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On Stević-Sharma operator from $F(p, q, s)$ space to Stević-type space

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Abstract. The boundedness and compactness of Stević-Sharma operator from the general function space $F(p, q, s)$ to Stević-type space are investigated in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the family of all holomorphic self-maps of $\mathbb{D}$. Denote by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $0 < p, s < \infty$, $-2 < q < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the general function space $F(p, q, s)$ if

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{w\in\mathbb{D}} \int_D |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_w(z)|^2)^s dA(z) < \infty,$$

where dA is the Lebesgue measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$ and $\phi_w(z) = (w - z)/(1 - \overline{w}z)$, $w \in \mathbb{D}$. The space $F(p, q, s)$ was introduced by Zhao in [32]. For some special values of the parameters p, q, s, we can get many classical holomorphic function spaces, such as $BMOA$ space, Q_p space, Bergman space, Hardy space, Bloch space. Since for $q + s \leq -1$, $F(p, q, s)$ is the space of constant functions, we assume that $q + s > -1$. For some results on $F(p, q, s)$ space see, for instance, [10, 13, 21, 22, 28, 29, 33].

Suppose that μ is a weight, namely a strictly positive continuous function on $\mathbb{D}$. We also assume that μ is radial, that is, $\mu(z) = \mu(|z|)$ for any $z \in \mathbb{D}$. Let $n \in \mathbb{N}_0$, Stević-type space (or the n-th weighted space), denoted by $\mathcal{W}^{(n)}_\mu$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\mu = \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty.$$

The space $\mathcal{W}^{(n)}_\mu$ was introduced by Stević in [18] (see also [19, 20]); for an n-dimensional counterpart see [23]). For $n = 1$, it becomes the Bloch-type space $\mathcal{B}_\mu$. In particular, when $\mu(z) = (1 - |z|^2)^\alpha$, $\mathcal{B}_\mu$ reduces to the

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α-Bloch space, which is denoted by $\mathcal{B}^\alpha$. For $n = 0$, $\mathcal{W}_\mu^{(n)}$ becomes the weighted-type space H_μ^∞ and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu$.

Let $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, the composition and multiplication operators on $H(\mathbb{D})$ are defined by

$$C_\varphi f(z) = f(\varphi(z)) \quad \text{and} \quad M_u f(z) = u(z)f(z),$$

respectively, where $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The product of these two operators is known as the weighted composition operator $W_{u,\varphi} = u(z)f(\varphi(z))$. It is important to provide function theoretic characterizations when φ and u induce a bounded or compact weighted composition operator on various function spaces. See [3, 34] for more research about the (weighted) composition operators acting on several spaces of analytic functions. The differentiation operator D, which is defined by $Df(z) = f'(z)$ for $f \in H(\mathbb{D})$, plays an important role in operator theory and many other different areas of mathematics.

The first papers on product-type operators including the differentiation operator dealt with the operators DC_φ and $C_\varphi D$ (see, for example, [8, 11, 12, 16, 17]). In [26, 27], Stević and co-workers introduced the so-called Stević-Sharma operator as follows

$$T_{u,v,\varphi} f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. By taking some specific choices of the involving symbols, we can easily get the general product-type operators:

$$\begin{aligned}
&M_uC_\varphi = T_{u,0,\varphi}, \quad C_\varphi M_u = T_{u\circ\varphi,0,\varphi}, \quad M_uD = T_{0,u,id}, \quad DM_u = T_{u',u,id}, \quad C_\varphi D = T_{0,1,\varphi},\\
&DC_\varphi = T_{0,\varphi',\varphi}, \quad M_uC_\varphi D = T_{0,u,\varphi}, \quad M_uDC_\varphi = T_{0,u\varphi',\varphi}, \quad C_\varphi M_uD = T_{0,u\circ\varphi,\varphi},\\
&DM_uC_\varphi = T_{u',u\varphi',\varphi}, \quad C_\varphi DM_u = T_{u'\circ\varphi,u\circ\varphi,\varphi}, \quad DC_\varphi M_u = T_{\varphi'(u'\circ\varphi),\varphi'(u\circ\varphi),\varphi}.
\end{aligned}$$

Recently, there has been an increasing interest in studying the Stević-Sharma operator $T_{u,v,\varphi}$ between various spaces of analytic function. For instance, Stević et al. in [26, 27] characterized the boundedness, compactness and essential norm of $T_{u,v,\varphi}$ on the weighted Bergman space under some assumptions. Liu et al. [14, 30] studied the boundedness and compactness of $T_{u,v,\varphi}$ from Hardy space to the Bloch-type space or Zygmund-type space. Guo and Shu in [6] investigated the boundedness and compactness of $T_{u,v,\varphi}$ from Hardy space to Stević-type space. Zhu et al. in [35] provided some necessary and sufficient conditions for $T_{u,v,\varphi}$ to be bounded or compact when considered as an operator from the analytic Besov space into Bloch space. Some more related results can be found (see, e.g.,[1, 4, 5, 7, 15] and the references therein). For a generalization of the Stević-Sharma operator see [28]. For some n-dimensional generalizations see, e.g., [24] and [25].

Inspired by the above results, the purpose of the paper is to study the boundedness and compactness of the Stević-Sharma operator $T_{u,v,\varphi}$ from the general function space $F(p, q, s)$ to Stević-type space.

Throughout this paper, for nonnegative quantities X and Y, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ if there exists a positive constant C independent of X and Y such that $X \leq CY$.

2. Preliminaries

In this section we formulate some auxiliary results which will be used in the proofs of our main results. The first lemma can be found in [32].

Lemma 2.1. *Let $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$. Then*

$$\|f\|_{\mathcal{B}^{\frac{2+q}{p}}} \lesssim \|f\|_{F(p,q,s)}$$

for each $f \in F(p, q, s)$.

Lemma 2.2. (see [34]) *Let $\alpha > 0$ and $f \in \mathcal{B}^\alpha$. Then*

$$|f(z)| \lesssim \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1, \\ \|f\|_{\mathcal{B}^\alpha} \ln \frac{2}{1-|z|^2}, & \alpha = 1, \\ \frac{1}{(1-|z|^2)^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha}, & \alpha > 1, \end{cases}$$

and

$$|f^{(n)}(z)| \lesssim \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+n-1}}$$

for each $n \in \mathbb{N}$.

Replacing n by $n+2$ in [18, Lemma 2.3], we can easily get the following lemma.

Lemma 2.3. *Let $a > 0$ and*

$$D_{n+2}(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n+1 \\ \vdots & \vdots & & \vdots \\ \prod_{j=0}^{n}(a+j) & \prod_{j=0}^{n}(a+j+1) & \cdots & \prod_{j=0}^{n}(a+j+n+1) \end{vmatrix}.$$

Then $D_{n+2}(a) = \prod_{j=1}^{n+1} j!$.

For any $w \in \mathbb{D}$ and $j \in \mathbb{N}_0$, set

$$f_{j,w}(z) = \frac{(1-|w|^2)^{j+1}}{(1-\overline{w}z)^{\frac{2+q}{p}+j}}, \quad z \in \mathbb{D}. \tag{1}$$

It is well known that $f_{j,w} \in F(p,q,s)$ and $\sup_{w\in\mathbb{D}} \|f_{j,w}\|_{F(p,q,s)} \lesssim 1$ for every $j \in \mathbb{N}_0$. Moreover, it is evident that $f_{j,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$.

Lemma 2.4. *Let $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$. For any $w \in \mathbb{D} \setminus \{0\}$ and $i, k \in \{0, 1, \cdots, n+1\}$, there exists a function $g_{k,w} \in F(p,q,s)$ such that*

$$g_{k,w}^{(i)}(w) = \frac{\overline{w}^k \delta_{ki}}{(1-|w|^2)^{\frac{2+q}{p}+k-1}},$$

where δ_{ki} is the Kronecker delta.

Proof. For any $w \in \mathbb{D} \setminus \{0\}$ and constants c_j, $j \in \{0, 1, \cdots, n+1\}$, let

$$g_w(z) = \sum_{j=0}^{n+1} c_j f_{j,w}(z),$$

where $f_{j,w}$ is defined in (1). For each $k \in \{0, 1, \cdots, n+1\}$, the system of linear equations

$$\begin{cases} g_w(w) = \frac{1}{(1-|w|^2)^{\frac{2+q}{p}-1}} \sum_{j=0}^{n+1} c_j = \frac{\delta_{k0}}{(1-|w|^2)^{\frac{2+q}{p}-1}} \\ g'_w(w) = \frac{\overline{w}}{(1-|w|^2)^{\frac{2+q}{p}}} \sum_{j=0}^{n+1} c_j(\frac{2+q}{p} + j) = \frac{\overline{w}\delta_{k1}}{(1-|w|^2)^{\frac{2+q}{p}}} \\ \cdots \\ g_w^{(i)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\frac{2+q}{p}-1+k}} \sum_{j=0}^{n+1} c_j \prod_{r=0}^{k-1}(\frac{2+q}{p} + j + r) = \frac{\overline{w}^k \delta_{ki}}{(1-|w|^2)^{\frac{2+q}{p}+k-1}} \\ \cdots \\ g_w^{(n+1)}(w) = \frac{\overline{w}^{n+1}}{(1-|w|^2)^{\frac{2+q}{p}+n}} \sum_{j=0}^{n+1} c_j \prod_{r=0}^{n}(\frac{2+q}{p} + j + r) = \frac{\overline{w}^{n+1}\delta_{k(n+1)}}{(1-|w|^2)^{\frac{2+q}{p}+n}} \end{cases}$$

has a unique solution $c_j^k, j \in \{0, 1, \cdots, n+1\}$ that is independent of w, since the determinant of coefficient matrix is not equal to zero by using Lemma 2.3. For such chosen numbers $c_j^k, j \in \{0, 1, \cdots, n+1\}$ the function

$$g_{k,w}(z) := \sum_{j=0}^{n+1} c_j^k f_{j,w}(z)$$

satisfies the desired conditions. □

For $k, l \in \mathbb{N}_0$ with $k \le l$, the partial Bell polynomials are defined as follows:

$$B_{l,k}(x_1, x_2, \cdots, x_{l-k+1}) = \sum \frac{l!}{j_1! j_2! \cdots j_{l-k+1}!} \Big(\frac{x_1}{1!}\Big)^{j_1} \Big(\frac{x_2}{2!}\Big)^{j_2} \cdots \Big(\frac{x_{l-k+1}}{(l-k+1)!}\Big)^{j_{l-k+1}},$$

where the sum is overall nonnegative integers $j_1, j_2, \cdots, j_{l-k+1}$ such that $j_1 + j_2 + \cdots + j_{l-k+1} = k$ and $j_1 + 2j_2 + \cdots + (l-k+1)j_{l-k+1} = l$. For more information about the Bell polynomials, see [2].

Lemma 2.5. [6, Lemma 5] *Let $n \in \mathbb{N}$, $u, v, f \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then*

$$(T_{u,v,\varphi} f)^{(n)}(z) = \sum_{k=0}^{n+1} f^{(k)}(\varphi(z))\Omega_{n,k}(z),$$

where

$$\Omega_{n,k}(z) = \begin{cases} u^{(n)}(z), & k = 0, \\ \sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}(\varphi'(z), \cdots, \varphi^{(l-k+1)}(z)) & \\ + \sum_{l=k-1}^{n} C_n^l v^{(n-l)}(z) B_{l,k-1}(\varphi'(z), \cdots, \varphi^{(l-k+2)}(z)), & k = 1, 2, \cdots, n, \\ v(z)\varphi'(z)^n, & k = n+1. \end{cases}$$

We also denote by $\Omega_{0,0}(z) = u(z)$ and $\Omega_{0,1}(z) = v(z)$ in the following. The lemma below can be proved in a standard way (see, e.g., [3, Proposition 3.11]) and we omit the details. One can consult [9] for more research.

Lemma 2.6. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is compact if and only if $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $F(p, q, s)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|T_{u,v,\varphi} f_k\|_{\mathcal{W}_\mu^{(n)}} \to 0$ as $k \to \infty$.*

Lemma 2.7. [31, Lemma 3.2] *Fix $0 < \alpha < 1$ and let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Then*

$$\lim_{k\to\infty} \sup_{z\in\mathbb{D}} |f_k(z)| = 0.$$

3. Main results

In this section, we characterize the boundedness and compactness of $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$.

Theorem 3.1. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.*

(i) *If $p > 2 + q$, then $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is bounded if and only if*

$$M_1 := \sup_{z\in\mathbb{D}} \mu(z)\left|u^{(n)}(z)\right| < \infty, \tag{2}$$

and

$$N_k := \sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1 - |\varphi(z)|^2\right)^{\frac{2+q}{p}+k-1}} < \infty, \tag{3}$$

where $k \in \{1, 2, \cdots, n+1\}$ and $\Omega_{n,k}(z)$ are the ones in Lemma 2.5. Moreover, the following asymptotic relations hold:

$$\begin{aligned} M_1 + \sum_{k=1}^{n+1} N_k &\lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \\ &\lesssim M_1 + \sum_{k=1}^{n+1} N_k + \sum_{j=0}^{n-1} \left|u^{(j)}(0)\right| + \sum_{k=1}^{n} \frac{\sum_{j=k-1}^{n-1} \left|\Omega_{j,k}(0)\right|}{\left(1 - |\varphi(0)|^2\right)^{\frac{2+q}{p}+k-1}}. \end{aligned} \tag{4}$$

(ii) *If $p < 2 + q$, then $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is bounded if and only if (3) holds and*

$$M_2 := \sup_{z\in\mathbb{D}} \frac{\mu(z)\left|u^{(n)}(z)\right|}{\left(1 - |\varphi(z)|^2\right)^{\frac{2+q}{p}-1}} < \infty. \tag{5}$$

Moreover, the following asymptotic relations hold:

$$\begin{aligned} M_2 + \sum_{k=1}^{n+1} N_k &\lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \\ &\lesssim M_2 + \sum_{k=1}^{n+1} N_k + \frac{\sum_{j=0}^{n-1} \left|u^{(j)}(0)\right|}{\left(1 - |\varphi(0)|^2\right)^{\frac{2+q}{p}-1}} + \sum_{k=1}^{n} \frac{\sum_{j=k-1}^{n-1} \left|\Omega_{j,k}(0)\right|}{\left(1 - |\varphi(0)|^2\right)^{\frac{2+q}{p}+k-1}}. \end{aligned} \tag{6}$$

(iii) *If $p = 2 + q$ and $s > 1$, then $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is bounded if and only if (3) holds and*

$$M_3 := \sup_{z\in\mathbb{D}} \mu(z)\left|u^{(n)}(z)\right| \ln \frac{2}{1 - |\varphi(z)|^2} < \infty. \tag{7}$$

Moreover, the following asymptotic relations hold:

$$\begin{aligned} M_3 + \sum_{k=1}^{n+1} N_k &\lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \\ &\lesssim M_3 + \sum_{k=1}^{n+1} N_k + \sum_{j=0}^{n-1} \left|u^{(j)}(0)\right| \ln \frac{2}{1 - |\varphi(0)|^2} + \sum_{k=1}^{n} \frac{\sum_{j=k-1}^{n-1} \left|\Omega_{j,k}(0)\right|}{\left(1 - |\varphi(0)|^2\right)^{k}}. \end{aligned} \tag{8}$$

Proof. (i) Suppose that $p > 2 + q$, (2) and (3) hold. Then by Lemmas 2.1, 2.2 and 2.5, for each $f \in F(p,q,s)$, we have

$$\begin{aligned}\mu(z)\left|(T_{u,v,\varphi}f)^{(n)}(z)\right| &\leq \mu(z)\sum_{k=0}^{n+1}\left|f^{(k)}(\varphi(z))\right|\left|\Omega_{n,k}(z)\right| \\ &\lesssim \left(\mu(z)\left|u^{(n)}(z)\right| + \sum_{k=1}^{n+1}\frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\frac{2+q}{p}+k-1}}\right)\|f\|_{F(p,q,s)} \\ &\lesssim \left(M_1 + \sum_{k=1}^{n+1}N_k\right)\|f\|_{F(p,q,s)}. \end{aligned} \tag{9}$$

On the other hand, we have

$$\begin{aligned}\sum_{j=0}^{n-1}\left|(T_{u,v,\varphi}f)^{(j)}(0)\right| &\leq \sum_{j=0}^{n-1}\sum_{k=0}^{j+1}\left|f^{(k)}(\varphi(0))\right|\left|\Omega_{j,k}(0)\right| \\ &= \left|f(\varphi(0))\right|\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right| + \sum_{k=1}^{n}\left|f^{(k)}(\varphi(0))\right|\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right| \\ &\lesssim \left(\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right| + \sum_{k=1}^{n}\frac{\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right|}{\left(1-|\varphi(0)|^2\right)^{\frac{2+q}{p}+k-1}}\right)\|f\|_{F(p,q,s)}. \end{aligned} \tag{10}$$

In view of (9) and (10), we conclude that $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is bounded and

$$\|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \lesssim M_1 + \sum_{k=1}^{n+1}N_k + \sum_{j=0}^{n-1}\left|u^{(j)}(0)\right| + \sum_{k=1}^{n}\frac{\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right|}{\left(1-|\varphi(0)|^2\right)^{\frac{2+q}{p}+k-1}}. \tag{11}$$

Conversely, assume that $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is bounded. Taking $f_0(z) = 1 \in F(p,q,s)$, then we have

$$M_1 = \sup_{z\in\mathbb{D}}\mu(z)\left|u^{(n)}(z)\right| \leq \|T_{u,v,\varphi}1\|_{\mathcal{W}_\mu^{(n)}} \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} < \infty. \tag{12}$$

That is, (2) holds. Now assume that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}}\mu(z)\left|\Omega_{n,i}(z)\right| \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} < \infty, \tag{13}$$

where $i \in \{1,2,\cdots,k-1\}$ and $k \in \{1,2,\cdots,n+1\}$. Taking the functions $f_k(z) = z^k \in F(p,q,s)$, $k \in \{1,2,\cdots,n+1\}$ and using the boundedness of $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$, we get (note that $\sum_{i=1}^{k-1}k(k-1)\cdots(k-i+1)\varphi(z)^{k-i}\Omega_{n,i}(z) = 0$ when $k = 1$ in the inequalities below)

$$\begin{aligned}\infty &> \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \gtrsim \|T_{u,v,\varphi}f_k\|_{\mathcal{W}_\mu^{(n)}} \\ &\geq \sup_{z\in\mathbb{D}}\mu(z)\left|\varphi(z)^k u^{(n)}(z) + \sum_{i=1}^{k-1}k(k-1)\cdots(k-i+1)\varphi(z)^{k-i}\Omega_{n,i}(z) + k!\Omega_{n,k}(z)\right| \\ &\geq k!\sup_{z\in\mathbb{D}}\mu(z)\left|\Omega_{n,k}(z)\right| - M_1\|\varphi\|_\infty^k - \sum_{i=1}^{k-1}k(k-1)\cdots(k-i+1)\|\varphi\|_\infty^{k-i}\sup_{z\in\mathbb{D}}\mu(z)\left|\Omega_{n,i}(z)\right|,\end{aligned}$$

which along with (12), the assumption (13) and the fact that $\|\varphi\|_\infty \le 1$ implies

$$L_k := \sup_{z\in\mathbb{D}} \mu(z)\left|\Omega_{n,k}(z)\right| \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} < \infty, \tag{14}$$

where $k \in \{1, 2, \cdots, n+1\}$. Therefore,

$$\sup_{|\varphi(z)|\le\frac{1}{2}} \frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+k-1}} \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} < \infty. \tag{15}$$

For $w \in \mathbb{D}$ such that $|\varphi(w)| > \frac{1}{2}$ and $k \in \{1, 2, \cdots, n+1\}$, $i \in \{0, 1, \cdots, n+1\}$, choose the corresponding family of functions $g_{k,\varphi(w)} \in F(p, q, s)$ defined in Lemma 2.4 satisfying

$$g_{k,\varphi(w)}^{(i)}(\varphi(w)) = \frac{\overline{\varphi(w)}^k \delta_{ki}}{(1-|\varphi(w)|^2)^{\frac{2+q}{p}+k-1}}.$$

From the boundedness of $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$, we obtain

$$\begin{aligned}
\infty &> \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \gtrsim \|T_{u,v,\varphi} g_{k,\varphi(w)}\|_{\mathcal{W}_\mu^{(n)}} \\
&\ge \sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{i=0}^{n+1} g_{k,\varphi(w)}^{(i)}(\varphi(z))\Omega_{n,i}(z)\right| \\
&\ge \frac{\mu(w)\left|\Omega_{n,k}(w)\right||\varphi(w)|^k}{(1-|\varphi(w)|^2)^{\frac{2+q}{p}+k-1}} \\
&> \frac{1}{2^k}\frac{\mu(w)\left|\Omega_{n,k}(w)\right|}{(1-|\varphi(w)|^2)^{\frac{2+q}{p}+k-1}},
\end{aligned}$$

which implies that

$$\sup_{|\varphi(w)|>\frac{1}{2}} \frac{\mu(w)\left|\Omega_{n,k}(w)\right|}{(1-|\varphi(w)|^2)^{\frac{2+q}{p}+k-1}} \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} < \infty. \tag{16}$$

From (15) and (16), we can see that (3) holds and

$$\sum_{k=1}^{n+1} N_k \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}}, \tag{17}$$

which along with (12) yields that

$$M_1 + \sum_{k=1}^{n+1} N_k \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}}. \tag{18}$$

Hence we obtain the asymptotic expression (4) by using (11) and (18).

(ii) Assume that $p < 2 + q$, (3) and (5) hold. Then by Lemmas 2.1, 2.2 and 2.5, for each $f \in F(p, q, s)$, we have

$$\begin{aligned}\mu(z)\left|(T_{u,v,\varphi}f)^{(n)}(z)\right| &\leq \mu(z)\sum_{k=0}^{n+1}\left|f^{(k)}(\varphi(z))\right|\left|\Omega_{n,k}(z)\right| \\ &\lesssim \Big(\frac{\mu(z)\left|u^{(n)}(z)\right|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}-1}} + \sum_{k=1}^{n+1}\frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\frac{2+q}{p}+k-1}}\Big)\|f\|_{F(p,q,s)} \\ &\lesssim \Big(M_2 + \sum_{k=1}^{n+1}N_k\Big)\|f\|_{F(p,q,s)}. \end{aligned} \tag{19}$$

On the other hand, we have

$$\begin{aligned}\sum_{j=0}^{n-1}\left|(T_{u,v,\varphi}f)^{(j)}(0)\right| &\leq \sum_{j=0}^{n-1}\sum_{k=0}^{j+1}\left|f^{(k)}(\varphi(0))\right|\left|\Omega_{j,k}(0)\right| \\ &= \left|f(\varphi(0))\right|\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right| + \sum_{k=1}^{n}\left|f^{(k)}(\varphi(0))\right|\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right| \\ &\lesssim \Big(\frac{\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right|}{\left(1-|\varphi(0)|^2\right)^{\frac{2+q}{p}-1}} + \sum_{k=1}^{n}\frac{\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right|}{\left(1-|\varphi(0)|^2\right)^{\frac{2+q}{p}+k-1}}\Big)\|f\|_{F(p,q,s)}. \end{aligned} \tag{20}$$

In view of (19) and (20), we conclude that $T_{u,v,\varphi}: F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is bounded and

$$\|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \lesssim M_2 + \sum_{k=1}^{n+1}N_k + \frac{\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right|}{\left(1-|\varphi(0)|^2\right)^{\frac{2+q}{p}-1}} + \sum_{k=1}^{n}\frac{\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right|}{\left(1-|\varphi(0)|^2\right)^{\frac{2+q}{p}+k-1}}. \tag{21}$$

Conversely, suppose that $T_{u,v,\varphi}: F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is bounded. Similar to the proof in (i), we can get that (3) and (17) hold. For $w \in \mathbb{D}$ such that $\varphi(w) \neq 0$ and $i \in \{0, 1, \cdots, n+1\}$, take the corresponding function $g_{0,\varphi(w)} \in F(p,q,s)$ defined in Lemma 2.4 satisfying

$$g_{0,\varphi(w)}^{(i)}(\varphi(w)) = \frac{\delta_{0i}}{(1-|\varphi(w)|^2)^{\frac{2+q}{p}-1}}.$$

From the boundedness of $T_{u,v,\varphi}: F(p,q,s) \to \mathcal{W}_\mu^{(n)}$, we obtain

$$\begin{aligned}\infty > \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} &\gtrsim \|T_{u,v,\varphi}g_{0,\varphi(w)}\|_{\mathcal{W}_\mu^{(n)}} \\ &\geq \sup_{z\in\mathbb{D}}\mu(z)\left|\sum_{i=0}^{n+1}g_{0,\varphi(w)}^{(i)}(\varphi(z))\Omega_{n,i}(z)\right| \\ &\geq \frac{\mu(w)\left|u^{(n)}(w)\right|}{(1-|\varphi(w)|^2)^{\frac{2+q}{p}-1}}, \end{aligned}$$

which implies that

$$\sup_{w\in\mathbb{D}}\frac{\mu(w)\left|u^{(n)}(w)\right|}{(1-|\varphi(w)|^2)^{\frac{2+q}{p}-1}} \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} < \infty. \tag{22}$$

That is, (5) holds. Note that when $\varphi(w)=0$, (22) becomes (12), which can be obtained as in the proof of (i). Combining (17) with (22) gives

$$M_2+\sum_{k=1}^{n+1}N_k \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}}. \tag{23}$$

Hence we obtain the asymptotic expression (6) by using (21) and (23).

(iii) Suppose that $p=2+q$, $s>1$, (3) and (7) hold. Then by Lemmas 2.1, 2.2 and 2.5, for each $f\in F(p,q,s)$, we have

$$\begin{aligned}\mu(z)\left|(T_{u,v,\varphi}f)^{(n)}(z)\right| &\leq \mu(z)\sum_{k=0}^{n+1}\left|f^{(k)}(\varphi(z))\right|\left|\Omega_{n,k}(z)\right|\\ &\lesssim \left(\mu(z)\left|u^{(n)}(z)\right|\ln\frac{2}{1-|\varphi(z)|^2}+\sum_{k=1}^{n+1}\frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^k}\right)\|f\|_{F(p,q,s)}\\ &\lesssim \left(M_3+\sum_{k=1}^{n+1}N_k\right)\|f\|_{F(p,q,s)}.\end{aligned} \tag{24}$$

On the other hand, we have

$$\begin{aligned}\sum_{j=0}^{n-1}\left|(T_{u,v,\varphi}f)^{(j)}(0)\right| &\leq \sum_{j=0}^{n-1}\sum_{k=0}^{j+1}\left|f^{(k)}(\varphi(0))\right|\left|\Omega_{j,k}(0)\right|\\ &= \left|f(\varphi(0))\right|\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right|+\sum_{k=1}^{n}\left|f^{(k)}(\varphi(0))\right|\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right|\\ &\lesssim \left(\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right|\ln\frac{2}{1-|\varphi(0)|^2}+\sum_{k=1}^{n}\frac{\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right|}{\left(1-|\varphi(0)|^2\right)^k}\right)\|f\|_{F(p,q,s)}.\end{aligned} \tag{25}$$

In view of (24) and (25), we conclude that $T_{u,v,\varphi}:F(p,q,s)\to\mathcal{W}_\mu^{(n)}$ is bounded and

$$\|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_\mu^{(n)}} \lesssim M_3+\sum_{k=1}^{n+1}N_k+\sum_{j=0}^{n-1}\left|u^{(j)}(0)\right|\ln\frac{2}{1-|\varphi(0)|^2}+\sum_{k=1}^{n}\frac{\sum_{j=k-1}^{n-1}\left|\Omega_{j,k}(0)\right|}{\left(1-|\varphi(0)|^2\right)^k}. \tag{26}$$

Conversely, assume that $T_{u,v,\varphi}:F(p,q,s)\to\mathcal{W}_\mu^{(n)}$ is bounded. Similar to the proof in (i), we can get that (3) and (17) hold. For a fixed $w\in\mathbb{D}$, set

$$h_w(z)=\ln\frac{2}{1-\overline{\varphi(w)}z}.$$

It is easy to see that $h_w\in F(p,q,s)$ and

$$h_w(\varphi(w))=\ln\frac{2}{1-|\varphi(w)|^2},\quad h_w^{(k)}(\varphi(w))=\frac{\overline{\varphi(w)}^k(k-1)!}{\left(1-|\varphi(w)|^2\right)^k},$$

where $k \in \{1, 2, \cdots, n+1\}$. From the boundedness of $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$, we have

$$\begin{aligned}
\infty > \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_{\mu}^{(n)}} &\gtrsim \|T_{u,v,\varphi}h_w\|_{\mathcal{W}_{\mu}^{(n)}} \\
&\geq \sup_{z\in\mathbb{D}} \mu(z)\Big|\sum_{i=0}^{n+1} h_w^{(i)}(\varphi(z))\Omega_{n,i}(z)\Big| \\
&\geq \mu(w)\Big|u^{(n)}(w)\Big| \ln\frac{2}{1-|\varphi(w)|^2} - \sum_{k=1}^{n+1}\frac{\mu(w)\Big|\Omega_{n,k}(w)\Big||\varphi(w)|^k(k-1)!}{\Big(1-|\varphi(w)|^2\Big)^k},
\end{aligned}$$

which along with (17) and the fact that $|\varphi(w)| \leq 1$ yields

$$\sup_{w\in\mathbb{D}} \mu(w)\Big|u^{(n)}(w)\Big| \ln\frac{2}{1-|\varphi(w)|^2} \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_{\mu}^{(n)}} < \infty. \tag{27}$$

That is, (7) holds. Combining (17) with (27) gives

$$M_3 + \sum_{k=1}^{n+1} N_k \lesssim \|T_{u,v,\varphi}\|_{F(p,q,s)\to\mathcal{W}_{\mu}^{(n)}}. \tag{28}$$

Hence we obtain the asymptotic expression (8) by using (26) and (28). □

Theorem 3.2. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.*

(i) *If $p > 2 + q$, then $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is compact if and only if $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\Big|\Omega_{n,k}(z)\Big|}{\Big(1-|\varphi(z)|^2\Big)^{\frac{2+q}{p}+k-1}} = 0, \tag{29}$$

where $k \in \{1, 2, \cdots, n+1\}$, and $\Omega_{n,k}(z)$ are the ones in Lemma 2.5.

(ii) *If $p < 2 + q$, then $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is compact if and only if $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is bounded,* (29) *holds and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\Big|u^{(n)}(z)\Big|}{\Big(1-|\varphi(z)|^2\Big)^{\frac{2+q}{p}-1}} = 0. \tag{30}$$

(iii) *If $p = 2 + q$ and $s > 1$, then $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is compact if and only if $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is bounded,* (29) *holds and*

$$\lim_{|\varphi(z)|\to 1} \mu(z)\Big|u^{(n)}(z)\Big|\ln\frac{2}{1-|\varphi(z)|^2} = 0. \tag{31}$$

Proof. (i) Suppose that $p > 2 + q$ and $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is compact. It is evident that $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}$ is bounded. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Without loss of generality, we can assume that $\varphi(z_j) \neq 0$ for all $j \in \mathbb{N}$. For $k \in \{1, 2, \cdots, n+1\}$, let $g_{k,\varphi(z_j)}$ be the corresponding family of functions defined in Lemma 2.4. Then $\{g_{k,\varphi(z_j)}\}_{j\in\mathbb{N}}$ is a bounded sequence in $F(p,q,s)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Moreover, we have

$$g_{k,\varphi(z_j)}^{(i)}(\varphi(z_j)) = \frac{\overline{\varphi(z_j)}^k \delta_{ki}}{\Big(1-|\varphi(z_j)|^2\Big)^{\frac{2+q}{p}+k-1}},$$

where $i \in \{0, 1, \cdots, n+1\}$. From Lemma 2.6 it follows that

$$\lim_{j\to\infty} \|T_{u,v,\varphi} g_{k,\varphi(z_j)}\|_{\mathcal{W}_\mu^{(n)}} = 0. \tag{32}$$

Then

$$\|T_{u,v,\varphi} g_{k,\varphi(z_j)}\|_{\mathcal{W}_\mu^{(n)}} \geq \sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{i=0}^{n+1} g^{(i)}_{k,\varphi(z_j)}(\varphi(z))\Omega_{n,i}(z)\right| \geq \frac{\mu(z_j)\left|\Omega_{n,k}(z_j)\right||\varphi(z_j)|^k}{\left(1-|\varphi(z_j)|^2\right)^{\frac{2+q}{p}+k-1}}. \tag{33}$$

Letting $j \to \infty$ in (33) and employing (32), we get (29).

Conversely, assume that $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is bounded and (29) holds, which implies that for any $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\frac{2+q}{p}+k-1}} < \epsilon,$$

whenever $\delta < |\varphi(z)| < 1$. Moreover, from the proof of Theorem 3.1 we obtain $M_1 < \infty$ and $L_k < \infty$, which are defined in (2) and (14), respectively. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $F(p, q, s)$ such that $\sup_{j\in\mathbb{N}} \|f_j\|_{F(p,q,s)} \lesssim 1$ and f_j converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Applying Lemmas 2.1, 2.2 and 2.5, we have

$$\begin{aligned}
&\|T_{u,v,\varphi} f_j\|_{\mathcal{W}_\mu^{(n)}}\\
&= \sum_{l=0}^{n-1} \left|(T_{u,v,\varphi} f_j)^{(l)}(0)\right| + \sup_{z\in\mathbb{D}} \mu(z)\left|(T_{u,v,\varphi} f_j)^{(n)}(z)\right|\\
&\leq \sum_{l=0}^{n-1}\sum_{k=0}^{l+1} \left|f_j^{(k)}(\varphi(0))\right|\left|\Omega_{l,k}(0)\right| + \sup_{z\in\mathbb{D}} \mu(z)\left|f_j(\varphi(z))\right|\left|u^{(n)}(z)\right|\\
&\quad + \sum_{k=1}^{n+1} \sup_{|\varphi(z)|\leq\delta} \mu(z)\left|f_j^{(k)}(\varphi(z))\right|\left|\Omega_{n,k}(z)\right| + \sum_{k=1}^{n+1} \sup_{\delta<|\varphi(z)|<1} \frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\frac{2+q}{p}+k-1}}\\
&\lesssim \sum_{l=0}^{n-1}\sum_{k=0}^{l+1} \left|f_j^{(k)}(\varphi(0))\right|\left|\Omega_{l,k}(0)\right| + M_1 \sup_{w\in\mathbb{D}} \left|f_j(w)\right| + \sum_{k=1}^{n+1} \sup_{|w|\leq\delta} L_k\left|f_j^{(k)}(w)\right| + (n+1)\epsilon.
\end{aligned} \tag{34}$$

Since $f_j \in F(p, q, s) \subset \mathcal{B}^{\frac{2+q}{p}}$, where $0 < \frac{2+q}{p} < 1$, and $f_j \to 0$ uniformly on compact subset of $\mathbb{D}$ as $j \to \infty$, we have $\lim_{j\to\infty} \sup_{w\in\mathbb{D}} |f_j(w)| = 0$ by using Lemma 2.7. Moreover, $f_j^{(k)}$ also converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$ by Cauchy's estimate. In particular, $\{\varphi(0)\}$ and $\{w : |w| \leq \delta\}$ are compact subsets of $\mathbb{D}$, hence letting $k \to \infty$ in (34) yields

$$\lim_{j\to\infty} \|T_{u,v,\varphi} f_j\|_{\mathcal{W}_\mu^{(n)}} \lesssim (n+1)\epsilon.$$

From the arbitrariness of ϵ it follows that $\lim_{j\to\infty} \|T_{u,v,\varphi} f_j\|_{\mathcal{W}_\mu^{(n)}} = 0$, from which by Lemma 2.6 we deduce that $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is compact.

(ii) Suppose that $p < 2 + q$ and $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is compact. Then $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is bounded and by the proof of (i), we can see that (29) holds. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Without loss of generality, we can assume that $\varphi(z_j) \neq 0$ for all $j \in \mathbb{N}$. Let $g_{0,\varphi(z_j)}$ be the corresponding function defined in Lemma 2.4. Then $\{g_{0,\varphi(z_j)}\}_{j\in\mathbb{N}}$ is a bounded sequence in $F(p,q,s)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. Moreover, we have

$$g^{(i)}_{0,\varphi(z_j)}(\varphi(z_j)) = \frac{\delta_{0i}}{\left(1-|\varphi(z_j)|^2\right)^{\frac{2+q}{p}-1}},$$

where $i\in\{0,1,\cdots,n+1\}$. From Lemma 2.6 it follows that

$$\lim_{j\to\infty}\|T_{u,v,\varphi}g_{0,\varphi(z_j)}\|_{\mathcal{W}^{(n)}_\mu} = 0. \tag{35}$$

Then

$$\|T_{u,v,\varphi}g_{0,\varphi(z_j)}\|_{\mathcal{W}^{(n)}_\mu} \ge \sup_{z\in\mathbb{D}}\mu(z)\left|\sum_{i=0}^{n+1} g^{(i)}_{0,\varphi(z_j)}(\varphi(z))\Omega_{n,i}(z)\right| \ge \frac{\mu(z_j)\left|u^{(n)}(z_j)\right|}{\left(1-|\varphi(z_j)|^2\right)^{\frac{2+q}{p}-1}}. \tag{36}$$

Letting $j\to\infty$ in (36) and employing (35), we get (30).

Conversely, assume that $T_{u,v,\varphi}: F(p,q,s)\to\mathcal{W}^{(n)}_\mu$ is bounded and (29), (30) holds, which implies that for any $\epsilon>0$, there exists $\delta\in(0,1)$ such that

$$\frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\frac{2+q}{p}+k-1}} < \epsilon, \quad k=0,1,\cdots,n+1,$$

whenever $\delta<|\varphi(z)|<1$. Moreover, from the proof of Theorem 3.1 we obtain $M_1<\infty$ and $L_k<\infty$, which are defined in (2) and (14), respectively. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $F(p,q,s)$ such that $\sup_{j\in\mathbb{N}}\|f_j\|_{F(p,q,s)}\lesssim 1$ and f_j converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. Applying Lemmas 2.1, 2.2 and 2.5, we have

$$\begin{aligned}
&\|T_{u,v,\varphi}f_j\|_{\mathcal{W}^{(n)}_\mu}\\
&=\sum_{l=0}^{n-1}\left|(T_{u,v,\varphi}f_j)^{(l)}(0)\right| + \sup_{z\in\mathbb{D}}\mu(z)\left|(T_{u,v,\varphi}f_j)^{(n)}(z)\right|\\
&\le\sum_{l=0}^{n-1}\sum_{k=0}^{l+1}\left|f_j^{(k)}(\varphi(0))\right|\left|\Omega_{l,k}(0)\right|\\
&\quad+\sum_{k=0}^{n+1}\sup_{|\varphi(z)|\le\delta}\mu(z)\left|f_j^{(k)}(\varphi(z))\right|\left|\Omega_{n,k}(z)\right| + \sum_{k=0}^{n+1}\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\frac{2+q}{p}+k-1}}\\
&\lesssim\sum_{l=0}^{n-1}\sum_{k=0}^{l+1}\left|f_j^{(k)}(\varphi(0))\right|\left|\Omega_{l,k}(0)\right| + M_1\sup_{|w|\le\delta}\left|f_j(w)\right| + \sum_{k=1}^{n+1}\sup_{|w|\le\delta}L_k\left|f_j^{(k)}(w)\right| + (n+2)\epsilon.
\end{aligned} \tag{37}$$

Since $f_j\to 0$ uniformly on compact subset of $\mathbb{D}$ as $j\to\infty$, we have $f_j^{(k)}$ also does by Cauchy's estimate. In particular, $\{\varphi(0)\}$ and $\{w:|w|\le\delta\}$ are compact subsets of $\mathbb{D}$, hence letting $k\to\infty$ in (37) yields

$$\lim_{j\to\infty}\|T_{u,v,\varphi}f_j\|_{\mathcal{W}^{(n)}_\mu}\lesssim(n+2)\epsilon.$$

From the arbitrariness of ϵ it follows that $\lim_{j\to\infty}\|T_{u,v,\varphi}f_j\|_{\mathcal{W}^{(n)}_\mu}=0$, from which by Lemma 2.6 we deduce that $T_{u,v,\varphi}: F(p,q,s)\to\mathcal{W}^{(n)}_\mu$ is compact.

(iii) Suppose that $p = 2 + q$, $s > 1$ and $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is compact. Then $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is bounded and by the proof of (i), we can see that (29) holds. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Set

$$h_j(z) = \Big(\ln \frac{2}{1 - \overline{\varphi(z_j)}z}\Big)^2 \Big(\ln \frac{2}{1 - |\varphi(z_j)|^2}\Big)^{-1}.$$

Then $\{h_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $F(p, q, s)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Moreover, we have

$$h_j(\varphi(z_j)) = \ln \frac{2}{1 - |\varphi(z_j)|^2},$$

and

$$h_j^{(i)}(\varphi(z_j)) = \frac{2(i-1)!\overline{\varphi(z_j)}^i}{(1 - |\varphi(z_j)|^2)^i} + \frac{2C_i\overline{\varphi(z_j)}^i}{(1 - |\varphi(z_j)|^2)^i}\Big(\ln \frac{2}{1 - |\varphi(z_j)|^2}\Big)^{-1}, \quad i = 1, 2, \cdots, n+1,$$

where $C_1 = 0$, $C_2 = 1$ and $C_i = (i-1)C_{i-1} + (i-2)!$ for $i > 2$. From Lemma 2.6 it follows that

$$\lim_{j\to\infty} \|T_{u,v,\varphi}h_j\|_{\mathcal{W}_\mu^{(n)}} = 0. \tag{38}$$

Then

$$\begin{aligned}
&\|T_{u,v,\varphi}h_j\|_{\mathcal{W}_\mu^{(n)}}\\
\geq &\sup_{z\in\mathbb{D}} \mu(z)\Big|\sum_{i=0}^{n+1} h_j^{(i)}(\varphi(z))\Omega_{n,i}(z)\Big|\\
\geq &\mu(z_j)\big|u^{(n)}(z_j)\big| \ln \frac{2}{1 - |\varphi(z_j)|^2}\\
&- \sum_{i=1}^{n+1} \mu(z_j)\big|\Omega_{n,i}(z_j)\big|\Big|\frac{2(i-1)!\overline{\varphi(z_j)}^i}{(1 - |\varphi(z_j)|^2)^i} + \frac{2C_i\overline{\varphi(z_j)}^i}{(1 - |\varphi(z_j)|^2)^i}\Big(\ln \frac{2}{1 - |\varphi(z_j)|^2}\Big)^{-1}\Big|.
\end{aligned} \tag{39}$$

Since $\Big(\ln \frac{2}{1-|\varphi(z_j)|^2}\Big)^{-1}$ is bounded, letting $j \to \infty$ in (39) and employing (29) and (38), we get (31).

Conversely, assume that $T_{u,v,\varphi} : F(p, q, s) \to \mathcal{W}_\mu^{(n)}$ is bounded and (29), (31) holds, which implies that for any $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\mu(z)\big|u^{(n)}(z)\big| \ln \frac{2}{1 - |\varphi(z)|^2} < \epsilon,$$

and

$$\frac{\mu(z)\big|\Omega_{n,k}(z)\big|}{\big(1 - |\varphi(z)|^2\big)^k} < \epsilon, \quad k = 1, 2, \cdots, n+1,$$

whenever $\delta < |\varphi(z)| < 1$. Moreover, from the proof of Theorem 3.1 we obtain $M_1 < \infty$ and $L_k < \infty$, which are defined in (2) and (14), respectively. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $F(p, q, s)$ such that $\sup_{j\in\mathbb{N}} \|f_j\|_{F(p,q,s)} \lesssim 1$ and f_j converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Applying Lemmas 2.1, 2.2 and 2.5, we have

$$\begin{aligned}
&\|T_{u,v,\varphi}f_j\|_{\mathcal{W}_\mu^{(n)}}\\
&=\sum_{l=0}^{n-1}\left|(T_{u,v,\varphi}f_j)^{(l)}(0)\right|+\sup_{z\in\mathbb{D}}\mu(z)\left|(T_{u,v,\varphi}f_j)^{(n)}(z)\right|\\
&\le\sum_{l=0}^{n-1}\sum_{k=0}^{l+1}\left|f_j^{(k)}(\varphi(0))\right|\left|\Omega_{l,k}(0)\right|+\sum_{k=0}^{n+1}\sup_{|\varphi(z)|\le\delta}\mu(z)\left|f_j^{(k)}(\varphi(z))\right|\left|\Omega_{n,k}(z)\right|\\
&\quad+\sup_{\delta<|\varphi(z)|<1}\mu(z)\left|u^{(n)}(z)\right|\ln\frac{2}{1-|\varphi(z)|^2}+\sum_{k=1}^{n+1}\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)\left|\Omega_{n,k}(z)\right|}{\left(1-|\varphi(z)|^2\right)^k}\\
&\lesssim\sum_{l=0}^{n-1}\sum_{k=0}^{l+1}\left|f_j^{(k)}(\varphi(0))\right|\left|\Omega_{l,k}(0)\right|+M_1\sup_{|w|\le\delta}\left|f_j(w)\right|+\sum_{k=1}^{n+1}\sup_{|w|\le\delta}L_k\left|f_j^{(k)}(w)\right|+(n+2)\epsilon. \qquad (40)
\end{aligned}$$

Since $f_j \to 0$ uniformly on compact subset of $\mathbb{D}$ as $j \to \infty$, we have $f_j^{(k)}$ also does by Cauchy's estimate. In particular, $\{\varphi(0)\}$ and $\{w : |w| \le \delta\}$ are compact subsets of $\mathbb{D}$, hence letting $k \to \infty$ in (40) yields

$$\lim_{j\to\infty}\|T_{u,v,\varphi}f_j\|_{\mathcal{W}_\mu^{(n)}}\lesssim(n+2)\epsilon.$$

From the arbitrariness of ϵ it follows that $\lim_{j\to\infty}\|T_{u,v,\varphi}f_j\|_{\mathcal{W}_\mu^{(n)}}=0$, from which by Lemma 2.6 we deduce that $T_{u,v,\varphi} : F(p,q,s) \to \mathcal{W}_\mu^{(n)}$ is compact. □

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