1-\iota}{ł}c^{\frac{1-\iota}{ł}-1} \quad \text{for some} \quad c \in [\vartheta, \sigma(\vartheta)].$$

Since $(1-\iota)/\text{ł} < 0$, we get $c^{\frac{1-\iota}{\text{ł}}-1} \geq (\sigma(\vartheta))^{\frac{1-\iota}{\text{ł}}-1}$ and

$$(u \circ v)^{\Delta}(\vartheta) \leq \frac{1-\iota}{\text{ł}} (\sigma(\vartheta))^{\frac{1-\iota}{\text{ł}}-1},$$

thus

$$\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{1-\iota}{\text{ł}}-1} \Delta\vartheta \leq \frac{\text{ł}}{1-\iota} \int_{\varphi}^{\varrho} (u \circ v)^{\Delta}(\vartheta)\Delta\vartheta = \frac{\text{ł}}{1-\iota} \left[(u \circ v)(\varrho) - (u \circ v)(\varphi)\right]$$
$$= \frac{\text{ł}}{\iota-1} \left[\varphi^{\frac{1-\iota}{\text{ł}}} - \varrho^{\frac{1-\iota}{\text{ł}}}\right] \leq \frac{\text{ł}}{\iota-1} \varphi^{\frac{1-\iota}{\text{ł}}}. \tag{67}$$

Substituting (67) into (66), we observe that

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left[(\sigma(\varphi))^{\frac{\iota-1}{\text{ł}}} - \varsigma^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+\text{ł}-\iota}{\text{ł}^*}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi$$
$$\leq \frac{\text{ł}}{\iota-1} \int_{\varsigma}^{\varrho} \left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{\text{ł}}} (\sigma(\varphi))^{\text{ł}} \Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi) \left[1 - \left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1} \Delta\varphi \tag{68}$$

Substituting (68) into (64), we see that

$$\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}} \Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}} \int_{\varsigma}^{\varrho} \left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{\text{ł}}} (\sigma(\varphi))^{\text{ł}} \Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi) \left[1 - \left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1} \Delta\varphi,$$

which is (55).
Case (b): $0 < (\iota-1)/\text{ł} \leq 1$. Note that

$$\Omega^{\sigma}(\varphi) = \int_{\varsigma}^{\sigma(\varphi)} \Theta(\vartheta)\Delta\vartheta = \int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-\text{ł}-1}{\text{łł}^*}} \left[\vartheta^{\frac{1+\text{ł}-\iota}{\text{łł}^*}} \Theta(\vartheta)\right] \Delta\vartheta. \tag{69}$$

Applying Lemma 2.3 on R.H.S. of (69), we get

$$\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-\text{ł}-1}{\text{łł}^*}} \left[\vartheta^{\frac{1+\text{ł}-\iota}{\text{łł}^*}} \Theta(\vartheta)\right] \Delta\vartheta \geq \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-\text{ł}-1}{\text{ł}}} \Delta\vartheta\right)^{\frac{1}{\text{ł}^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\text{ł}}}. \tag{70}$$

From (69) and (70), we see that

$$\Omega^{\sigma}(\varphi) \geq \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-\text{ł}-1}{\text{ł}}} \Delta\vartheta\right)^{\frac{1}{\text{ł}^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\text{ł}}}. \tag{71}$$

Using (33), since $0 < (\iota-1)/\text{ł} \leq 1$, then $c^{\frac{\iota-\text{ł}-1}{\text{ł}}} \leq \vartheta^{\frac{\iota-\text{ł}-1}{\text{ł}}}$ and

$$(u \circ v)^{\Delta}(\vartheta) \leq \frac{\iota-1}{\text{ł}} \vartheta^{\frac{\iota-\text{ł}-1}{\text{ł}}},$$

thus

$$\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-\text{ł}-1}{\text{ł}}} \Delta\vartheta \geq \frac{\text{ł}}{\iota-1} \int_{\varsigma}^{\sigma(\varphi)} (u \circ v)^{\Delta}(\vartheta)\Delta\vartheta = \frac{\text{ł}}{\iota-1} \left[(\sigma(\varphi))^{\frac{\iota-1}{\text{ł}}} - \varsigma^{\frac{\iota-1}{\text{ł}}}\right]. \tag{72}$$

Since $\text{ł}^* > 0$, then by substituting (72) into (71), we see that

$$\Omega^{\sigma}(\varphi) \geq \left(\frac{\text{ł}}{\iota-1}\right)^{\frac{1}{\text{ł}^*}} \left[(\sigma(\varphi))^{\frac{\iota-1}{\text{ł}}} - \varsigma^{\frac{\iota-1}{\text{ł}}}\right]^{\frac{1}{\text{ł}^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}} \Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\text{ł}}},$$

and then we have for $ł < 0$ that

$$[\Omega^{\sigma}(\varphi)]^{ł} \leq \left(\frac{ł}{\iota-1}\right)^{ł-1}\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\vartheta)\Delta\vartheta.$$

Multiplying the last inequality by $\varpi^{-\iota}(\varphi)$ and then integrating over φ from ς to ϱ, we observe that

$$\begin{aligned}&\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{ł}\Delta\varphi\\ \leq\ &\left(\frac{ł}{\iota-1}\right)^{ł-1}\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi. \end{aligned} \tag{73}$$

Applying (16) on R.H.S. of (73) with

$$u(\varphi) = -\int_{\varphi}^{\varrho}\varpi^{-\iota}(\vartheta)\left[(\sigma(\vartheta))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta \quad \text{and} \quad v(\varphi) = \int_{\varsigma}^{\varphi}\vartheta^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\vartheta)\Delta\vartheta$$

we see that

$$\begin{aligned}&\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi\\ =\ &\int_{\varsigma}^{\varrho}u^{\Delta}(\varphi)v^{\sigma}(\varphi)\Delta\varphi = u(\varrho)v(\varrho) - u(\varsigma)v(\varsigma) - \int_{\varsigma}^{\varrho}u(\varphi)v^{\Delta}(\varphi)\Delta\varphi\\ =\ &-\int_{\varsigma}^{\varrho}\varphi^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\varphi)u(\varphi)\Delta\varphi\\ =\ &\int_{\varsigma}^{\varrho}\varphi^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\varphi)\left(\int_{\varphi}^{\varrho}\varpi^{-\iota}(\vartheta)\left[(\sigma(\vartheta))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta\right)\Delta\varphi\\ =\ &\int_{\varsigma}^{\varrho}\varphi^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\varphi)\left[\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{(\iota-1)(ł-1)}{ł}-\iota}\left(\frac{\sigma(\vartheta)}{\varpi(\vartheta)}\right)^{\iota}\left[1-\left(\frac{\varsigma}{\sigma(\vartheta)}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta\right]\Delta\varphi. \end{aligned} \tag{74}$$

Since $\sigma(\vartheta)/\varpi(\vartheta)$ and $\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}$ is nonincreasing on (ς, ϱ), we have from (74) that

$$\begin{aligned}&\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi\\ \leq\ &\int_{\varsigma}^{\varrho}\varphi^{\frac{1+ł-\iota}{ł^*}}\Theta^{ł}(\varphi)\left[\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{(\iota-1)(ł-1)}{ł}-\iota}\left(\frac{\sigma(\vartheta)}{\varpi(\vartheta)}\right)^{\iota}\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta\right]\Delta\varphi\\ \leq\ &\int_{\varsigma}^{\varrho}\varphi^{\frac{1+ł-\iota}{ł^*}}(\sigma(\varphi))^{\iota}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\left[\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{\iota-1}{ł^*}-\iota}\Delta\vartheta\right]\Delta\varphi. \end{aligned} \tag{75}$$

Applying (15) with $v(\vartheta) = \vartheta$ and $u(\varphi) = \varphi^{\frac{1-\iota}{ł}}$, we get

$$(u\circ v)^{\Delta}(\vartheta) = u'(v(c))v^{\Delta}(\vartheta) = \frac{1-\iota}{ł}c^{\frac{1-\iota}{ł}-1} \quad \text{for some} \quad c\in[\vartheta, \sigma(\vartheta)], \tag{76}$$

thus for $(1-\iota)/ł < 0$, we obtain $c^{\frac{1-\iota}{ł}-1} \geq (\sigma(\vartheta))^{\frac{1-\iota}{ł}-1}$ and

$$\frac{ł}{1-\iota}(u\circ v)^{\Delta}(\vartheta) \geq (\sigma(\vartheta))^{\frac{1-\iota}{ł}-1} = (\sigma(\vartheta))^{\frac{\iota-1}{ł^*}-\iota},$$

therefore

$$\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{\iota-1}{\text{ł}}-\iota}\Delta\vartheta \leq \frac{\text{ł}}{1-\iota}\int_{\varphi}^{\varrho}(u\circ v)^{\Delta}(\vartheta)\Delta\vartheta = \frac{\text{ł}}{\iota-1}\left[\varphi^{\frac{1-\iota}{\text{ł}}}-\varrho^{\frac{1-\iota}{\text{ł}}}\right] \leq \frac{\text{ł}}{\iota-1}\varphi^{\frac{1-\iota}{\text{ł}}}. \tag{77}$$

Substituting (77) into (75), we obtain

$$\begin{aligned}&\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{\text{ł}}}-\varsigma^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\left(\int_{\varsigma}^{\sigma(\varphi)}\vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}}}\Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\ \leq\ & \frac{\text{ł}}{\iota-1}\int_{\varsigma}^{\varrho}\left(\frac{\sigma(\varphi)}{\varphi}\right)^{\iota}(\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi. \end{aligned}\tag{78}$$

Substituting (78) into (73), we observe that

$$\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}}\Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}\left(\frac{\sigma(\varphi)}{\varphi}\right)^{\iota}(\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi,$$

which is (56). □

By similar steps than before, one can easily obtain the following.

Lemma 3.12. *Assume that* $\varsigma \in \mathbb{T}$, $0 \leq \varsigma < \infty$, $\text{ł} < 0$, $0 \leq \iota < 1$ *and* $\Theta, \varpi \in C_{\text{rd}}([\varsigma,\infty)_{\mathbb{T}},\mathbb{R}^{+})$ *such that* $\sigma(\varphi)/\varpi(\varphi)$ *is nonincreasing on* (ς,∞).
(a) If $(\iota-1)/\text{ł} \geq 1$, *then*

$$\int_{\varsigma}^{\infty}\varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}}\Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\infty}\left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{\text{ł}}}(\sigma(\varphi))^{\text{ł}}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi. \tag{79}$$

(b) If $0 < (\iota-1)/\text{ł} \leq 1$, *then*

$$\int_{\varsigma}^{\infty}\varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}}\Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\infty}\left(\frac{\sigma(\varphi)}{\varphi}\right)^{\iota}(\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi, \tag{80}$$

where $\Omega(\varphi) = \int_{\varsigma}^{\varphi}\Theta(\vartheta)\Delta\vartheta$.

Remark 3.13. *In Lemma 3.12, if* $\mathbb{T} = \mathbb{R}$ *and* $\varsigma = 0$, *then* (79) *and* (80) *reduce to* (7).

Now, we aim to get the time scale version of (7) using the chain rule for $\iota < 0$. Similarly to the previous result, we have to consider two cases: $(\iota-1)/\text{ł} \geq 1$ and $0 < (\iota-1)/\text{ł} \leq 1$.

Lemma 3.14. *Assume that* $\varsigma, \varrho \in \mathbb{T}$, $0 \leq \varsigma < \varrho < \infty$, $\text{ł}, \iota < 0$ *and* $\Theta, \varpi \in C_{\text{rd}}([\varsigma,\varrho)_{\mathbb{T}},\mathbb{R}^{+})$ *such that* $\varphi/\varpi(\varphi)$ *is nondecreasing on* (ς,ϱ).
(a) If $0 < (\iota-1)/\text{ł} \leq 1$, *then*

$$\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}}\Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}(\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi. \tag{81}$$

(b) If $(\iota-1)/\text{ł} \geq 1$, *then*

$$\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}}\Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}\left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{\text{ł}}+\iota}(\sigma(\varphi))^{\text{ł}}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi, \tag{82}$$

where $\Omega(\varphi) = \int_{\varsigma}^{\varphi}\Theta(\vartheta)\Delta\vartheta$.

Proof. We consider the following two cases to prove this theorem.

Case (a): $0 < (\iota - 1)/ł \leq 1$.

Note that

$$\Omega^{\sigma}(\varphi) = \int_{\varsigma}^{\sigma(\varphi)} \Theta(\vartheta)\Delta\vartheta = \int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-ł-1}{łł^*}} \left[\vartheta^{\frac{1+ł-\iota}{łł^*}} \Theta(\vartheta)\right] \Delta\vartheta, \tag{83}$$

where $ł^* = ł/(ł-1)$. Applying Lemma 2.3 on R.H.S. of (83), we get

$$\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-ł-1}{łł^*}} \left[\vartheta^{\frac{1+ł-\iota}{łł^*}} \Theta(\vartheta)\right] \Delta\vartheta \geq \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-ł-1}{ł}} \Delta\vartheta\right)^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}}. \tag{84}$$

From (83) and (84), we see that

$$\Omega^{\sigma}(\varphi) \geq \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-ł-1}{ł}} \Delta\vartheta\right)^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}}. \tag{85}$$

Since $0 < (\iota - 1)/ł \leq 1$, then by using (33), we have that $\frac{ł}{\iota-1}(u \circ v)^{\Delta}(\vartheta) \leq \vartheta^{\frac{\iota-ł-1}{ł}}$, thus

$$\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{\iota-ł-1}{ł}} \Delta\vartheta \geq \frac{ł}{\iota-1} \int_{\varsigma}^{\sigma(\varphi)} (u \circ v)^{\Delta}(\vartheta)\Delta\vartheta = \frac{ł}{\iota-1}\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]. \tag{86}$$

Substituting (86) into (85), we see that

$$\Omega^{\sigma}(\varphi) \geq \left(\frac{ł}{\iota-1}\right)^{\frac{1}{ł^*}} \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}},$$

and then we have for $ł < 0$, that

$$[\Omega^{\sigma}(\varphi)]^{ł} \leq \left(\frac{ł}{\iota-1}\right)^{ł-1} \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta.$$

Multiplying the last inequality by $\varpi^{-\iota}(\varphi)$ and then integrating over φ from ς to ϱ, we observe that

$$\begin{aligned} &\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{ł} \Delta\varphi \\ \leq\ &\left(\frac{ł}{\iota-1}\right)^{ł-1} \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi. \end{aligned} \tag{87}$$

Applying (16) on R.H.S. of (87) with

$$u(\varphi) = -\int_{\varphi}^{\varrho} \varpi^{-\iota}(\vartheta)\left[(\sigma(\vartheta))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \Delta\vartheta \quad \text{and} \quad v(\varphi) = \int_{\varsigma}^{x} \vartheta^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta,$$

we observe that

$$
\begin{aligned}
&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{\text{ł}}} - \varsigma^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
&= \int_{\varsigma}^{\varrho} u^{\Delta}(\varphi)v^{\sigma}(\varphi)\Delta\varphi = u(\varrho)v(\varrho) - u(\varsigma)v(\varsigma) - \int_{\varsigma}^{\varrho} u(\varphi)v^{\Delta}(\varphi)\Delta\varphi \\
&= -\int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)u(\varphi)\Delta\varphi \\
&= \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(\int_{\varphi}^{\varrho} \varpi^{-\iota}(\vartheta)\left[(\sigma(\vartheta))^{\frac{\iota-1}{\text{ł}}} - \varsigma^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\vartheta\right)\Delta\varphi \\
&= \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{\iota-1}{\text{ł}^*}-\iota}\varpi^{-\iota}(\vartheta)(\sigma(\vartheta))^{\iota}\left[1-\left(\frac{\varsigma}{\sigma(\vartheta)}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\vartheta\right)\Delta\varphi \\
&\leq \int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\varphi)\left(\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{\iota-1}{\text{ł}^*}-\iota}\left(\frac{\vartheta}{\varpi(\vartheta)}\right)^{\iota}\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\vartheta\right)\Delta\varphi. \qquad (88)
\end{aligned}
$$

Since $(\vartheta/\varpi(\vartheta))^{\iota}$ and $\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}$ are nonincreasing on (ς, ϱ), then (88) becomes

$$
\begin{aligned}
&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{\text{ł}}} - \varsigma^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
\leq\ &\int_{\varsigma}^{\varrho} \varphi^{\frac{1+\text{ł}-\iota}{\text{ł}^*}+\iota}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\left(\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{\iota-1}{\text{ł}^*}-\iota}\Delta\vartheta\right)\Delta\varphi. \qquad (89)
\end{aligned}
$$

Since $\iota < 0$ and $\text{ł} < 0$, then by using (76), we have that $\frac{\text{ł}}{1-\iota}(u \circ v)^{\Delta}(\vartheta) \geq (\sigma(\vartheta))^{\frac{1-\iota}{\text{ł}}-1} = (\sigma(\vartheta))^{\frac{\iota-1}{\text{ł}^*}-\iota}$, and then

$$
\int_{\varphi}^{\varrho} (\sigma(\vartheta))^{\frac{\iota-1}{\text{ł}^*}-\iota}\Delta\vartheta \leq \frac{\text{ł}}{1-\iota}\int_{\varphi}^{\varrho}(u \circ v)^{\Delta}(\vartheta)\Delta\vartheta = \frac{\text{ł}}{\iota-1}\left[\varphi^{\frac{1-\iota}{\text{ł}}} - \varrho^{\frac{1-\iota}{\text{ł}}}\right] \leq \frac{\text{ł}}{\iota-1}\varphi^{\frac{1-\iota}{\text{ł}}}. \qquad (90)
$$

Substituting (90) into (89), we obtain

$$
\begin{aligned}
&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{\text{ł}}} - \varsigma^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\left(\int_{\varsigma}^{\sigma(\varphi)} \vartheta^{\frac{1+\text{ł}-\iota}{\text{ł}^*}}\Theta^{\text{ł}}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
\leq\ &\frac{\text{ł}}{\iota-1}\int_{\varsigma}^{\varrho} (\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi. \qquad (91)
\end{aligned}
$$

Substituting (91) into (87), we observe that

$$
\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}}\Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho} (\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi,
$$

which is (81).

Case (b): $(\iota-1)/\text{ł} \geq 1$.

Note that

$$
\Omega^{\sigma}(\varphi) = \int_{\varsigma}^{\sigma(\varphi)} \Theta(\vartheta)\Delta\vartheta = \int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-\text{ł}-1}{\text{łł}^*}}\left[(\sigma(\vartheta))^{\frac{1+\text{ł}-\iota}{\text{łł}^*}}\Theta(\vartheta)\right]\Delta\vartheta, \qquad (92)
$$

where $ł^* = ł/(ł-1)$. Applying Lemma 2.3 on R.H.S. of (92), we get

$$\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-ł-1}{łł^*}} \left[(\sigma(\vartheta))^{\frac{1+ł-\iota}{łł^*}} \Theta(\vartheta)\right] \Delta\vartheta \geq \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-ł-1}{ł}} \Delta\vartheta\right)^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}}. \tag{93}$$

Substituting (93) into (92), we obtain

$$\Omega^{\sigma}(\varphi) \geq \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-ł-1}{ł}} \Delta\vartheta\right)^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}}. \tag{94}$$

Since $(\iota-1)/ł \geq 1$, then by using (33), we have

$$(u \circ v)^{\Delta}(\vartheta) \leq \frac{\iota-1}{ł} (\sigma(\vartheta))^{\frac{\iota-1-ł}{ł}},$$

therefore

$$\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{\iota-ł-1}{ł}} \Delta\vartheta \geq \frac{ł}{\iota-1} \int_{\varsigma}^{\sigma(\varphi)} (u \circ v)^{\Delta}(\vartheta)\Delta\vartheta = \frac{ł}{\iota-1}\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]. \tag{95}$$

Substituting (95) into (94), since $ł^* > 0$, we observe that

$$\Omega^{\sigma}(\varphi) \geq \left(\frac{ł}{\iota-1}\right)^{\frac{1}{ł^*}} \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{\frac{1}{ł^*}} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)^{\frac{1}{ł}},$$

and then we have for $ł < 0$, that

$$[\Omega^{\sigma}(\varphi)]^{ł} \leq \left(\frac{ł}{\iota-1}\right)^{ł-1} \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta. \tag{96}$$

Multiplying (96) with $\varpi^{-\iota}(\varphi)$ and then integrating over φ from ς to ϱ, we see that

$$\begin{aligned} &\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) [\Omega^{\sigma}(\varphi)]^{ł} \Delta\varphi \\ \leq\ &\left(\frac{ł}{\iota-1}\right)^{ł-1} \int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi) \left[(\sigma(\varphi))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \left(\int_{\varsigma}^{\sigma(\varphi)} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi. \end{aligned} \tag{97}$$

Applying (16) on R.H.S. of (97) with

$$u(\varphi) = -\int_{\varphi}^{\varrho} \varpi^{-\iota}(\vartheta) \left[(\sigma(\vartheta))^{\frac{\iota-1}{ł}} - \varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1} \Delta\vartheta \quad \text{and} \quad v(\varphi) = \int_{\varsigma}^{\varphi} (\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^*}} \Theta^{ł}(\vartheta)\Delta\vartheta,$$

we see that

$$
\begin{aligned}
&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}}-\varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)}(\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
&=\int_{\varsigma}^{\varrho} u^{\Delta}(\varphi)v^{\sigma}(\varphi)\Delta\varphi=u(\varrho)v(\varrho)-u(\varsigma)v(\varsigma)-\int_{\varsigma}^{\varrho} u(\varphi)v^{\Delta}(\varphi)\Delta\varphi \\
&=-\int_{\varsigma}^{\varrho}(\sigma(\varphi))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\varphi)u(\varphi)\Delta\varphi \\
&=\int_{\varsigma}^{\varrho}(\sigma(\varphi))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\varphi)\left(\int_{\varphi}^{\varrho}\varpi^{-\iota}(\vartheta)\left[(\sigma(\vartheta))^{\frac{\iota-1}{ł}}-\varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta\right)\Delta\varphi \\
&=\int_{\varsigma}^{\varrho}(\sigma(\varphi))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\varphi)\left(\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{\iota-1}{ł^{*}}-\iota}(\sigma(\vartheta))^{\iota}\varpi^{-\iota}(\vartheta)\left[1-\left(\frac{\varsigma}{\sigma(\vartheta)}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta\right)\Delta\varphi \\
&\leq\int_{\varsigma}^{\varrho}(\sigma(\varphi))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\varphi)\left(\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{\iota-1}{ł^{*}}-\iota}\left(\frac{\vartheta}{\varpi(\vartheta)}\right)^{\iota}\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\vartheta\right)\Delta\varphi. \qquad (98)
\end{aligned}
$$

Since $(\vartheta/\varpi(\vartheta))^{\iota}$ and $\left[1-\left(\frac{\varsigma}{\vartheta}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}$ are nonincreasing on (ς,ϱ), then (98) becomes

$$
\begin{aligned}
&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}}-\varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)}(\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
\leq\ &\int_{\varsigma}^{\varrho}\varphi^{\iota}(\sigma(\varphi))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{\iota-1}{ł^{*}}-\iota}\Delta\vartheta\right)\Delta\varphi. \qquad (99)
\end{aligned}
$$

Since $\iota<0$ and $ł<0$, then by using (76), we have

$$
\frac{ł}{1-\iota}(u\circ v)^{\Delta}(\vartheta)\geq(\sigma(\vartheta))^{\frac{1-\iota}{ł}-1}=(\sigma(\vartheta))^{\frac{\iota-1}{ł^{*}}-\iota},
$$

and then

$$
\int_{\varphi}^{\varrho}(\sigma(\vartheta))^{\frac{\iota-1}{ł^{*}}-\iota}\Delta\vartheta\leq\frac{ł}{1-\iota}\int_{\varphi}^{\varrho}(u\circ v)^{\Delta}(\vartheta)\Delta\vartheta=\frac{ł}{\iota-1}\left[\varphi^{\frac{1-\iota}{ł}}-\varrho^{\frac{1-\iota}{ł}}\right]\leq\frac{ł}{\iota-1}\varphi^{\frac{1-\iota}{ł}}. \qquad (100)
$$

Substituting (100) into (99), (note $\frac{1-\iota}{ł}+\iota<0$), we observe that

$$
\begin{aligned}
&\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi))^{\frac{\iota-1}{ł}}-\varsigma^{\frac{\iota-1}{ł}}\right]^{ł-1}\left(\int_{\varsigma}^{\sigma(\varphi)}(\sigma(\vartheta))^{\frac{1+ł-\iota}{ł^{*}}}\Theta^{ł}(\vartheta)\Delta\vartheta\right)\Delta\varphi \\
\leq\ &\frac{ł}{\iota-1}\int_{\varsigma}^{\varrho}\left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{ł}+\iota}(\sigma(\varphi))^{ł}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\varphi. \qquad (101)
\end{aligned}
$$

Substituting (101) into (97), we see that

$$
\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{ł}\Delta\varphi\leq\left(\frac{ł}{\iota-1}\right)^{ł}\int_{\varsigma}^{\varrho}\left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{ł}+\iota}(\sigma(\varphi))^{ł}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\varphi,
$$

which is (82). □

By similar steps of those before, one can easily get the following.

Lemma 3.15. *Assume that $\varsigma \in \mathbb{T}$, $0 \leq \varsigma < \infty$, ł, $\iota < 0$ and $\Theta, \varpi \in C_{\mathrm{rd}}([\varsigma,\infty)_{\mathbb{T}}, \mathbb{R}^{+})$ such that $\varphi/\varpi(\varphi)$ is nondecreasing on (ς,∞).*
(a) If $0 < (\iota-1)/\text{ł} \leq 1$, then

$$\int_{\varsigma}^{\infty} \varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}} \Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}} \int_{\varsigma}^{\infty} \left(\varphi\Theta(\varphi)\right)^{\text{ł}} \varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1} \Delta\varphi. \tag{102}$$

(b) If $(\iota-1)/\text{ł} \geq 1$, then

$$\int_{\varsigma}^{\infty} \varpi^{-\iota}(\varphi)\left[\Omega^{\sigma}(\varphi)\right]^{\text{ł}} \Delta\varphi \leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}} \int_{\varsigma}^{\infty} \left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{\text{ł}}+\iota} (\sigma(\varphi))^{\text{ł}}\,\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1} \Delta\varphi, \tag{103}$$

where $\Omega(\varphi) = \int_{\varsigma}^{\varphi}\Theta(\vartheta)\Delta\vartheta$.

Corollary 3.16. *In Lemma 3.15, if $\mathbb{T}=\mathbb{R}$ and $\varsigma = 0$, then* (102) *and* (103) *reduce to* (7).

Now we are prepared to state our main results. Combining Lemmas 3.2 and 3.11, we get the following theorem.

Theorem 3.17. *Assume that $\varsigma, \varrho \in \mathbb{T}$, $0 \leq \varsigma < \varrho < \infty$, $\text{ł} < 0$, $\eta > 0$, $0 \leq \iota < 1$ and $\Theta, \varpi \in C_{\mathrm{rd}}([\varsigma,\varrho)_{\mathbb{T}}, \mathbb{R}^{+})$ such that $\sigma(\varphi)/\varpi(\varphi)$ is nonincreasing on (ς,ϱ) and $\Theta(\varphi)$ is nondecreasing on (ς,ϱ).*
(a) If $(\iota-1)/\text{ł} \geq 1$, then

$$\begin{aligned}&\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-\text{ł}}{\eta}}\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\sigma(\varphi)-\varsigma)\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{\text{ł}}{\eta}}\\ &\leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}\left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{\text{ł}}}(\sigma(\varphi))^{\text{ł}}\,\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi.\end{aligned} \tag{104}$$

(b) If $0 < (\iota-1)/\text{ł} \leq 1$, then

$$\begin{aligned}&\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-\text{ł}}{\eta}}\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\sigma(\varphi)-\varsigma)\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{\text{ł}}{\eta}}\\ &\leq \left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}\left(\frac{\sigma(\varphi)}{\varphi}\right)^{\iota}(\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\Delta\varphi.\end{aligned} \tag{105}$$

Corollary 3.18. *If $\mathbb{T}=\mathbb{R}$, then both* (104) *and* (105) *reduce to the classical integral inequality* (12) *proved by Azzouz et al. [5].*

Example 3.19. *Let $\mathbb{T}=\mathbb{R}, \varsigma=0, \eta=1, \text{ł}=\frac{-1}{8}, \iota=\frac{1}{6}, \Theta(\varphi)=\varphi^{2}$ and $\varpi(\varphi)=\varphi^{3}$. Then, we observe that*

$$\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\mathrm{d}\varphi\right)^{\frac{\eta-\text{ł}}{\eta}}\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\varphi-\varsigma)\Theta(\varphi)\right]^{\eta}\mathrm{d}\varphi\right)^{\frac{\text{ł}}{\eta}} = 2(7\varrho)^{\frac{1}{8}},$$

and

$$\left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\int_{\varsigma}^{\varrho}(\varphi\Theta(\varphi))^{\text{ł}}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}\mathrm{d}\varphi = 8\left(\frac{20\varrho}{3}\right)^{\frac{1}{8}}.$$

Hence, we have verified that the inequality (12) *holds.*

Corollary 3.20. *Assume that* $\varsigma, \varrho \in h\mathbb{N}_0$ *for* $h > 0$, $0 \leq \varsigma < \varrho < \infty$, $ł < 0$, $\eta > 0$, $0 \leq \iota < 1$ *and* Θ, ϖ *are positive sequences such that* $(\varphi + h)/\varpi(\varphi)$ *is nonincreasing on* (ς, ϱ) *and* $\Theta(\varphi)$ *is nondecreasing on* (ς, ϱ).
(a) If $(\iota - 1)/ł \geq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho-h} h\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-ł}{\eta}} \left(\sum_{\varphi=\varsigma}^{\varrho-h} h\varpi^{-\iota}(\varphi)\left[(\varphi+h-\varsigma)\Theta(\varphi)\right]^{\eta}\right)^{\frac{ł}{\eta}}$$
$$\leq \left(\frac{ł}{\iota-1}\right)^{ł} \sum_{\varphi=\varsigma}^{\varrho-h} h\left(\frac{\varphi}{\varphi+h}\right)^{\frac{1-\iota}{ł}} (\varphi+h)^{ł}\,\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}.$$

(b) If $0 < (\iota - 1)/ł \leq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho-h} h\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-ł}{\eta}} \left(\sum_{\varphi=\varsigma}^{\varrho-h} h\varpi^{-\iota}(\varphi)\left[(\varphi+h-\varsigma)\Theta(\varphi)\right]^{\eta}\right)^{\frac{ł}{\eta}}$$
$$\leq \left(\frac{ł}{\iota-1}\right)^{ł} \sum_{\varphi=\varsigma}^{\varrho-h} h\left(\frac{\varphi+h}{\varphi}\right)^{\iota} (\varphi\Theta(\varphi))^{ł}\,\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}.$$

Corollary 3.21. *Assume that* $\mathbb{T} = q^{\mathbb{N}_0}$ *for* $q > 1$, $\varsigma, \varrho \in \mathbb{T}$, $\varsigma < \varrho < \infty$, $ł < 0$, $\eta > 0$, $0 \leq \iota < 1$ *and* Θ, ϖ *are positive sequences such that* $q\varphi/\varpi(\varphi)$ *is nonincreasing on* (ς, ϱ) *and* $\Theta(\varphi)$ *is nondecreasing on* (ς, ϱ).
(a) If $(\iota - 1)/ł \geq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho/q} (q-1)\varphi\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-ł}{\eta}} \left(\sum_{\varphi=\varsigma}^{\varrho/q} (q-1)\varphi\varpi^{-\iota}(\varphi)\left[(q\varphi-\varsigma)\Theta(\varphi)\right]^{\eta}\right)^{\frac{ł}{\eta}}$$
$$\leq \left(\frac{ł}{\iota-1}\right)^{ł} q^{\frac{\iota-1}{ł}+ł}(q-1) \sum_{\varphi=\varsigma}^{\varrho/q} \varphi^{ł+1}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}.$$

(b) If $0 < (\iota - 1)/ł \leq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho/q} (q-1)\varphi\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-ł}{\eta}} \left(\sum_{\varphi=\varsigma}^{\varrho/q} (q-1)\varphi\varpi^{-\iota}(\varphi)\left[(q\varphi-\varsigma)\Theta(\varphi)\right]^{\eta}\right)^{\frac{ł}{\eta}}$$
$$\leq \left(\frac{ł}{\iota-1}\right)^{ł} q^{\iota}(q-1) \sum_{\varphi=\varsigma}^{\varrho/q} \varphi\,(\varphi\Theta(\varphi))^{ł}\,\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}.$$

Combining Lemmas 3.2 and 3.14, we get the following theorem.

Theorem 3.22. *Assume that* $\varsigma, \varrho \in \mathbb{T}$, $0 \leq \varsigma < \varrho < \infty$, $ł, \iota < 0$, $\eta > 0$ *and* $\Theta, \varpi \in C_{\mathrm{rd}}([\varsigma, \varrho)_{\mathbb{T}}, \mathbb{R}^+)$ *such that* $\Theta(\varphi)$ *and* $\varphi/\varpi(\varphi)$ *are nondecreasing on* (ς, ϱ).
(a) If $0 < (\iota - 1)/ł \leq 1$, *then*

$$\left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-ł}{\eta}} \left(\int_{\varsigma}^{\varrho} \varpi^{-\iota}(\varphi)\left[(\sigma(\varphi)-\varsigma)\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{ł}{\eta}}$$
$$\leq \left(\frac{ł}{\iota-1}\right)^{ł} \int_{\varsigma}^{\varrho} (\varphi\Theta(\varphi))^{ł}\,\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\varphi. \tag{106}$$

(b) If $(\iota-1)/ł \geq 1$, *then*

$$\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\Delta\varphi\right)^{\frac{\eta-ł}{\eta}}\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\sigma(\varphi)-\varsigma)\Theta(\varphi)\right]^{\eta}\Delta\varphi\right)^{\frac{ł}{\eta}}$$
$$\leq\left(\frac{ł}{\iota-1}\right)^{ł}\int_{\varsigma}^{\varrho}\left(\frac{\varphi}{\sigma(\varphi)}\right)^{\frac{1-\iota}{ł}+\iota}(\sigma(\varphi))^{ł}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\Delta\varphi. \tag{107}$$

Corollary 3.23. *If* $\mathbb{T}=\mathbb{R}$ *and* $\iota<0$, *then* (106) *and* (107) *reduce to the classical integral inequality* (12) *proved by Azzouz et al. [5].*

Example 3.24. *If* $\mathbb{T}=\mathbb{R}, \varsigma=0, \eta=1, ł=\frac{-1}{3}, \iota=\frac{-1}{2}, \Theta(\varphi)=\varphi^{2}$ *and* $\varpi(\varphi)=\varphi^{\frac{1}{2}}$, *then* (12) *holds. Here we see that*

$$\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\mathrm{d}\varphi\right)^{\frac{\eta-ł}{\eta}}\left(\int_{\varsigma}^{\varrho}\varpi^{-\iota}(\varphi)\left[(\varphi-\varsigma)\Theta(\varphi)\right]^{\eta}\mathrm{d}\varphi\right)^{\frac{ł}{q}}=1.2029556*\varrho^{\frac{1}{4}}$$

and

$$\left(\frac{ł}{\iota-1}\right)^{ł}\int_{\varsigma}^{\varrho}(\varphi\Theta(\varphi))^{ł}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}\mathrm{d}\varphi=6.6038544*\varrho^{\frac{1}{4}}.$$

Corollary 3.25. *Assume that* $\varsigma, \varrho\in h\mathbb{N}_0$ *for* $h>0$, $0\leq\varsigma<\varrho<\infty$, $ł, \iota<0$, $\eta>0$ *and* Θ, ϖ *are positive sequences such that* $\Theta(\varphi)$ *and* $\varphi/\varpi(\varphi)$ *are nondecreasing on* (ς, ϱ).
(a) If $0<(\iota-1)/ł\leq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho-h}h\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-ł}{\eta}}\left(\sum_{\varphi=\varsigma}^{\varrho-h}h\varpi^{-\iota}(\varphi)\left[(\varphi+h-\varsigma)\Theta(\varphi)\right]^{\eta}\right)^{\frac{ł}{\eta}}$$
$$\leq\left(\frac{ł}{\iota-1}\right)^{ł}\sum_{\varphi=\varsigma}^{\varrho-h}h(\varphi\Theta(\varphi))^{ł}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}.$$

(b) If $(\iota-1)/ł\geq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho-h}h\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-ł}{\eta}}\left(\sum_{\varphi=\varsigma}^{\varrho-h}h\varpi^{-\iota}(\varphi)\left[(\varphi+h-\varsigma)\Theta(\varphi)\right]^{\eta}\right)^{\frac{ł}{\eta}}$$
$$\leq\left(\frac{ł}{\iota-1}\right)^{ł}\sum_{\varphi=\varsigma}^{\varrho-h}h\left(\frac{\varphi}{\varphi+h}\right)^{\frac{1-\iota}{ł}+\iota}(\varphi+h)^{ł}\Theta^{ł}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}.$$

Corollary 3.26. *Assume that* $\mathbb{T}=q^{\mathbb{N}_0}$ *for* $q>1$, $\varsigma, \varrho\in\mathbb{T}$, $\varsigma<\varrho<\infty$, $ł, \iota<0$, $\eta>0$ *and* Θ, ϖ *are positive sequences such that* $\Theta(\varphi)$ *and* $\varphi/\varpi(\varphi)$ *are nondecreasing on* (ς, ϱ).
(a) If $0<(\iota-1)/ł\leq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)\varphi\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-ł}{\eta}}\left(\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)\varphi\varpi^{-\iota}(\varphi)\left[(q\varphi-\varsigma)\Theta(\varphi)\right]^{\eta}\right)^{\frac{ł}{\eta}}$$
$$\leq\left(\frac{ł}{\iota-1}\right)^{ł}\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)\varphi(\varphi\Theta(\varphi))^{ł}\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{ł}}\right]^{ł-1}.$$

(b) If $(\iota-1)/\text{ł} \geq 1$, *then*

$$\left(\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)\varphi\varpi^{-\iota}(\varphi)\right)^{\frac{\eta-\text{ł}}{\eta}}\left(\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)\varphi\varpi^{-\iota}(\varphi)\left[(q\varphi-\varsigma)\,\Theta(\varphi)\right]^{\eta}\right)^{\frac{\text{ł}}{\eta}}$$
$$\leq\left(\frac{\text{ł}}{\iota-1}\right)^{\text{ł}}\sum_{\varphi=\varsigma}^{\varrho/q}(q-1)\varphi^{\text{ł}+1}q^{\frac{\iota-1}{\text{ł}}+\text{ł}-\iota}\Theta^{\text{ł}}(\varphi)\varpi^{-\iota}(\varphi)\left[1-\left(\frac{\varsigma}{\varphi}\right)^{\frac{\iota-1}{\text{ł}}}\right]^{\text{ł}-1}.$$

4. Conclusion and discussion

In this paper, we established new Hardy-type dynamic inequalities involving negative parameters on time scales. Our results involve, as a special case, classical integral inequalities proved by Benaissa, Sarikaya and Azzouz et al. Also, as another particular cases of our results, we stated new inequalities in difference and quantum calculi, which are essentially new. Finally, we presented some numerical examples which confirm our achievement. For a future work, we proposed a related open problem, whether it is possible to replace $\left(\int_{\varsigma}^{\varphi}\Theta(\vartheta)\Delta\vartheta\right)^{\text{ł}}$ for $\text{ł}>1$ with $\phi\left(\int_{\varsigma}^{\varphi}\Theta(\vartheta)\Delta\vartheta\right)$, where ϕ is a convex function.

Declarations

- **Ethical Approval**: Not applicable.
- **Availability of data and materials**: Not applicable.
- **Competing interests**: The authors declare that they have no competing interests.
- **Authors' contributions**: Authors declare that they have contributed equally to this paper. All authors have read and approved this version.

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References

[1] R. P. Agarwal, D. O'Regan, S. H. Saker, *Hardy type inequalities on time scales.*, Springer, 2016.

[2] M. Anwar, R. Bibi, M. Bohner, J. Pečarić, *Integral inequalities on time scales via the theory of isotonic linear functionals.*, Abstract and Applied Analysis, (2011) 1–16.

[3] E. Awwad, A. Saied, *Some new multidimensional Hardy-type inequalities with general kernels on time scales*, J. Math. Inequal, **16**, (2022), 393–412.

[4] E. Awwad, A. Saied, *Some weighted dynamic inequalities of hardy type with kernels on time scales nabla calculus*, J. Math. Inequal., **18**, (2024), 457–475.

[5] N. Azzouz, B. Benaissa, A. Senouci, *Some Hardy type integral inequalities with two parameters of summation*, Annales Universitatis Paedagogicae Cracoviensis. Studia Mathematica, **22(1)**, (2023), 99–109.

[6] B. Benaissa, M. Z. Sarikaya, *Generalization of some Hardy-type integral inequality with negative parameter*, Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science, (2020), 69–76.

[7] Y. Bicheng, *On a new Hardy type integral inequalities*, int. In Math, Forum, **2**, (2007), 3317–3322.

[8] M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Springer Science, Business Media, 2001.

[9] G. H. Hardy, *Note on a theorem of Hilbert*, Mathematische Zeitschrift, **6(3)**, (1920), 314–317.

[10] G. H. Hardy, *Notes on some points in the integral calculus (lxit)*, Messenger of Math., **57**, (1928), 12–16.

[11] G. H. Hardy, J. E. Littlewood, G. P´olya, *Inequalities*, Cambridge university press, 1952.

[12] S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results in mathematics, **18(1)**, (1990), 18–56.

[13] L. Leindler, *Generalization of inequalities of Hardy and Littlewood*, Acta Sci. Math.(Szeged), **31(3-4)**, (1970), 279–285.

[14] J. A. Oguntuase, L.-E. Persson, *Time scales Hardy-type inequalities via superquadracity*, Annals of Functional Analysis, **5(2)**, (2014), 61–73.

[15] P. Řehák, *Hardy inequality on time scales and its application to half-linear dynamic equations*, Journal of Inequalities and Applications, (2005), 1–13.

[16] S. Saker, *Hardy-Leindler type inequalities on time scales* , Applied Mathematics, Information Sciences, **8(6)**, (2014) 2975–2981.

[17] S. H. Saker, J. Alzabut, A. I. Saied, D. O'Regan, *New characterizations of weights on dynamic inequalities involving a Hardy operator* , J. Inequal. Appl., **73**, (2021), 1–24.

[18] S. H. Saker, A. I. Saied, D. R. Anderson, *Some new characterizations of weights in dynamic inequalities involving monotonic functions*, Qual. Theory Dyn. Syst., **20(2)**, Paper No. 49, (2021), 1–22.

[19] S. H. Saker, A. I. Saied, M. Krnić, *Some new weighted dynamic inequalities for monotone functions involving kernels*, Mediterr. J. Math., **17(2)**, 39, (2020), 1–18.