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Numerical radius bounds for certain operator matrices

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Abstract. Due to the unavoidable role of operator matrices in advancing the study of the numerical radius, we further investigate upper and lower bounds for the numerical radii of certain operator matrices.

The obtained results are generalizations of some known results in the literature.

We compare our results with existing results using a rigorous approach and numerical examples to show the value of our findings.

1. Introduction

For a complex Hilbert space $\mathbb{H}$, with inner product $\langle \cdot, \cdot \rangle$, the set of all bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ forms a C^*-algebra, that we denote $\mathbb{B}(\mathbb{H})$. It is customary to use upper case letters to denote elements of $\mathbb{B}(\mathbb{H})$, while lower case letters are reserved for scalars.

Comparisons among elements of $\mathbb{B}(\mathbb{H})$ are usually done through scalar quantities associated with each element of this C^*-algebra. Among those important quantities, the operator norm (or usual operator norm) and the numerical radius have received considerable attention.

We recall that if $A \in \mathbb{B}(\mathbb{H})$, the operator norm and the numerical radius of A are defined, respectively, by

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \text{ and } \omega(A) = \sup |\langle Ax, x \rangle|.$$

It is evident that $\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle|$. Due to this, the relation $\omega(A) \leq \|A\|$ becomes clear.

The numerical radius defines a norm when $\mathbb{H}$ is a complex Hilbert space, and this norm happens to be equivalent to the operator norm via the relation [13, Theorem 1.3-1]

$$\frac{\|A\|}{2} \leq \omega(A) \leq \|A\|. \tag{1}$$

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One significance of this relation is that calculations of the operator norm are usually easier than those for the numerical radius. Thus, (1) provides an interval that contains the numerical radius for sure. However, this interval can be so wide that the error in approximating $\omega(A)$ becomes large. This is one reason researchers have devoted considerable effort to tightening the two bounds in (1).

Among those celebrated results, we cite the following three results from [18, 19, 26], respectively,

$$\omega(A) \leq \frac{1}{2} \left\| |A| + |A^*| \right\|, \tag{2}$$

$$\omega^2(A) \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|, \tag{3}$$

and

$$\omega(A) \leq \frac{1}{2} \left(\|A\| + \omega(\widetilde{A}) \right),$$

where A^* is the adjoint operator of A, $|A|$ is the absolute value of A defined as $|A| = (A^*A)^{\frac{1}{2}}$ and $\widetilde{A}$ is the Aluthge transform of A. We recall that the Aluthge transform appeared in [4] for the first time, where if $A = U|A|$ is the polar decomposition of A, then $\widetilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$.

As seen in the cited references, the above three inequalities sharpen the second inequality in (1). In fact, (2) is sharper than (3), but the significance of (3) was in having a lower bound for $\omega^2(A)$ related to the upper bound in (3).

Discussing numerical radius bounds can take so long that we refer the reader to [1–3, 5–10, 12, 14, 15, 18–20, 23] as a sample of a wider list of references treating this idea.

In (1), if $A^2 = O$, the zero operator, then $\omega(A) = \frac{\|A\|}{2}$, and $\omega(A) = \|A\|$ if A is normal. In the case where $\omega(A) = \frac{\|A\|}{2}$, it was shown in [22] that

$$\left\| \mathfrak{R}(A) \right\| = \left\| \mathfrak{I}(A) \right\| = \frac{\|A\|}{2},$$

where $\mathfrak{R}(A) = \frac{A+A^*}{2}$ and $\mathfrak{I}(A) = \frac{A-A^*}{2i}$ are the real and imaginary parts of A, respectively.

Techniques used to obtain numerical radius bounds include inner product properties, Cauchy-Schwarz type inequalities, and operator matrices, to name a few.

We recall here that given $A, B, C, D \in \mathbb{B}(\mathbb{H})$, the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is referred to as an operator matrix, which is an element of $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$. Operator matrices have played a significant role in advancing our understanding of the numerical radius, as one can see in [1, 6, 14, 15, 20, 21, 25]. Actually, operator matrices have become a powerful tool in this field.

Some basic properties of the numerical radii of certain 2×2 operator matrices are listed below, as found in [11] and [14].

$$\omega\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix} \right) = \max(\omega(A), \omega(B))$$

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) = \omega\left(\begin{bmatrix} O & B \\ A & O \end{bmatrix} \right)$$

$$\omega\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) = \max(\omega(A+B), \omega(A-B)).$$

In particular,

$$\omega\left(\begin{bmatrix} O & B \\ B & O \end{bmatrix} \right) = \omega(B).$$

Among the most interesting upper bounds for the numerical radius of an operator matrix, the following was shown in [1], for an $n \times n$ operator matrix $T = [T_{ij}] \in \mathbb{B}(\oplus_{k=1}^{n}\mathbb{H})$:

$$\omega(T) \leq \omega\left([t_{ij}]\right), \text{ where } t_{ij} = \begin{cases} \omega(T_{ij}), & i = j \\ \|T_{ij}\|, & i \neq j \end{cases}. \tag{4}$$

The significance of this result is due to the replacement of operators in T by scalars in $[t_{ij}]$. In fact, this result is related to the following celebrated lemma [16].

Lemma 1.1. *Let $T_{ij} \in \mathbb{B}(\mathbb{H})$, for $i, j = 1, \ldots, n$, and let $T = [T_{ij}]$. Then $\|T\| \leq \left\|[\|T_{ij}\|]\right\|$.*

In this paper, we prove new forms of numerical radius inequalities for certain 2×2, 3×3, 4×4, and $n \times n$ operator matrices. The obtained results will be compared with (4) and other bounds. We will see that the newly obtained bounds form a new set of independent bounds that could be used as alternatives to existing ones.

For example, we will show that if $A, B, C, D \in \mathbb{B}(\mathbb{H})$, and if $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then

$$\omega^2(T) \leq \frac{1}{2}\left\|\begin{bmatrix} A^*A + C^*C + AA^* + BB^* & A^*B + C^*D + AC^* + BD^* \\ B^*A + D^*C + CA^* + DB^* & B^*B + D^*D + CC^* + DD^* \end{bmatrix}\right\|.$$

Upon letting $B = C = D = O$, this reduces to (3). So, this presents a generalization of (3) to operator matrices. Furthermore, we deduce that if $T = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$, then

$$\omega(T) \leq \sqrt{\frac{1}{2}\left(\|AA^* + BB^*\| + \|B^*B + D^*D\| + \max\{\|AA^*\|, \|DD^*\|\}\right)}. \tag{5}$$

After that, we discuss 3×3 and 4×4 operator matrices and find new bounds that present extensions of previous results in the literature.

For example, we show that if $T = \begin{bmatrix} O & A & O \\ B & O & C \\ O & D & O \end{bmatrix}$, then

$$\omega(T) \leq \frac{1}{\sqrt{2}}\sqrt{\left\|\begin{bmatrix} B^*B + AA^* & O & B^*C + AD^* \\ O & A^*A + D^*D + BB^* + CC^* & O \\ C^*B + DA^* & O & C^*C + DD^* \end{bmatrix}\right\|}$$

and that if $T = \begin{bmatrix} A & O & B \\ O & C & O \\ B & O & A \end{bmatrix}$, then

$$\omega(T) = \max\{\omega(A + B), \omega(A - B), \omega(C)\}.$$

More forms are discussed in the sequel, with comparisons and the relation with existing results.

Before proceeding, we list some lemmas that we will need. The first result is the so-called mixed Cauchy-Schwarz inequality [17].

Lemma 1.2. *Let $A \in \mathbb{B}(\mathbb{H})$ and let $x, y \in \mathbb{H}$. If $0 \leq t \leq 1$, then*

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2t}x, x\right\rangle \left\langle |T^*|^{2(1-t)}y, y\right\rangle.$$

The second lemma is the operator Jensen inequality [24, Theorem 1.2].

Lemma 1.3. *Let $f : [m, M] \to \mathbb{R}$ be a concave function, and let $A \in \mathbb{B}(\mathbb{H})$ be a self-adjoint operator with spectrum in $[m, M]$. If $x \in \mathbb{H}$ is a unit vector, then*

$$f(\langle Ax, x\rangle) \geq \langle f(A)x, x\rangle.$$

if f is convex, the inequality is reversed.

2. Results for 2 × 2 operator matrices

We begin our results by presenting the following two upper bounds for the numerical radius of a 2×2 operator matrix.

Theorem 2.1. *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$ and let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then for any $0 \le t \le 1$,*

$$\omega^2(T) \le \left\| \begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix} \right\|^t \left\| \begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix} \right\|^{1-t}.$$

Moreover,

$$\omega^2(T) \le \left\| \begin{bmatrix} K & L \\ M & N \end{bmatrix} \right\|.$$

where,

$$K = t(A^*A + C^*C) + (1-t)(AA^* + BB^*),$$

$$L = t(A^*B + C^*D) + (1-t)(AC^* + BD^*),$$

$$M = t(B^*A + D^*C) + (1-t)(CA^* + DB^*)$$

and

$$N = t(B^*B + D^*D) + (1-t)(CC^* + DD^*).$$

In particular, upon letting $t = \frac{1}{2}$,

$$\omega^2(T) \le \left\| \begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix} \right\|^{\frac{1}{2}} \left\| \begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix} \right\|^{\frac{1}{2}}, \tag{6}$$

and

$$\omega^2(T) \le \frac{1}{2} \left\| \begin{bmatrix} A^*A + C^*C + AA^* + BB^* & A^*B + C^*D + AC^* + BD^* \\ B^*A + D^*C + CA^* + DB^* & B^*B + D^*D + CC^* + DD^* \end{bmatrix} \right\|. \tag{7}$$

Proof. Let $x \in \mathbb{H} \oplus \mathbb{H}$ be a unit vector. Applying Lemma 1.2, then Lemma 1.3, we have

$$\begin{aligned}
&\left| \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} x, x \right\rangle \right|^2 \\
\le\ & \left\langle \left| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right|^{2t} x, x \right\rangle \left\langle \left| \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \right|^{2(1-t)} x, x \right\rangle \\
=\ & \left\langle \begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix}^t x, x \right\rangle \left\langle \begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix}^{1-t} x, x \right\rangle \\
\le\ & \left\langle \begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix} x, x \right\rangle^t \left\langle \begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix} x, x \right\rangle^{1-t}.
\end{aligned}$$

We deduce the first inequality after taking the supremum over all unit vectors. For the second inequality, applying the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
&\left\langle \begin{bmatrix} A^*A+C^*C & A^*B+C^*D \\ B^*A+D^*C & B^*B+D^*D \end{bmatrix} x,x \right\rangle^t \left\langle \begin{bmatrix} AA^*+BB^* & AC^*+BD^* \\ CA^*+DB^* & CC^*+DD^* \end{bmatrix} x,x \right\rangle^{1-t} \\
\leq\ & t\left\langle \begin{bmatrix} A^*A+C^*C & A^*B+C^*D \\ B^*A+D^*C & B^*B+D^*D \end{bmatrix} x,x \right\rangle + (1-t)\left\langle \begin{bmatrix} AA^*+BB^* & AC^*+BD^* \\ CA^*+DB^* & CC^*+DD^* \end{bmatrix} x,x \right\rangle \\
=\ & \left\langle \begin{bmatrix} K & L \\ M & N \end{bmatrix} x,x \right\rangle.
\end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathbb{H} \oplus \mathbb{H}$ implies the second inequality. □

We discuss some special cases of Theorem 2.1.

- If we let $B = C = D = O$ in (6), we obtain

  $$\omega^2(A) \leq \|A^*A\|^{\frac{1}{2}} \|AA^*\|^{\frac{1}{2}} = \|A\|^2,$$

  which is the right inequality in (1).

- If we let $B = C = D = O$ in (7), we obtain

  $$\omega^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|,$$

  which is (3).

Corollary 2.2. *Let $A, B, D \in \mathbb{B}(\mathbb{H})$ and let $T = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$. Then for any $0 \leq t \leq 1$,*

$$\omega^2(T) \leq \left\| \begin{bmatrix} tA^*A + (1-t)(AA^* + BB^*) & tA^*B + (1-t)BD^* \\ tB^*A + (1-t)DB^* & t(B^*B + D^*D) + (1-t)DD^* \end{bmatrix} \right\|.$$

In particular,

$$\omega^2(T) \leq \frac{1}{2} \left\| \begin{bmatrix} A^*A + AA^* + BB^* & A^*B + BD^* \\ B^*A + DB^* & B^*B + D^*D + DD^* \end{bmatrix} \right\|$$

Remark 2.3. *In this remark, we give an example to show that Corollary 2.2 can be better than* (4)*. For this purpose, let*

$$A = \begin{bmatrix} -6 & -5 \\ -4 & -9 \end{bmatrix}, B = \begin{bmatrix} -1 & 3 \\ -7 & -9 \end{bmatrix}, D = \begin{bmatrix} -6 & -6 \\ 5 & -3 \end{bmatrix}.$$

Then

$$\omega(T) \approx 14.4393, \omega\left(\begin{bmatrix} \omega(A) & \|B\| \\ 0 & \omega(D) \end{bmatrix}\right) \approx 16.0302,$$

and

$$\frac{1}{\sqrt{2}} \left\| \begin{bmatrix} A^*A + AA^* + BB^* & A^*B + BD^* \\ B^*A + DB^* & B^*B + D^*D + DD^* \end{bmatrix} \right\|^{\frac{1}{2}} \approx 15.7117.$$

Although the following bound is weaker than that in Corollary 2.2, we present it as an alternative easier form.

Corollary 2.4. *Let $A, B, D \in \mathbb{B}(\mathbb{H})$ and let $T = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$. Then*

$$\omega^2(T) \le \frac{1}{2}\left(\|AA^* + BB^*\| + \|D^*D + B^*B\| + \max\left\{\|A\|^2, \|D\|^2\right\}\right).$$

Proof. It follows from (7) that

$$\begin{aligned}
& w^2(T) \\
& \le \frac{1}{2}\left\|\begin{bmatrix} A^*A + AA^* + BB^* & A^*B + BD^* \\ B^*A + DB^* & D^*D + DD^* + B^*B \end{bmatrix}\right\| \\
& = \frac{1}{2}\left\|\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} + \begin{bmatrix} AA^* & O \\ O & D^*D \end{bmatrix} + \begin{bmatrix} BB^* & BD^* \\ DB^* & DD^* \end{bmatrix}\right\| \\
& \le \frac{1}{2}\left(\left\|\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix}\right\| + \left\|\begin{bmatrix} BB^* & BD^* \\ DB^* & DD^* \end{bmatrix}\right\| + \left\|\begin{bmatrix} AA^* & O \\ O & D^*D \end{bmatrix}\right\|\right) \\
& = \frac{1}{2}\left(\left\|\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix}\right\| + \left\|\begin{bmatrix} BB^* & BD^* \\ DB^* & DD^* \end{bmatrix}\right\| + \max\{\|AA^*\|, \|D^*D\|\}\right) \\
& = \frac{1}{2}\left(\left\|\begin{bmatrix} A^* & O \\ B^* & O \end{bmatrix}\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right\| + \left\|\begin{bmatrix} B & O \\ D & O \end{bmatrix}\begin{bmatrix} B^* & D^* \\ O & O \end{bmatrix}\right\| + \max\{\|AA^*\|, \|D^*D\|\}\right) \\
& = \frac{1}{2}\left(\left\|\begin{bmatrix} A & B \\ O & O \end{bmatrix}\begin{bmatrix} A^* & O \\ B^* & O \end{bmatrix}\right\| + \left\|\begin{bmatrix} B^* & D^* \\ O & O \end{bmatrix}\begin{bmatrix} B & O \\ D & O \end{bmatrix}\right\| + \max\{\|AA^*\|, \|D^*D\|\}\right) \\
& = \frac{1}{2}\left(\left\|\begin{bmatrix} AA^* + BB^* & O \\ O & O \end{bmatrix}\right\| + \left\|\begin{bmatrix} B^*B + D^*D & O \\ O & O \end{bmatrix}\right\| + \max\{\|AA^*\|, \|D^*D\|\}\right) \\
& = \frac{1}{2}\left(\|AA^* + BB^*\| + \|D^*D + B^*B\| + \max\{\|AA^*\|, \|D^*D\|\}\right) \\
& = \frac{1}{2}\left(\|AA^* + BB^*\| + \|D^*D + B^*B\| + \max\left\{\|A\|^2, \|D\|^2\right\}\right),
\end{aligned}$$

as required. □

Remark 2.5. *In this remark, we give an example to show that the bound found in Corollary 2.4 can be better than* (4). *However, we point out that for most examples,* (4) *is better. If we let*

$$A = \begin{bmatrix} 2 & 8 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & -4 \\ -9 & 6 \end{bmatrix}, D = \begin{bmatrix} 3 & -3 \\ -7 & -7 \end{bmatrix},$$

numerical calculations show that $\omega(T) \approx 11.7178$,

$$\left\{\frac{1}{2}\left(\|AA^* + BB^*\| + \|D^*D + B^*B\| + \max\left\{\|A\|^2, \|D\|^2\right\}\right)\right\}^{\frac{1}{2}} \approx 14.1823,$$

and

$$\omega\left(\begin{bmatrix} \omega(A) & \|B\| \\ 0 & \omega(D) \end{bmatrix}\right) \approx 14.6684.$$

Another consequence of Corollary 2.2 is stated next. The coming remark explains its connection with (4).

Corollary 2.6. *Let $A, B, D \in \mathbb{B}(\mathbb{H})$ and let $T = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$. Then for any $0 \le t \le 1$,*

$$\begin{aligned}
&\omega^2(T)\\
&\le \frac{1}{2}\left(\|tA^*A + (1-t)(AA^* + BB^*)\| + \|t(D^*D + B^*B) + (1-t)DD^*\|\right)\\
&\quad + \frac{1}{2}\sqrt{\left(\|tA^*A + (1-t)(AA^* + BB^*)\| - \|t(D^*D + B^*B) + (1-t)DD^*\|\right)^2 + 4\|tA^*B + (1-t)BD^*\|^2}.
\end{aligned}$$

Proof. It follows from Corollary 2.2 that

$$\begin{aligned}
&\omega^2(T)\\
&\le \left\|\begin{bmatrix} tA^*A + (1-t)(AA^* + BB^*) & tA^*B + (1-t)BD^* \\ tB^*A + (1-t)DB^* & t(D^*D + B^*B) + (1-t)DD^* \end{bmatrix}\right\|\\
&\le \left\|\begin{bmatrix} \|tA^*A + (1-t)(AA^* + BB^*)\| & \|tA^*B + (1-t)BD^*\| \\ \|tB^*A + (1-t)DB^*\| & \|t(D^*D + B^*B) + (1-t)DD^*\| \end{bmatrix}\right\| \text{(by Lemma 1.1)}\\
&= \frac{1}{2}\left(\|tA^*A + (1-t)(AA^* + BB^*)\| + \|t(D^*D + B^*B) + (1-t)DD^*\|\right)\\
&\quad + \frac{1}{2}\sqrt{\left(\|tA^*A + (1-t)(AA^* + BB^*)\| - \|t(D^*D + B^*B) + (1-t)DD^*\|\right)^2 + 4\|tA^*B + (1-t)BD^*\|^2},
\end{aligned}$$

as required. □

Remark 2.7. *Let $T = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$.*

(i) If we put $t = 0$, in Corollary 2.6, then we obtain

$$\omega^2(T) \le \frac{1}{2}\left(\|D\|^2 + \|AA^* + BB^*\| + \sqrt{\left(\|D\|^2 - \|AA^* + BB^*\|\right)^2 + 4\|BD^*\|^2}\right). \tag{8}$$

If we let

$$A = \begin{bmatrix} 9 & -8 \\ 0 & 7 \end{bmatrix}, D = \begin{bmatrix} -7 & 7 \\ -3 & -2 \end{bmatrix}, B = \begin{bmatrix} 6 & -4 \\ 10 & 2 \end{bmatrix},$$

then numerical calculations show that

$$\frac{1}{2}\left(\|D\|^2 + \|AA^* + BB^*\| + \sqrt{\left(\|D\|^2 - \|AA^* + BB^*\|\right)^2 + 4\|BD^*\|^2}\right) \approx 254.552,$$

$$\omega^2\left(\begin{bmatrix} \omega(A) & \|B\| \\ 0 & \omega(D) \end{bmatrix}\right) \approx 260.863, \omega^2\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right) \approx 156.639.$$

This gives an example where (8) can be better than (4), which provides a sense of (8).

(ii) If we put $t = 1$, in Corollary 2.6, then we get

$$w^2(T) \le \frac{1}{2}\left(\|A\|^2 + \|D^*D + B^*B\| + \sqrt{\left(\|A\|^2 - \|D^*D + B^*B\|\right)^2 + 4\|A^*B\|^2}\right).$$

Remark 2.8. *It follows from Corollaries 2.4 and 2.6, and parts (i) and (ii) in Remark 2.7 that*

$$w\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \le \frac{\sqrt{2}}{2}\sqrt{\|AA^* + BB^*\| + \|B\|^2 + \|A\|^2},$$

$$\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \leq \frac{1}{2}\sqrt{\|A^*A + AA^* + BB^*\| + \|B\|^2 + \sqrt{\left(\|A^*A + AA^* + BB^*\| - \|B\|^2\right)^2 + 4\|A^*B\|^2}},$$

$$\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \leq \|AA^* + BB^*\|^{\frac{1}{2}},$$

and

$$\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \leq \frac{\sqrt{2}}{2}\sqrt{\|A\|^2 + \|B\|^2 + \sqrt{\left(\|A\|^2 - \|B\|^2\right)^2 + 4\|A^*B\|^2}}.$$

Consequently, from the above inequalities, we infer that

$$\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \leq \min\{\alpha, \beta, \lambda, \mu\}$$

where

$$\alpha = \frac{\sqrt{2}}{2}\sqrt{\|AA^* + BB^*\|^{\frac{1}{2}} + \|B\| + \|A\|^2},$$

$$\beta = \frac{1}{2}\sqrt{\|A^*A + AA^* + BB^*\| + \|B\|^2 + \sqrt{\left(\|A^*A + AA^* + BB^*\| - \|B\|^2\right)^2 + 4\|A^*B\|^2}},$$

$$\lambda = \|AA^* + BB^*\|^{\frac{1}{2}},$$

$$\mu = \frac{\sqrt{2}}{2}\sqrt{\|A\|^2 + \|B\|^2 + \sqrt{\left(\|A\|^2 - \|B\|^2\right)^2 + 4\|A^*B\|^2}}.$$

Remark 2.9. *Replacing $B = D = O$ in inequality (5) we get the right side of inequality (1)*

$$\omega(A) \leq \sqrt{\frac{1}{2}\left(\|AA^*\| + \|AA^*\|\right)} = \sqrt{\|AA^*\|} = \sqrt{\|A\|^2} = \|A\|.$$

By letting $A = D = O$ in (7), we obtain the following upper bound for the numerical radius of an off-diagonal 2×2 operator matrix.

Corollary 2.10. *Let $B, C \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega^2\left(\begin{bmatrix} O & B \\ C & O \end{bmatrix}\right) \leq \max\left(\|BB^* + C^*C\|, \|B^*B + CC^*\|\right).$$

Remark 2.11. *Among the most known upper bound for $\omega\left(\begin{bmatrix} O & B \\ C & O \end{bmatrix}\right)$, we have [15]*

$$\omega\left(\begin{bmatrix} O & B \\ C & O \end{bmatrix}\right) \leq \frac{\|B\| + \|C\|}{2}. \tag{9}$$

If we let $B = \begin{bmatrix} 0 & -3 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$, we find that

$$\omega^2\left(\begin{bmatrix} O & B \\ C & O \end{bmatrix}\right) \approx 11.3021, \left(\frac{\|B\| + \|C\|}{2}\right)^2 \approx 16.1523,$$

and

$$\max\left(\|BB^* + C^*C\|, \|B^*B + CC^*\|\right) \approx 15.8009,$$

showing that the bound in Corollary 2.10 can be better than that in (9).

3. Results for 3 × 3 operator matrices

This section presents upper bounds for the numerical radii of certain 3×3 operator matrices.

Theorem 3.1. *Let $A,B,C,D\in\mathbb{B}(\mathbb{H})$ and let $T=\begin{bmatrix} O & A & O\\ B & O & C\\ O & D & O\end{bmatrix}$. Then for any $0\le t\le 1$,*

$$\omega^2(T)\le \left\|\begin{bmatrix} B^*B & O & B^*C\\ O & A^*A+D^*D & O\\ C^*B & O & C^*C\end{bmatrix}\right\|^{t}\left\|\begin{bmatrix} AA^* & O & AD^*\\ O & BB^*+CC^* & O\\ DA^* & O & DD^*\end{bmatrix}\right\|^{1-t},$$

and

$$w^2(T)\le \left\|\begin{bmatrix} tB^*B+(1-t)AA^* & O & tB^*C+(1-t)AD^*\\ O & t(A^*A+D^*D)+(1-t)(BB^*+CC^*) & O\\ tC^*B+(1-t)DA^* & O & tC^*C+(1-t)DD^*\end{bmatrix}\right\|.$$

In particular, when $t=\frac{1}{2}$,

$$\omega^2(T)\le \left\|\begin{bmatrix} B^*B & O & B^*C\\ O & A^*A+D^*D & O\\ C^*B & O & C^*C\end{bmatrix}\right\|^{\frac{1}{2}}\left\|\begin{bmatrix} AA^* & O & AD^*\\ O & BB^*+CC^* & O\\ DA^* & O & DD^*\end{bmatrix}\right\|^{\frac{1}{2}},$$

and

$$w^2(T)\le \frac{1}{2}\left\|\begin{bmatrix} AA^*+B^*B & O & AD^*+B^*C\\ O & A^*A+BB^*+CC^*+D^*D & O\\ DA^*+C^*B & O & C^*C+DD^*\end{bmatrix}\right\|. \tag{10}$$

Proof. Applying Lemma 1.2, then Lemma 1.3, we have for any unit vector $x\in\oplus_{k=1}^{3}\mathbb{H}$,

$$\begin{aligned}
&\left|\left\langle\begin{bmatrix} O & A & O\\ B & O & C\\ O & D & O\end{bmatrix}x,x\right\rangle\right|^2\\
&\le\left\langle\left|\begin{bmatrix} O & A & O\\ B & O & C\\ O & D & O\end{bmatrix}\right|^{2t}x,x\right\rangle\left\langle\left|\begin{bmatrix} O & A & O\\ B & O & C\\ O & D & O\end{bmatrix}^*\right|^{2(1-t)}x,x\right\rangle\\
&=\left\langle\begin{bmatrix} B^*B & O & B^*C\\ O & A^*A+D^*D & O\\ C^*B & O & C^*C\end{bmatrix}^{t}x,x\right\rangle\left\langle\begin{bmatrix} AA^* & O & AD^*\\ O & BB^*+CC^* & O\\ DA^* & O & DD^*\end{bmatrix}^{1-t}x,x\right\rangle\\
&\le\left\langle\begin{bmatrix} B^*B & O & B^*C\\ O & A^*A+D^*D & O\\ C^*B & O & C^*C\end{bmatrix}x,x\right\rangle^{t}\left\langle\begin{bmatrix} AA^* & O & AD^*\\ O & BB^*+CC^* & O\\ DA^* & O & DD^*\end{bmatrix}x,x\right\rangle^{1-t}.
\end{aligned}$$

Now take the supremum for all unit vectors x to get the first desired result. To get the second inequality, we apply the arithmetic-geometric mean inequality to obtain

$$\begin{aligned}
&\left\langle \begin{bmatrix} B^*B & O & B^*C \\ O & A^*A+D^*D & O \\ C^*B & O & C^*C \end{bmatrix} x,x \right\rangle^{t} \left\langle \begin{bmatrix} AA^* & O & AD^* \\ O & BB^*+CC^* & O \\ DA^* & O & DD^* \end{bmatrix} x,x \right\rangle^{1-t} \\
&\leq t\left\langle \begin{bmatrix} B^*B & O & B^*C \\ O & A^*A+D^*D & O \\ C^*B & O & C^*C \end{bmatrix} x,x \right\rangle + (1-t)\left\langle \begin{bmatrix} AA^* & O & AD^* \\ O & BB^*+CC^* & O \\ DA^* & O & DD^* \end{bmatrix} x,x \right\rangle \\
&= \left\langle \begin{bmatrix} tB^*B+(1-t)AA^* & O & tB^*C+(1-t)AD^* \\ O & t(A^*A+D^*D)+(1-t)(BB^*+CC^*) & O \\ tC^*B+(1-t)DA^* & O & tC^*C+(1-t)DD^* \end{bmatrix} x,x \right\rangle \\
&\leq \left\| \begin{bmatrix} tB^*B+(1-t)AA^* & O & tB^*C+(1-t)AD^* \\ O & t(A^*A+D^*D)+(1-t)(BB^*+CC^*) & O \\ tC^*B+(1-t)DA^* & O & tC^*C+(1-t)DD^* \end{bmatrix} \right\|.
\end{aligned}$$

Taking the supremum over such unit vectors x yields the desired result and completes the proof. □

Remark 3.2. *In this remark, we compare between* (10) *in Theorem 3.1 and* (4)*. It turns out that the two presented bounds for $\omega(T)$ are incomparable. For this conclusion, let*

$$A = \begin{bmatrix} 4 & 4 \\ 0 & -5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & -4 \\ 1 & -5 \end{bmatrix}, D = \begin{bmatrix} -3 & 4 \\ 3 & 0 \end{bmatrix}.$$

Then numerical calculations show that

$$\omega(T) \approx 5.35182, \omega\left(\begin{bmatrix} 0 & \|A\| & 0 \\ \|B\| & 0 & \|C\| \\ 0 & \|D\| & 0 \end{bmatrix} \right) \approx 7.55894,$$

and

$$\frac{\sqrt{2}}{2} \left\| \begin{bmatrix} AA^*+B^*B & O & AD^*+B^*C \\ O & A^*A+BB^*+CC^*+D^*D & O \\ DA^*+C^*B & O & C^*C+DD^* \end{bmatrix} \right\|^{\frac{1}{2}} \approx 6.92274,$$

showing that Theorem 3.1 can be better than (4)*.*

We point out that in some numerical examples, (4) *is better than Theorem 3.1.*

The following theorem is a numerical radius inequality for another 3×3 operator matrix. The proof is similar to that of Theorem 3.1 and is left to the reader.

Theorem 3.3. *Let $E, F, G, H, I \in \mathbb{B}(\mathbb{H})$ and let $T = \begin{bmatrix} E & O & F \\ O & G & O \\ H & O & I \end{bmatrix}$. Then for any $0 \leq t \leq 1$,*

$$\begin{aligned}
&\omega^2(T) \\
&\leq \left\| \begin{bmatrix} t(E^*E+H^*H)+(1-t)(EE^*+FF^*) & O & t(E^*F+H^*I)+(1-t)(EH^*+FI^*) \\ O & tG^*G+(1-t)GG^* & O \\ t(F^*E+I^*H)+(1-t)(HE^*+IF^*) & O & t(F^*F+I^*I)+(1-t)(HH^*+II^*) \end{bmatrix} \right\|.
\end{aligned}$$

In particular,

$$\omega^2(T) \leq \frac{1}{2} \left\| \begin{bmatrix} E^*E+H^*H+EE^*+FF^* & O & E^*F+H^*I+EH^*+FI^* \\ O & G^*G+GG^* & O \\ (F^*E+I^*H)+(HE^*+IF^*) & O & F^*F+I^*I+HH^*+II^* \end{bmatrix} \right\|.$$

Now we employ Theorems 3.1 and 3.3 to obtain the following general bound for any 3×3 operator matrix.

Theorem 3.4. *Let* $T = \begin{bmatrix} E & A & F \\ B & G & C \\ H & D & I \end{bmatrix} \in \mathbb{B}(\oplus_{k=1}^{3}\mathbb{H})$. *Then*

$$\begin{aligned}\omega(T) \le\ & \frac{\sqrt{2}}{2}\left\|\begin{bmatrix} AA^*+B^*B & O & AD^*+B^*C \\ O & A^*A+BB^*+CC^*+D^*D & O \\ DA^*+C^*B & O & C^*C+DD^* \end{bmatrix}\right\|^{1/2} \\ & +\frac{\sqrt{2}}{2}\left\|\begin{bmatrix} E^*E+H^*H+EE^*+FF^* & O & E^*F+H^*I+EH^*+FI^* \\ O & G^*G+GG^* & O \\ (F^*E+I^*H)+(HE^*+IF^*) & O & F^*F+I^*I+HH^*+II^* \end{bmatrix}\right\|^{1/2}.\end{aligned}$$

Proof. We have

$$T = \begin{bmatrix} O & A & O \\ B & O & C \\ O & D & O \end{bmatrix} + \begin{bmatrix} E & O & F \\ O & G & O \\ H & O & I \end{bmatrix}.$$

This implies

$$\begin{aligned}\omega(T) &= w\left(\begin{bmatrix} O & A & O \\ B & O & C \\ O & D & O \end{bmatrix} + \begin{bmatrix} E & O & F \\ O & G & O \\ H & O & I \end{bmatrix}\right) \\ &\le w\begin{bmatrix} O & A & O \\ B & O & C \\ O & D & O \end{bmatrix} + w\begin{bmatrix} E & O & F \\ O & G & O \\ H & O & I \end{bmatrix} \\ &\le \frac{\sqrt{2}}{2}\left\|\begin{bmatrix} AA^*+B^*B & O & AD^*+B^*C \\ O & A^*A+BB^*+CC^*+D^*D & O \\ DA^*+C^*B & O & C^*C+DD^* \end{bmatrix}\right\|^{1/2} \\ &\quad +\frac{\sqrt{2}}{2}\left\|\begin{bmatrix} E^*E+H^*H+EE^*+FF^* & O & E^*F+H^*I+EH^*+FI^* \\ O & G^*G+GG^* & O \\ (F^*E+I^*H)+(HE^*+IF^*) & O & F^*F+I^*I+HH^*+II^* \end{bmatrix}\right\|^{1/2},\end{aligned}$$

where we have used Theorems 3.1 and 3.3 to obtain the last inequality. This completes the proof. □

While the above results provide upper bounds for the numerical radii of certain operator matrices, the following result provides an identity for the numerical radius of a certain 3×3 operator matrix.

Theorem 3.5. *Let* $T = \begin{bmatrix} A & O & B \\ O & C & O \\ B & O & A \end{bmatrix}$. *Then*

$$\omega(T) = \max(\omega(A+B), \omega(A-B), \omega(C))$$

Proof. Let $\mathcal{I}$ be the identity operator in $\mathbb{B}(\mathbb{H})$, and let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathcal{I} & O & \mathcal{I} \\ O & \sqrt{2}\mathcal{I} & O \\ -\mathcal{I} & O & \mathcal{I} \end{bmatrix}$. Then it can be easily seen that U is unitary, and

$$UTU^* = \begin{bmatrix} A+B & O & O \\ O & C & O \\ O & O & A-B \end{bmatrix}.$$

Since $\omega(UTU^*) = \omega(T)$, we immediately reach the desired result. □

Remark 3.6. *Letting $A = C = O$ in Theorem 3.5 implies*

$$\omega \begin{bmatrix} O & O & B \\ O & B & O \\ B & O & O \end{bmatrix} = \omega(B)$$

4. More advanced forms

We conclude this work by presenting an identity for the numerical radius of a certain 4×4 operator matrix and a lower bound for a certain $n \times n$ operator matrix.

Theorem 4.1. *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$ and let $T = \begin{bmatrix} A & O & O & B \\ O & C & O & O \\ O & O & D & O \\ B & O & O & A \end{bmatrix}$. Then*

$$\omega(T) = \max(\omega(A+B), \omega(A-B), \omega(C), \omega(D))$$

Proof. With $\mathcal{I}$ being the identity operator, $U = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathcal{I} & O & O & -\mathcal{I} \\ O & \sqrt{2}\mathcal{I} & O & O \\ O & O & \sqrt{2}\mathcal{I} & O \\ \mathcal{I} & O & O & \mathcal{I} \end{bmatrix}$ is a unitary operator, and

$$U^*TU = \begin{bmatrix} A+B & O & O & O \\ O & C & O & O \\ O & O & D & O \\ O & O & O & A-B \end{bmatrix}.$$

Then the result follows immediately noting that $\omega(U^*TU) = \omega(T)$. □

At the end of this paper, we present the following theorem, which gives a lower bound for the numerical radius of an $n \times n$ off-diagonal operator matrix. When $n = 2$, this result was shown in [15, Theorem 2.3], and has been discussed recently in [20].

Theorem 4.2. *Let $A_1, A_2, \ldots, A_n \in \mathbb{B}(\mathbb{H})$ and let $T = \begin{bmatrix} O & O & O & A_1 \\ O & O & A_2 & O \\ O & \ddots & O & O \\ A_n & O & O & O \end{bmatrix}$. Then, for any natural number k,*

$$\omega(T) \geq \max_{i=1,\ldots,n} \sqrt[2k]{\omega\left((A_i A_{n-i+1})^k\right)} \tag{11}$$

Proof. Let $T = \begin{bmatrix} O & O & O & A_1 \\ O & O & A_2 & O \\ O & \ddots & O & O \\ A_n & O & O & O \end{bmatrix}$, where n is even integer. Then

$T^2 = \begin{bmatrix} A_1A_n & O & O & O & O & O & O & O \\ O & A_2A_{n-1} & O & O & O & O & O & O \\ O & O & \ddots & O & O & O & O & O \\ O & O & O & A_{\frac{n}{2}}A_{\frac{n}{2}+1} & O & O & O & O \\ O & O & O & O & A_{\frac{n}{2}+1}A_{\frac{n}{2}} & O & O & O \\ O & O & O & O & O & \ddots & O & O \\ O & O & O & O & O & O & A_{n-1}A_2 & O \\ O & O & O & O & O & O & O & A_nA_1 \end{bmatrix}$. Thus, for any natural number k,

$T^{2k} = \begin{bmatrix} (A_1A_n)^k & O & O & O & O & O & O & O \\ O & (A_2A_{n-1})^k & O & O & O & O & O & O \\ O & O & \ddots & O & O & O & O & O \\ O & O & O & \left(A_{\frac{n}{2}}A_{\frac{n}{2}+1}\right)^k & O & O & O & O \\ O & O & O & O & \left(A_{\frac{n}{2}+1}A_{\frac{n}{2}}\right)^k & O & O & O \\ O & O & O & O & O & \ddots & O & O \\ O & O & O & O & O & O & (A_{n-1}A_2)^k & O \\ O & O & O & O & O & O & O & (A_nA_1)^k \end{bmatrix}$. This implies that

$\max_{i=1,\dots,n} \omega\left((A_iA_{n-i+1})^k\right) = \omega(T^{2k}) \leq \omega^{2k}(T)$. This proves (11) when n is even.

Now, consider $T = \begin{bmatrix} O & O & O & A_1 \\ O & O & A_2 & O \\ O & \ddots & O & O \\ A_n & O & O & O \end{bmatrix}$ where n is odd. Then

$T^2 = \begin{bmatrix} A_1A_n & O & O & O & O & O & O \\ O & A_2A_{n-1} & O & O & O & O & O \\ O & O & \ddots & O & O & O & O \\ O & O & O & \left(A_{\frac{n+1}{2}}\right)^2 & O & O & O \\ O & O & O & O & \ddots & O & O \\ O & O & O & O & O & A_{n-1}A_2 & O \\ O & O & O & O & O & 0 & A_nA_1 \end{bmatrix}$. Then for any natural number k, we have

$T^{2k} = \begin{bmatrix} (A_1A_n)^k & O & O & O & O & O & O \\ O & (A_2A_{n-1})^k & O & O & O & O & O \\ O & O & \ddots & O & O & O & O \\ O & O & O & \left(A_{\frac{n+1}{2}}\right)^{2k} & O & O & O \\ O & O & O & O & \ddots & O & O \\ O & O & O & O & O & (A_{n-1}A_2)^k & O \\ O & O & O & O & O & O & (A_nA_1)^k \end{bmatrix}$. This implies that

$\max_{i=1,\dots,n,} \omega\left((A_iA_{n-i+1})^k\right) = \omega(T^{2k}) \leq \omega^{2k}(T)$, which is exactly (11). This completes the proof. □

Remark 4.3. *We know that [14]*

$$\omega\left(\begin{bmatrix} O & A_1 \\ A_2 & O \end{bmatrix}\right) \geq \frac{1}{2}\|A_1 + A_2^*\|. \tag{12}$$

Theorem 4.2 asserts that for any natural number k,

$$\omega\left(\begin{bmatrix} O & A_1 \\ A_2 & O \end{bmatrix}\right) \geq \max\left(\sqrt[2k]{\omega\left((A_1A_2)^k\right)}, \sqrt[2k]{\omega\left((A_2A_1)^k\right)}\right). \tag{13}$$

In this remark, we give a numerical example to show that (13) *can be better than* (12). *However, we point out that other examples show the opposite conclusion. This means that Theorem 4.2 provides a new independent lower bound to the existing literature.*
If we let $A_1 = \begin{bmatrix} -3 & -3 \\ -2 & 3 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 3 & 3 \\ 2 & 0 \end{bmatrix}$, *we find that, for* $k = 1$,

$$\omega\left(\begin{bmatrix} O & A_1 \\ A_2 & O \end{bmatrix}\right) \approx 4.11341, \frac{1}{2}\|A_1 + A_2^*\| \approx 1.65139,$$

and

$$\max\left(\sqrt[2k]{\omega\left((A_1A_2)^k\right)}, \sqrt[2k]{\omega\left((A_2A_1)^k\right)}\right) \approx 4.10658,$$

which indicates that (13) *is much sharper than* (12) *in this example.*

Remark 4.4. *Due to the power inequality* $\omega(T^k) \leq \omega^k(T)$, *for* $T \in \mathbb{B}(\mathbb{H})$ *and* $k \in \mathbb{N}$, *it follows that*

$$\sqrt[2k]{\omega\left((A_iA_{n-i+1})^k\right)} \leq \sqrt{\omega(A_iA_{n-i+1})}.$$

Thus, the best bound in Theorem 4.2 is attained when $k = 1$. *This means that*

$$\omega(T) \geq \max_{i=1,\ldots,n} \sqrt{\omega(A_iA_{n-i+1})}.$$

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