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On Păltănea operator linking Hermite polynomials

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Abstract. In this article, we introduce a new variant of the Păltănea operator based on modified Hermite polynomials of two variables. We establish several approximation properties for this operator including Voronovskaja-type theorem in weighted space and illustrate its convergence both numerically and graphically. Additionally, we capture a new interesting operator based on a composition method and then establish an asymptotic formula for the composition operator. We also study its convergence in terms of first and second order modulus of continuity and present a theorem based on difference estimates.

1. Introduction

In this article, we deal with the operators associated with the modified Hermite polynomials [8] of two variables denoted by

$$H_k(n,\alpha) = k!\sum_{s=0}^{[\frac{k}{2}]}\frac{n^{k-2s}\alpha^s}{(k-2s)!s!}, \qquad \alpha \geq 0, n, k \in \mathbb{N}.$$

These polynomials have a high importance in the field of mathematics and are named after the French mathematician Charles Hermite (1822–1901). Recently, we have introduced and examined operators based on Hermite polynomials (see [20, 22]). Here, we introduce a new variant of Păltănea operator based on the Hermite polynomials, which for $f \in C[0,\infty)$ (the class of all continuous functions), $\rho, \alpha, x \geq 0$ and $n \in \mathbb{N}$ is defined as follows:

$$(\mathcal{P}^{\alpha}_{n,\rho} f)(x) := \int_0^{\infty} \phi^{\rho}_{n,k}(x,y) f(y) dy, \tag{1}$$

where

$$\phi^{\rho}_{n,k}(x,y) = n\rho \sum_{k=1}^{\infty} q^{\alpha}_{n,k}(x) \cdot p^{\rho}_{n,k}(y) + \delta(y) \cdot q^{\alpha}_{n,0}(x),$$

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with $q^{\alpha}_{n,k}(x) = e^{-nx-\alpha x^2} H_k(n,\alpha)\frac{x^k}{k!}$ and $p^{\rho}_{n,k}(y) = e^{-n\rho y}\frac{(n\rho y)^{k\rho-1}}{\Gamma(k\rho)}$.
If $\alpha = 0$ then $H_k(n,0) = n^k$ and this operator reduces to the well known Szász-Mirakyan-Păltănea operator (see [19])

$$(\mathcal{P}^{0}_{n,\rho}f)(x) = n\rho\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} p^{\rho}_{n,k}(y)f(y)dy + e^{-nx}f(0), \tag{2}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$ and $p^{\rho}_{n,k}(y)$ is defined above.
Further, if $\rho = 1$ this operator reduces to the well-known Phillips operator (see [27]).

$$(\mathcal{P}^{0}_{n,1}f)(x) = n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k-1}(y)f(y)dy + e^{-nx}f(0).$$

Over several decades, researchers have been interested in studying the Păltănea operator, and many studies have been conducted in this field (see [14, 18, 30, 32]). In this article, we have introduced a new variant of Păltănea operator and study their approximation properties including Voronovskaja-type theorem in weighted space. Recently, the study of operator compositions has garnered significant interest among researchers. In this context, we have introduced a novel operator, denoted as $\mathcal{Q}^{\alpha}_{n,\rho}$, which is formulated using the composition method. Subsequently, we examine several approximation characteristics of this operator. Throughout this article, we denote $\exp_{\gamma}(y) = e^{\gamma y}$ and $e_m(y) = y^m, m = 0, 1, 2, \ldots$.

2. Moments Estimation

The following Lemmas are important for obtaining the main results.

Lemma 2.1. *The moment generating function for the operator $\mathcal{P}^{\alpha}_{n,\rho}$ is given by*

$$\left(\mathcal{P}^{\alpha}_{n,\rho}\exp_{\gamma}(y)\right)(x) = \exp\left(\alpha x^2\left(\frac{(n\rho)^{2\rho}}{(n\rho-\gamma)^{2\rho}} - 1\right) + nx\left(\frac{(n\rho)^{\rho}}{(n\rho-\gamma)^{\rho}} - 1\right)\right). \tag{3}$$

The m-th moment is denoted by $\mathcal{P}^{\alpha}_{n,\rho,m}$, i.e., $\mathcal{P}^{\alpha}_{n,\rho,m}(x) := \left(\mathcal{P}^{\alpha}_{n,\rho}e_m\right)(x)$ where $m = 0, 1, 2, \ldots$, then we have

$$\begin{aligned}
\mathcal{P}^{\alpha}_{n,\rho,0}(x) =&\ 1\\
\mathcal{P}^{\alpha}_{n,\rho,1}(x) =&\ x + \frac{2\alpha x^2}{n}\\
\mathcal{P}^{\alpha}_{n,\rho,2}(x) =&\ x^2 + \frac{nx + n\rho x + 2\alpha x^2 + 4\alpha\rho x^2 + 4\alpha n\rho x^3 + 4\alpha^2\rho x^4}{n^2\rho}\\
\mathcal{P}^{\alpha}_{n,\rho,3}(x) =&\ x^3 + \frac{2nx + 3n\rho x + n\rho^2 x + 4\alpha x^2 + 12\alpha\rho x^2 + 3n^2\rho x^2 + 8\alpha\rho^2 x^2 + 3n^2\rho^2 x^2 + 12\alpha n\rho x^3}{n^3\rho^2}\\
&+ \frac{18\alpha n\rho^2 x^3 + 12\alpha^2\rho x^4 + 24\alpha^2\rho^2 x^4 + 6\alpha n^2\rho^2 x^4 + 12\alpha^2 n\rho^2 x^5 + 8\alpha^3\rho^2 x^6}{n^3\rho^2}\\
\mathcal{P}^{\alpha}_{n,\rho,4}(x) =&\ x^4 + \frac{6nx + 11n\rho x + 6n\rho^2 x + n\rho^3 x + 12\alpha x^2 + 44\alpha\rho x^2 + 11n^2\rho x^2 + 48\alpha\rho^2 x^2 + 18n^2\rho^2 x^2}{n^4\rho^3}\\
&+ \frac{16\alpha\rho^3 x^2 + 7n^2\rho^3 x^2 + 44\alpha n\rho x^3 + 108\alpha n\rho^2 x^3 + 6n^3\rho^2 x^3 + 64\alpha n\rho^3 x^3 + 6n^3\rho^3 x^3}{n^4\rho^3}\\
&+ \frac{44\alpha^2\rho x^4 + 144\alpha^2\rho^2 x^4 + 36\alpha n^2\rho^2 x^4 + 112\alpha^2\rho^3 x^4 + 48\alpha n^2\rho^3 x^4 + 72\alpha^2 n\rho^2 x^5}{n^4\rho^3}\\
&+ \frac{120\alpha^2 n\rho^3 x^5 + 8\alpha n^3 r^3 x^5 + 48\alpha^3\rho^2 x^6 + 96\alpha^3\rho^3 x^6 + 24\alpha^2 n^2\rho^3 x^6 + 32\alpha^3 n\rho^3 x^7 + 16\alpha^4\rho^3 x^8}{n^4\rho^3}.
\end{aligned}$$

Proof. By using equation (1), we have

$$\begin{aligned}\left(\mathcal{P}_{n,\rho}^{\alpha}\exp_{\gamma}(y)\right)(x) &= \frac{1}{e^{\alpha x^2+nx}}\sum_{k=1}^{\infty}H_k(n,\alpha)\frac{x^k}{k!}\int_0^{\infty}\frac{n\rho}{\Gamma(k\rho)}e^{-n\rho y}(n\rho y)^{k\rho-1}e^{\gamma y}dy+\frac{H_0(\alpha,n)}{e^{\alpha x^2+nx}}\\ &= \frac{1}{e^{\alpha x^2+nx}}\sum_{k=1}^{\infty}\frac{x^k}{k!}H_k(n,\alpha)\frac{(n\rho)^{k\rho}}{\Gamma(k\rho)}\int_0^{\infty}e^{-(n\rho-\gamma)y}y^{k\rho-1}dy+\frac{H_0(\alpha,n)}{e^{\alpha x^2+nx}}\\ &= \frac{1}{e^{\alpha x^2+nx}}\sum_{k=0}^{\infty}H_k(n,\alpha)\frac{x^k}{k!}\frac{(n\rho)^{k\rho}}{(n\rho-\gamma)^{k\rho}}\\ &= \exp\left(\alpha x^2\left(\frac{(n\rho)^{2\rho}}{(n\rho-\gamma)^{2\rho}}-1\right)+nx\left(\frac{(n\rho)^{\rho}}{(n\rho-\gamma)^{\rho}}-1\right)\right).\end{aligned}$$

The moment generating function are related with the m-th moment by the following relation

$$\mathcal{P}_{n,\rho,m}^{\alpha}(x)=\left[\frac{\partial^m}{\partial\gamma^m}\left(\exp\left(\alpha x^2\left(\frac{(n\rho)^{2\rho}}{(n\rho-\gamma)^{2\rho}}-1\right)+nx\left(\frac{(n\rho)^{\rho}}{(n\rho-\gamma)^{\rho}}-1\right)\right)\right)\right]_{\gamma=0}.$$

□

Lemma 2.2. *The m-th central moments for the operator $\mathcal{P}_{n,\rho}^{\alpha}$ is denoted by $\mu_{n,\rho,m}^{\alpha}(x):=(\mathcal{P}_{n,\rho,m}^{\alpha}(e_1-xe_0)^m)(x)$ where $m=0,1,2,...$, then we have*

$$\mu_{n,\rho,m}^{\alpha}(x)=\left[\frac{\partial^m}{\partial\gamma^m}\left(\exp\left(\alpha x^2\left(\frac{(n\rho)^{2\rho}}{(n\rho-\gamma)^{2\rho}}-1\right)+nx\left(\frac{(n\rho)^{\rho}}{(n\rho-\gamma)^{\rho}}-1\right)-\gamma x\right)\right)\right]_{\gamma=0}.$$

In particular, we have

$$\begin{aligned}\mu_{n,\rho,0}^{\alpha}(x) &=1\\ \mu_{n,\rho,1}^{\alpha}(x) &=\frac{2\alpha x^2}{n}\\ \mu_{n,\rho,2}^{\alpha}(x) &=\frac{nx+n\rho x+2\alpha x^2+4\alpha\rho x^2+4\alpha^2\rho x^4}{n^2\rho}\\ \mu_{n,\rho,3}^{\alpha}(x) &=\frac{2nx+3n\rho x+n\rho^2x+4\alpha x^2+12\alpha\rho x^2+8\alpha\rho^2x^2+6\alpha n\rho x^3}{n^3\rho^2}\\ &\quad+\frac{6\alpha n\rho^2x^3+12\alpha^2\rho x^4+24\alpha^2\rho^2x^4+8\alpha^3\rho^2x^6}{n^3\rho^2}\\ \mu_{n,\rho,4}^{\alpha}(x) &=\frac{6nx+11n\rho x+6n\rho^2x+n\rho^3x+n\rho^3x+12\alpha x^2+44\alpha\rho x^2+3n^2\rho x^2+48\alpha\rho^2x^2+6n^2\rho^2x^2}{n^4\rho^3}\\ &\quad+\frac{16\alpha\rho^3x^2+3n^2\rho^3x^2+28\alpha n\rho x^3+60\alpha n\rho^2x^3+32\alpha n\rho^3x^3+44\alpha^2\rho x^4}{n^4\rho^3}\\ &\quad+\frac{144\alpha^2\rho^2x^4+112\alpha^2\rho^3x^4+24\alpha^2n\rho^2x^5+24\alpha^2n\rho^3x^5+48\alpha^3r^2x^6+96\alpha^3\rho^3x^6+16\alpha^4\rho^3x^8}{n^4\rho^3}\\ \mu_{n,\rho,6}^{\alpha}(x) &=\frac{120nx+274n\rho x+225n\rho^2x+85n\rho^3x+15n\rho^4x+n\rho^5x+240\alpha x^2+1096\alpha\rho x^2+130n^2\rho x^2}{n^6\rho^5}\end{aligned}$$

$$+\frac{1800\alpha\rho^2x^2+375n^2\rho^2x^2+1360\alpha\rho^3x^2+385n^2\rho^3x^2+480\alpha\rho^4x^2+165n^2\rho^4x^2+64\alpha\rho^5x^2}{n^6\rho^5}$$
$$+\frac{25n^2\rho^5x^2+808\alpha n\rho x^3+2850\alpha n\rho^2x^3+15n^3\rho^2x^3+3760\alpha n\rho^3x^3+45n^3\rho^3x^3+2190\alpha n\rho^4x^3}{n^6\rho^5}$$
$$+\frac{45n^3\rho^4x^3+472\alpha n\rho^5x^3+15n^3\rho^5x^3+1096\alpha^2\rho x^4+5400\alpha^2\rho^2x^4+330\alpha n^2\rho^2x^4+9520\alpha^2\rho^3x^4}{n^6\rho^5}$$
$$+\frac{960\alpha n^2\rho^3x^4+7200\alpha^2\rho^4x^4+930\alpha n^2\rho^4x^4+1984\alpha^2\rho^5x^4+300\alpha n^2\rho^5x^4+1500\alpha^2n\rho^2x^5}{n^6\rho^5}$$
$$+\frac{5160\alpha^2n\rho^3x^5+5880\alpha^2n\rho^4x^5+2220\alpha^2n\rho^5x^5+1800\alpha^3\rho^2x^6+8160\alpha^3\rho^3x^6+180\alpha^2n^2\rho^3x^6}{n^6\rho^5}$$
$$+\frac{12000\alpha^3\rho^4x^6+360\alpha^2n^2\rho^4x^6+5760\alpha^3\rho^5x^6+180\alpha^2n^2\rho^5x^6+1040\alpha^3n\rho^3x^7+2640\alpha^3n\rho^4x^7}{n^6\rho^5}$$
$$+\frac{1600\alpha^3n\rho^5x^7+1360\alpha^4\rho^3x^8+4800\alpha^4\rho^4x^8+4160\alpha^4\rho^5x^8+240\alpha^4n\rho^4x^9+240\alpha^4n\rho^5x^9}{n^6\rho^5}$$
$$+\frac{480\alpha^5\rho^4x^{10}+960\alpha^5\rho^5x^{10}+64\alpha^6\rho^5x^{12}}{n^6\rho^5}.$$

3. Convergence Analysis

Theorem 3.1. *For any continuous and bounded function f defined on $\mathbb{R}^+$, $b\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$, then*

$$\lim_{n\to\infty}(\mathcal{P}^{\alpha}_{bn,\rho}f(ny))\left(\frac{x}{n}\right)=(\mathcal{P}^{0}_{b,\rho}f)(x),$$

and

$$\lim_{\rho\to\infty}(\mathcal{P}^{\alpha}_{n,\rho}f)(x)=(G^{\alpha}_{n}f)(x),$$

where G^{α}_{n} is the generalization of Szász-Mirakyan operator defined by Krech [26].

Proof. For the operator $\mathcal{P}^{\alpha}_{n,\rho}$ defined in equation (3) for $s\in\mathbb{R}$, $i=\sqrt{-1}$, we have

$$\begin{aligned}\lim_{n\to\infty}(\mathcal{P}^{\alpha}_{bn,\rho}\exp_{isn}(y))\left(\frac{x}{n}\right)&=\lim_{n\to\infty}\exp\left(\frac{\alpha x^2}{n^2}\left(\frac{(bn\rho)^{2\rho}}{(bn\rho-isn)^{2\rho}}-1\right)\right)\\&\quad\times\exp\left(\frac{bnx}{n}\left(\frac{(bn\rho)^{\rho}}{(bn\rho-isn)^{\rho}}-1\right)\right)\\&=\exp\left(bx\left(\left(\frac{b\rho}{b\rho-is}\right)^{\rho}-1\right)\right)\\&=(\mathcal{P}^{0}_{b,\rho}f)(x).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\lim_{\rho\to\infty}(\mathcal{P}^{\alpha}_{n,\rho}\exp_{is}(y))(x)&=\exp\left(\alpha x^2\left(\frac{(n\rho)^{2\rho}}{(n\rho-is)^{2\rho}}-1\right)+nx\left(\frac{(n\rho)^{\rho}}{(n\rho-is)^{\rho}}-1\right)\right)\\&=\exp\left(nx\left(e^{\frac{is}{n}}-1\right)+\alpha x^2\left(e^{\frac{2is}{n}}-1\right)\right)\\&=(G^{\alpha}_{n}\exp_{is}(y))(x).\end{aligned}$$

Thus by using equation (3) and by Theorem 1 of [6], we get the desired result. □

Let $C_E[0,\infty)$ denote the space of all continuous functions $h(x)$ defined on $\mathbb{R}^+$ such that $|h(x)| \leq E(1+x^2)$ for any fixed $E \geq 0$ and $\lim_{n\to\infty} \frac{h(x)}{1+x^2}$ exists and is finite. In [25], the weighted modulus of continuity is denoted by

$$\Omega(h,\delta) = \sup_{\substack{0\leq\xi\leq\delta \\ t\in[0,\infty)}} \frac{|h(t+\xi)-h(t)|}{(1+\xi^2)(1+t^2)}, \quad h \in C_E[0,\infty).$$

Voronovskaja-type theorems are fundamental results for understanding the limits and capabilities of positive linear operators in approximating functions. They quantify the rate of convergence and provide precise error estimates. Voronovskaja-type theorems have been established for classical operators such as the Bernstein, Szász-Mirakjan, Baskakov, Meyer-König and Zeller operators, as well as their variants. For results in weighted spaces, we refer to the following articles (see [1–3, 5, 9, 11, 13, 31]).

Theorem 3.2. *Let f be any polynomial function and $f'' \in C_E[0,\infty)$ where $x \in [0,\infty)$, the following inequality holds*

$$\left|(\mathcal{P}^{\alpha}_{n,\rho}f)(x) - f(x) - \frac{2\alpha x^2}{n}f'(x) - \frac{nx+n\rho x+2\alpha x^2+4\alpha\rho x^2+4\alpha^2\rho x^4}{2n^2\rho}f''(x)\right|$$
$$\leq 16(1+x^2)\left(\frac{nx+n\rho x+2\alpha x^2+4\alpha\rho x^2+4\alpha^2\rho x^4}{n^2\rho}\right)\Omega\left(f'', \left(\frac{\mu^{\alpha}_{n,\rho,6}(x)}{\mu^{\alpha}_{n,\rho,2}(x)}\right)^{\frac{1}{4}}\right).$$

Proof. By employing the operator $\mathcal{P}^{\alpha}_{n,\rho}$ on a Taylor's expansion of the function f, we get

$$(\mathcal{P}^{\alpha}_{n,\rho}f(t))(x) - f(x) = \mu^{\alpha}_{n,\rho,1}(x)f'(x) + \mu^{\alpha}_{n,\rho,2}(x)\frac{f''(x)}{2} + \left(\mathcal{P}^{\alpha}_{n,\rho}\frac{(t-x)^2}{2!}\cdot(f''(\xi)-f''(x))\right)(x),$$

where $x < \xi < t$ and $f''(\xi) - f''(x) \to 0$ as $t \to x$. Now, by using Lemma 2.2, we get

$$\left|(\mathcal{P}^{\alpha}_{n,\rho}f)(x) - f(x) - \frac{2\alpha x^2}{n}f'(x) - \frac{nx+n\rho x+2\alpha x^2+4\alpha\rho x^2+4\alpha^2\rho x^4}{2n^2\rho}f''(x)\right|$$
$$\leq \left(\mathcal{P}^{\alpha}_{n,\rho}\left|\frac{f''(\xi)-f''(x)}{2}\right|(t-x)^2\right)(x). \tag{4}$$

From [4], for any $\delta > 0$ and (t,x) belongs to first quadrant then the following holds:

$$\frac{|f''(\xi)-f''(x)|}{4} \leq \left(\frac{\delta^4+(t-x)^4}{\delta^4}\right)(x^2+1)(1+\delta^2)^2\Omega(f'',\delta), \tag{5}$$

by multiply with the factor $(t-x)^2$ in equation (5) and then applying the operator $\mathcal{P}^{\alpha}_{n,\rho}$, we get

$$\left(\mathcal{P}^{\alpha}_{n,\rho}\left|\frac{f''(\xi)-f''(x)}{2}\right|(t-x)^2\right)(x) \leq 8\mu^{\alpha}_{n,\rho,2}(x)(x^2+1)\left(1+\frac{\mu^{\alpha}_{n,\rho,6}(x)}{\delta^4\mu^{\alpha}_{n,\rho,2}(x)}\right)\Omega(f'',\delta).$$

Choose $\delta^4 := \dfrac{\mu^{\alpha}_{n,\rho,6}(x)}{\mu^{\alpha}_{n,\rho,2}(x)} \leq 1$, then by using Lemma 2.2 and equation (4), we get required result. □

Corollary 3.3. *Let f be any polynomial function and $f'' \in C_E[0,\infty)$ where $x \in [0,\infty)$, then the following holds*

$$\lim_{n\to\infty} n[(\mathcal{P}^{\alpha}_{n,\rho}f - f)(x)] = 2\alpha x^2 f'(x) + \left(\frac{x+\rho x}{2\rho}\right)f''(x).$$

Proof. The estimate given in Theorem 3.2 is multiplied by n and as the limit n tends to infinity, the required relation holds. □

Theorem 3.4. *Let $f \in C_B[0,\infty)$ ($C_B[0,\infty)$ be the space of all continuous and bounded functions defined on $[0,\infty)$), ω denote the first order modulus of continuity, then*

$$\|\mathcal{P}^{\alpha}_{n,\rho}f - f\| \leq 2\omega\left(f, \sqrt{\frac{nx + n\rho x + 2\alpha x^2 + 4\alpha\rho x^2 + 4\alpha^2\rho x^4}{n^2\rho}}\right).$$

Proof. The proof follows directly by applying [29, Theorem 1] and using Lemma 2.2. We omit the details. □

4. Numerical and Graphical Analysis

In this section, we provide the numerical interpretation for upper bound of error when $\mathcal{P}^{\alpha}_{n,\rho}$ applied to any function $f \in C_B[0,\infty)$, for various values of n, α, ρ also includes graphical interpretation of certain functions.

The numerical data from a Table 1 indicate the effect of n, ρ on the convergence of our operator $\mathcal{P}^{\alpha}_{n,\rho}$. The analysis of the numerical Table 1 reveals that as both n and ρ increases it leads to a reduction in upper

n \ ρ	1	50	250	500	750
50	4.07922	4.04544	4.04489	4.04482	4.04479
250	0.85411	0.82901	0.82859	0.82854	0.82852
500	0.44988	0.42663	0.42624	0.42619	0.42617
750	0.31439	0.29228	0.29190	0.29186	0.29184
1000	0.24617	0.22495	0.22459	0.22454	0.22452

Table 1: Table comparing the upper bound of error for $\mathcal{P}^{\alpha}_{n,\rho}$ with any continuous and bounded function f, for $\alpha = 1$ and $x \in [0,10]$.

bound of error. Moreover, when ρ and α are fixed, a rise in n decreases the error bound. Furthermore, for a fixed value of n, increasing ρ leads to a more gradual reduction in the error bound compared to the scenario where ρ is fixed and n is increases.

n \ α	1	2	3	4	5
100	2.06398	4.0398	6.03158	8.02754	10.025
250	0.94385	1.80917	2.6899	3.57464	4.46108
625	0.36869	0.66683	0.97888	1.29481	1.6125
1000	0.24617	0.42567	0.6179	0.81388	1.01143
1250	0.2049	0.34520	0.49754	0.65355	0.81112

Table 2: Table comparing the upper bound of error for $\mathcal{P}^{\alpha}_{n,\rho}$ with any continuous and bounded function f, for $\rho = 1$ and $x \in [0,10]$.

The numerical data from a Table 2 indicate the effect of n, α on the convergence of our operator $\mathcal{P}^{\alpha}_{n,\rho}$. The analysis of numerical Table 2 reveals that when both n and α are increases simultaneously, the upper bound of error decreases. Moreover, for fixed values of n and ρ, an increase in α leads to a higher upper bound of error. Furthermore, when n and α are held fixed, a rise in ρ results in a reduction in the upper bound of error.

Remark 4.1. *We present the convergence of the operator $\mathcal{P}^{\alpha}_{n,\rho}$ in Figure 1 for the function $f(x) = x^2e^{-x} - 0.5e^{-x} + 0.07x^2 - 0.5x + 1$ with increasing values of n where $\alpha = \rho = 1$ be fixed on interval $x \in [0,7]$ and in Figure 2 represents the convergence for the function $f(x) = x^2e^{-x} + e^{-2x} - e^x + 7x^2 + x + 5$ with increasing values of n where $\alpha = \rho = 0.5$ be fixed on interval $x \in [0,5]$.*

Graphical analysis indicates that as n increases, the operator $\mathcal{P}^{\rho}_{n,\alpha}$ converges more rapidly to the given functions.

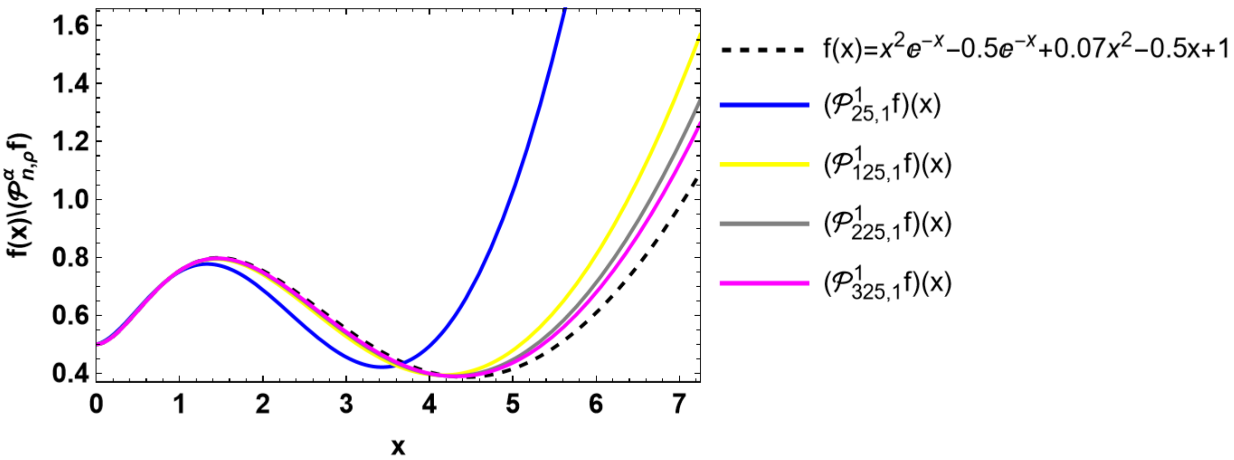

Figure 1: Convergence of operator $\mathcal{P}_{n,\alpha}^{\rho}$ for $f(x) = x^2e^{-x} - 0.5e^{-x} + 0.07x^2 - 0.5x + 1$

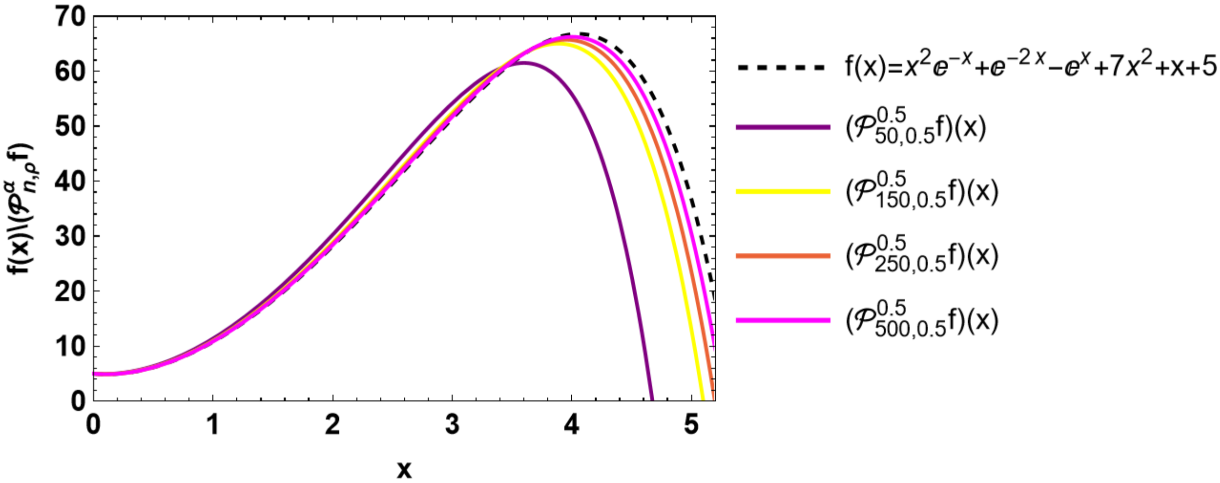

Figure 2: Convergence of operator $\mathcal{P}_{n,\alpha}^{\rho}$ for $f(x) = x^2e^{-x} + e^{-2x} - e^{x} + 7x^2 + x + 5$.

5. Approximation by Composition Operator

Recent advancements in the composition of positive linear operators focus on enhancing approximation efficiency through combinations of novel operators that refined asymptotic and error analysis (see [21, 23]). For $f \in C[0,\infty)$, the Szász-Mirakyan operators are defined by (see [28])

$$(S_n f)(x) = \sum_{v=0}^{\infty} s_{n,v}(x) f\left(\frac{v}{n}\right), \text{ where } s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}.$$

From the observation we get

$$(S_n \exp_\gamma(y))(x) = e^{nx(e^{\frac{\gamma}{n}}-1)}. \tag{6}$$

Also, they satisfies the following asymptotic formula

$$\lim_{n\to\infty} n[(S_n f - f)(x)] = \frac{x}{2} f''(x).$$

Now, we consider the following composition operator $Q_{n,\rho}^{\alpha} := \mathcal{P}_{n,\rho}^{\alpha} \circ S_n$.

Theorem 5.1. *The operator $Q_{n,\rho}^{\alpha}$ takes the following form*

$$(Q_{n,\rho}^{\alpha} f)(x) = \sum_{j=0}^{\infty} r_{n,j}^{\rho,\alpha} f\left(\frac{j}{n}\right) + \frac{f(0)}{e^{x(\alpha x+n)}}, \tag{7}$$

where

$$r_{n,j}^{\rho,\alpha} = \frac{n^j}{j!} \sum_{k=1}^{\infty} q_{n,k}^{\alpha}(x) \frac{(n\rho)^{k\rho}}{(n\rho+n)^{k\rho+j}} \frac{\Gamma(k\rho+j)}{\Gamma(k\rho)}.$$

For $j = 0$, we have

$$(Q_{n,\rho}^{\alpha} f)(x) = \exp\left(nx\left(\frac{\rho^{\rho}}{(\rho+1)^{\rho}} - 1\right) + \alpha x^2 \left(\frac{\rho^{2\rho}}{(\rho+1)^{2\rho}} - 1\right)\right) f(0).$$

Proof. By using definition of $\mathcal{P}_{n,\rho}^{\alpha}$ and S_n, we get

$$\begin{aligned}
(Q_{n,\rho}^{\alpha} f)(x) =& n\rho \sum_{k=1}^{\infty} q_{n,k}^{\alpha}(x) \int_0^{\infty} p_{n,k}^{\rho}(y) \sum_{j=0}^{\infty} s_{n,j}(y) f\left(\frac{j}{n}\right) dy + \frac{(S_n f)(0)}{e^{x(\alpha x+n)}} \\
=& n\rho \sum_{j=0}^{\infty} f\left(\frac{j}{n}\right) \frac{n^j}{j!} \sum_{k=1}^{\infty} q_{n,k}^{\alpha}(x) \int_0^{\infty} e^{-n\rho y} \frac{(n\rho y)^{k\rho-1}}{\Gamma(k\rho)} e^{-ny} y^j dy + \frac{f(0)}{e^{x(\alpha x+n)}} \\
=& \sum_{j=0}^{\infty} \frac{n^j}{j!} f\left(\frac{j}{n}\right) \sum_{k=1}^{\infty} q_{n,k}^{\alpha}(x) \frac{(n\rho)^{k\rho}}{\Gamma(k\rho)} \int_0^{\infty} e^{-ny(\rho+1)} y^{j+k\rho-1} dy + \frac{f(0)}{e^{x(\alpha x+n)}} \\
=& \sum_{j=0}^{\infty} \frac{n^j}{j!} f\left(\frac{j}{n}\right) \sum_{k=1}^{\infty} q_{n,k}^{\alpha}(x) \frac{(n\rho)^{k\rho}}{\Gamma(k\rho)} \frac{\Gamma(j+k\rho)}{(n\rho+n)^{k\rho+j}} + \frac{f(0)}{e^{x(\alpha x+n)}} \\
=& \sum_{j=0}^{\infty} r_{n,j}^{\rho,\alpha} f\left(\frac{j}{n}\right) + \frac{f(0)}{e^{x(\alpha x+n)}}.
\end{aligned}$$

For $j = 0$, we have

$$\begin{aligned}
r_{n,0}^{\rho,\alpha} + e^{-x(\alpha x+n)} =& \sum_{k=1}^{\infty} \frac{H_k(n,\alpha)}{k!} x^k e^{-(nx+\alpha x^2)} \frac{\rho^{k\rho}}{(\rho+1)^{k\rho}} + e^{-x(\alpha x+n)} \\
=& e^{-x(\alpha x+n)} \sum_{k=0}^{\infty} \frac{H_k(n,\alpha)}{k!} \left(\frac{x\rho^{\rho}}{(\rho+1)^{\rho}}\right)^k \\
=& \exp\left(nx\left(\frac{\rho^{\rho}}{(\rho+1)^{\rho}} - 1\right) + \alpha x^2 \left(\frac{\rho^{2\rho}}{(\rho+1)^{2\rho}} - 1\right)\right).
\end{aligned}$$

□

Remark 5.2. *For $\alpha = 0$, $\rho = 1$, the operator $Q_{n,\rho}^{\alpha}$ obtained the following form*

$$(Q_{n,1}^{0} f)(x) = \sum_{j=0}^{\infty} \frac{e^{-nx} nx}{2^{j+1}} \, {}_1F_1\left(j+1; 2; \frac{nx}{2}\right) f\left(\frac{j}{n}\right) + e^{-nx} f(0).$$

Proof. By using equation (7) for operator $Q_{n,\rho}^{\alpha}$ and substituting $\alpha = 0$, $\rho = 1$, then by simple computations we get hold of a desired results. □

Lemma 5.3. *The moment generating function for the operator $Q^{\alpha}_{n,\rho}$ is given by*

$$(Q^{\alpha}_{n,\rho}\exp_{\gamma}(y))(x) \quad = \quad \exp\left(\alpha x^2\left(\frac{\rho^{2\rho}}{(\rho-e^{\frac{\gamma}{n}}+1)^{2\rho}}-1\right)+nx\left(\frac{\rho^{\rho}}{(\rho-e^{\frac{\gamma}{n}}+1)^{\rho}}-1\right)\right).$$

Proof. *Using definition of operator $Q^{\alpha}_{n,\rho}$, we have*

$$(Q^{\alpha}_{n,\rho}\exp_{\gamma}(y))(x)=(\mathcal{P}^{\alpha}_{n,\rho}\circ S_n\exp_{\gamma}(y))(x).$$

Then by employing the equation (3) and equation (6), we get

$$\begin{aligned}(Q^{\alpha}_{n,\rho}\exp_{\gamma}(y))(x) &= \left(\mathcal{P}^{\alpha}_{n,\rho}\exp\left(ny(e^{\frac{\gamma}{n}}-1)\right)\right)(x)\\ &= \exp\left(\alpha x^2\left(\frac{\rho^{2\rho}}{(\rho-e^{\frac{\gamma}{n}}+1)^{2\rho}}-1\right)+nx\left(\frac{\rho^{\rho}}{(\rho-e^{\frac{\gamma}{n}}+1)^{\rho}}-1\right)\right).\end{aligned}$$

□

Lemma 5.4. *The m-th central moments for the operator $Q^{\alpha}_{n,\rho}$ is denoted by $\eta^{\alpha}_{n,\rho,m}(x) := (Q^{\alpha}_{n,\rho}(e_1-xe_0)^m)(x)$ where $m=0,1,2,...$ then we have*

$$\eta^{\alpha}_{n,\rho,m}(x)=\left[\frac{\partial^m}{\partial\gamma^m}\left(\exp\left(\alpha x^2\left(\frac{\rho^{2\rho}}{(\rho-e^{\frac{\gamma}{n}}+1)^{2\rho}}-1\right)+nx\left(\frac{\rho^{\rho}}{(\rho-e^{\frac{\gamma}{n}}+1)^{\rho}}-1\right)-\gamma x\right)\right)\right]_{\gamma=0}.$$

In particular, we have

$$\begin{aligned}
\eta^{\alpha}_{n,\rho,0}(x) =& 1\\
\eta^{\alpha}_{n,\rho,1}(x) =& \frac{2\alpha x^2}{n}\\
\eta^{\alpha}_{n,\rho,2}(x) =& \frac{nx+2n\rho x+2\alpha x^2+6\alpha\rho x^2+4\alpha^2\rho x^4}{n^2\rho}\\
\eta^{\alpha}_{n,\rho,3}(x) =& \frac{2nx+6n\rho x+5n\rho^2x+4\alpha x^2+18\alpha\rho x^2+22\alpha\rho^2x^2+6\alpha n\rho x^3+12\alpha n\rho^2x^3}{n^3\rho^2}\\
&+\frac{12\alpha^2\rho x^4+36\alpha^2\rho^2x^4+8\alpha^3\rho^2x^6}{n^3\rho^2}\\
\eta^{\alpha}_{n,\rho,4}(x) =& \frac{6nx+23n\rho x+31n\rho^2x+15n\rho^3x+12\alpha x^2+68\alpha\rho x^2+3n^2\rho x^2+134\alpha\rho^2x^2+12n^2\rho^2x^2}{n^4\rho^3}\\
&+\frac{94\alpha\rho^3x^2+12n^2\rho^3x^2+28\alpha n\rho x^3+108\alpha n\rho^2x^3+112\alpha n\rho^3x^3+44\alpha^2\rho x^4+216\alpha^2\rho^2x^4}{n^4\rho^3}\\
&+\frac{284\alpha^2\rho^3x^4+24\alpha^2n\rho^2x^5+48\alpha^2n\rho^3x^5+48\alpha^3r^2x^6+144\alpha^3\rho^3x^6+16\alpha^4\rho^3x^8}{n^4\rho^3}
\end{aligned}$$

$$\eta^{\alpha}_{n,\rho,6}(x) = \frac{120nx + 634n\rho x + 1365n\rho^2 x + 1505n\rho^3 x + 856n\rho^4 x + 203n\rho^5 x + 240\alpha x^2 + 1816\alpha\rho x^2}{n^6\rho^5}$$
$$+ \frac{130n^2\rho x^2 + 5580\alpha\rho^2 x^2 + 765n^2\rho^2 x^2 + 8780\alpha\rho^3 x^2 + 1715n^2\rho^3 x^2 + 7142\alpha\rho^4 x^2 + 1755n^2\rho^4 x^2}{n^6\rho^5}$$
$$+ \frac{2430\alpha\rho^5 x^2 + 700n^2\rho^5 x^2 + 808\alpha n\rho x^3 + 5130\alpha n\rho^2 x^3 + 15n^3\rho^2 x^3 + 12830\alpha n\rho^3 x^3 + 90n^3\rho^3 x^3}{n^6\rho^5}$$
$$+ \frac{15030\alpha n\rho^4 x^3 + 180n^3\rho^4 x^3 + 6994\alpha n\rho^5 x^3 + 120n^3\rho^5 x^3 + 1096\alpha^2\rho x^4 + 8400\alpha^2\rho^2 x^4}{n^6\rho^5}$$
$$+ \frac{330\alpha n^2\rho^2 x^4 + 24980\alpha^2\rho^3 x^4 + 1830\alpha n^2\rho^3 x^4 + 34440\alpha^2\rho^4 x^4 + 3480\alpha n^2\rho^4 x^4 + 18748\alpha^2\rho^5 x^4}{n^6\rho^5}$$
$$+ \frac{2280\alpha n^2\rho^5 x^4 + 1500\alpha^2 n\rho^2 x^5 + 8820\alpha^2 n\rho^3 x^5 + 18120\alpha^2 n\rho^4 x^5 + 13020\alpha^2 n\rho^5 x^5 + 1800\alpha^3\rho^2 x^6}{n^6\rho^5}$$
$$+ \frac{12360\alpha^3\rho^3 x^6 + 180\alpha^2 n^2\rho^3 x^6 + 29520\alpha^3\rho^4 x^6 + 720\alpha^2 n^2\rho^4 x^6 + 24720\alpha^3\rho^5 x^6 + 720\alpha^2 n^2\rho^5 x^6}{n^6\rho^5}$$
$$+ \frac{1040\alpha^3 n\rho^3 x^7 + 4560\alpha^3 n\rho^4 x^7 + 5120\alpha^3 n\rho^5 x^7 + 1360\alpha^4\rho^3 x^8 + 7200\alpha^4\rho^4 x^8 + 10000\alpha^4\rho^5 x^8}{n^6\rho^5}$$
$$+ \frac{240\alpha^4 n\rho^4 x^9 + 480\alpha^4 n\rho^5 x^9 + 480\alpha^5\rho^4 x^{10} + 1440\alpha^5\rho^5 x^{10} + 64\alpha^6\rho^5 x^{12}}{n^6\rho^5}.$$

Theorem 5.5. *Let f be any polynomial function and $f'' \in C_E[0,\infty)$ where $x \in [0,\infty)$, then the following inequality holds*

$$\left|(Q^{\alpha}_{n,\rho}f)(x) - f(x) - \frac{2\alpha x^2}{n}f'(x) - \frac{nx + 2n\rho x + 2\alpha x^2 + 6\alpha\rho x^2 + 4\alpha^2\rho x^4}{2n^2\rho}f''(x)\right|$$
$$\leq 16(1+x^2)\left(\frac{nx + 2n\rho x + 2\alpha x^2 + 6\alpha\rho x^2 + 4\alpha^2\rho x^4}{n^2\rho}\right)\Omega\left(f'', \left(\frac{\eta^{\alpha}_{n,\rho,6}(x)}{\eta^{\alpha}_{n,\rho,2}(x)}\right)^{\frac{1}{4}}\right).$$

Proof. The proof of the preceding theorem follows a similar approach to that of Theorem 3.2, incorporating Lemma 5.4. For brevity, we omit the detailed calculations. □

Corollary 5.6. *For a function $f \in C_B[0,\infty)$ and $x \in [0,\infty)$, then the following relation holds*

$$\lim_{n\to\infty} n[(Q^{\alpha}_{n,\rho}f - f)(x)] = 2\alpha x^2 f'(x) + \left(\frac{x + 2\rho x}{2\rho}\right)f''(x).$$

By observations, we have

$$\lim_{n\to\infty} n[(Q^{\alpha}_{n,\rho}f - f)(x)] = \lim_{n\to\infty} n[(S_n f + \mathcal{P}^{\alpha}_{n,\rho}f - 2f)(x)].$$

Theorem 5.7. *For $f \in C_B[0,\infty)$, then there exists a constant $B > 0$, such that*

$$\left|(Q^{\alpha}_{n,\rho}f)(x) - f(x)\right| \leq B\omega_2\left(f, \sqrt{\frac{(n + 2n\rho + 2\alpha x + 6\alpha\rho x + 8\alpha^2 x^3\rho)x}{4n^2\rho}}\right) + \omega\left(f, \frac{2\alpha x^2}{n}\right),$$

where ω_2 is the second order modulus of continuity.

Proof. Let us consider the operator $\tilde{Q}^{\alpha}_{n,\rho}$, defined for $f \in C_B[0,\infty)$, as follows

$$(\tilde{Q}^{\alpha}_{n,\rho}f)(x) = (Q^{\alpha}_{n,\rho}f)(x) + f(x) - f\left(x + \frac{2\alpha x^2}{n}\right). \tag{8}$$

Then by Taylor's series expansion, for $g \in C_B[0,\infty)$ and then applying the operator $\tilde{Q}^{\alpha}_{n,\rho}$ for $x, y, h \in [0,\infty)$, we get

$$\begin{aligned} |(\tilde{Q}^{\alpha}_{n,\rho}g)(x) - g(x)| &\leq \left(\tilde{Q}^{\alpha}_{n,\rho}\left|\int_x^h (h-y)g''(y)dy\right|\right)(x) \\ &= \left(Q^{\alpha}_{n,\rho}\left|\int_x^h (h-y)g''(y)dy\right|\right)(x) \\ &\quad + \int_x^{x+\frac{2\alpha x^2}{n}} \left(x + \frac{2\alpha x^2}{n} - y\right) g''(y)dy. \end{aligned}$$

By using supremum norm property with the definition of $\tilde{Q}^{\alpha}_{n,\rho}$ and $Q^{\alpha}_{n,\rho}$, we get

$$\left\|\tilde{Q}^{\alpha}_{n,\rho}f\right\| \leq \left\|Q^{\alpha}_{n,\rho}f\right\| + 2\|f\| \leq 3\|f\|, \quad f \in C_B[0,\infty), \tag{9}$$

by using this relation $\left|\int_x^h (h-y)g''(y)dy\right| \leq (h-x)^2 \left\|g''\right\|$, we get

$$\left|\int_x^{x+\frac{2\alpha x^2}{n}} \left(x + \frac{2\alpha x^2}{n} - y\right) g''(y)dy\right| \leq \frac{4\alpha^2 x^4}{n^2}\|g''\|. \tag{10}$$

And

$$\left(Q^{\alpha}_{n,\rho}\left|\int_x^h (h-y)g''(y)dy\right|\right)(x) \leq Q^{\alpha}_{n,\rho}(h-x)^2\|g''\|.$$

Then by using Lemma 5.4 and equation (10), we get

$$|(\tilde{Q}^{\alpha}_{n,\rho}g)(x) - g(x)| \leq \frac{(n + 2n\rho + 2\alpha x + 6\alpha\rho x + 8\alpha^2 x^3 \rho)x}{n^2\rho}\|g''\|. \tag{11}$$

Using equations (8), (9) and (11), we obtain

$$\begin{aligned} \left|(Q^{\alpha}_{n,\rho}f)(x) - f(x)\right| &\leq \left|(\tilde{Q}^{\alpha}_{n,\rho}(f-g))(x) - (f-g)(x)\right| + \left|(\tilde{Q}^{\alpha}_{n,\rho}g)(x) - g(x)\right| \\ &\quad + \left|f(x) - f\left(x + \frac{2\alpha x^2}{n}\right)\right| \\ &\leq 4\left\{\|f-g\| + \frac{(n + 2n\rho + 2\alpha x + 6\alpha\rho x + 8\alpha^2 x^3\rho)x\left\|g''\right\|}{4n^2\rho}\right\} \\ &\quad + \omega\left(f, \frac{2\alpha x^2}{n}\right). \end{aligned}$$

By definition of second order K-functional

$$K^2(f,\delta) = \{\inf \|f-g\| + \delta\|g''\| : f, g, g', g'' \in C_B[0,\infty)\}.$$

Following [12, Theorem 2.4], their exists a relation between second order K-functional and ω_2 be the second order modulus of continuity for $\delta > 0$, $\exists$ a constant $B > 0$, such that $K^2(f,\delta) \leq B\omega_2(f, \sqrt{\delta})$. By taking infimum over g and using above relation, we get the assertion of the theorem. □

The study of difference estimates for operators is fundamental in analyzing the convergence behavior of various operators. These estimates provide quantitative bounds for the operators. In recent years, significant advances have been made in this area, leading to refined bounds and improved theoretical frameworks, see references [7, 10, 15–17, 24]. Here, we present a theorem based on difference of operators.

Theorem 5.8. *For $x \in [0,\infty)$, $f \in C_B[0,\infty)$ then, we have*

$$\left|\left(\mathcal{Q}^{\alpha}_{n,\rho}f - \mathcal{P}^{\alpha}_{n,\rho}f\right)(x)\right| \leq 2\omega\left(f, \sqrt{\frac{x}{n} + \frac{2\alpha x^2}{n^2}}\right).$$

Proof. By using the definition of operator $\mathcal{Q}^{\alpha}_{n,\rho}$, we have

$$\left|\left(\mathcal{Q}^{\alpha}_{n,\rho}f - \mathcal{P}^{\alpha}_{n,\rho}f\right)(x)\right| \leq \int_0^\infty \phi^{\rho}_{n,k}(x,y)\left|S_nf(y) - f(y)\right| dy.$$

Following [29], and by using property of well known Szász-Mirakyan operator, we have

$$\left|\left(S_nf - f\right)(x)\right| \leq \left|1 + \frac{1}{\delta^2}\left(\frac{x}{n}\right)\right|\omega(f,\delta),$$

substituting this estimates in the above inequality, we get

$$\begin{aligned}\left|\left(\mathcal{Q}^{\alpha}_{n,\rho}f - \mathcal{P}^{\alpha}_{n,\rho}f\right)(x)\right| &\leq n\rho\sum_{k=1}^{\infty} q^{\alpha}_{n,k}(x)\int_0^\infty p^{\rho}_{n,k}(y)\left(1 + \frac{1}{\delta^2}\left(\frac{y}{n}\right)\right)\omega(f,\delta)dy\\ &= \left(1 + \frac{1}{\delta^2}\left(\frac{x}{n} + \frac{2\alpha x^2}{n^2}\right)\right)\omega(f,\delta).\end{aligned}$$

Choosing $\delta = \sqrt{\frac{x}{n} + \frac{2\alpha x^2}{n^2}}$, we obtained the desired result. □

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Conflict of interest

The authors declare that they have no Conflict of interest.

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