Filomat 39:13 (2025), 4371–4381
https://doi.org/10.2298/FIL2513371A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some notable results on approximate pseudospectrum

Aymen Ammar[a], Jadav Ganesh[b], S. Veeramani[c,*]

[a]*Department of Mathematics, University of Sfax, Soukra Road Km 3.5, B. P. 1171, 3000, Sfax, Tunisia*
[b]*Department of Mathematics, GITAM University, Rudraram, Patancheru Mandal, Hyderabad, 502329, Telangana, India*
[c]*Department of Mathematics, Vellore Institute of Technology, Tiruvalam Road, Katpadi, Vellore, 632014, Tamil Nadu, India*

Abstract. The concept of approximate pseudospectrum is studied in this note. We first prove that any open ball of an element in the approximate spectrum is a subset of approximate pseudospectrum. We also have derived some results related to mapping theorem of approximate pseudospectrum. Examples are given to throw light on the established outcomes.

1. Introduction

Let $B(X)$ be the space of all bounded linear operators acting on a complex Banach space X. The identity operator defined on X is denoted by I. The spectrum of an element $T \in B(X)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(X)\}.$$

The complement of $\sigma(T)$ is known as the resolvent of T which is denoted as $\rho(T)$. The spectrum of an operator reveals the nature of the underlying operator. The spectrum of the complex normal matrices ensures the diagonalization of that matrix. Similarly, in some particular kind of normal operators the corresponding spectrum set characterizes its nature. However, the spectrum of a not normal matrices and operators might not be very informative. As a result, the concept of pseudospectrum has been introduced.

Let $T \in B(X)$ and $\varepsilon > 0$. The ε-pseudospectrum of T is denoted by $\Lambda_\varepsilon(T)$ and is defined as

$$\Lambda_\varepsilon(T) = \left\{\lambda \in \mathbb{C} : \|(T - \lambda I)^{-1}\| \geq \frac{1}{\varepsilon}\right\}$$

by convention $\|(T - \lambda I)^{-1}\| = +\infty$ for all $\lambda \in \sigma(T)$. A novel technique for learning about matrices and linear operators is the theory of the pseudospectrum. Pseudospectrum offers the analytical and graphical methods to study the problems that involve matrices and operators which are non normal. Pseudospectrum has nonetheless proven to be a useful tool for studying them. For more information on the concept of pseudo spectra and how they are used in engineering and research, see the book [11].

2020 *Mathematics Subject Classification*. Primary 46L05; Secondary 47A05.
Keywords. pseudospectrum, approximate pseudospectrum, spectral mapping theorem
Received: 16 July 2024; Revised: 26 January 2025; Accepted: 08 March 2025
Communicated by Dijana Mosić
* Corresponding author: S. Veeramani
Email addresses: ammar_aymen84@yahoo.fr (Aymen Ammar), jadav.ganesh@gmail.com (Jadav Ganesh), veeramani.s@vit.ac.in (S. Veeramani)
ORCID iDs: https://orcid.org/0000-0001-6728-3728 (Aymen Ammar), https://orcid.org/0000-0002-4127-728X (Jadav Ganesh), https://orcid.org/0009-0008-6155-701X (S. Veeramani)

J. M. Varah [12] first proposed the concept of pseudospectrum in 1967. The present interpretation of pseudospectrum was introduced in 1986 by J. H. Wilkinson [10], who defined it as any arbitrary matrix norm induced by a vector norm. L. N. Trefethen pioneered the research of pseudospectrum for matrices and operators during the 1990s (see [10],[9] and [6]). He also discussed approximate eigenvalues and pseudospectrum and applied this idea to intriguing problems in mathematical physics.

The main aim of this article is to investigate the mapping theorem of approximate pseudospectrum sets. The articles [3] and [4] serve as the impetus for discussing the mapping theorem. For $\varepsilon > 0$, consider the following set:

$$\sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T) \cup \left\{ \lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| < \varepsilon \right\}. \tag{1}$$

where $\sigma_{ap}(T) = \left\{ \lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| = 0 \right\}$.

The above set was introduced by Nevanlinna, Olavi in the book [5] (see Definition 2.2.5). In [5], the author studies the relationship between the sets $\sigma_{ap}(T)$ and $\sigma_{ap,\varepsilon}(T)$ and their continuity property. In [1], A. Ammar, A. Jeribi, and K. Mahfoudhi investigated the concept of the set $\sigma_{ap,\varepsilon}(T)$ of an unbounded, closed, and densely defined operator T. The articles [1] and [14] contain various other important and interesting properties of $\sigma_{ap,\varepsilon}(T)$.

In this article, we focus on the $\varepsilon-$approximate pseudospectrum set which is defined in [14] by M. P. H. Wolff. The definition is as follows:

Definition 1.1. *[Page 4 in [14]] Let $T \in B(X)$ and $\varepsilon > 0$. The $\varepsilon-$approximate pseudospectrum is defined by*

$$\Sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T) \cup \left\{ \lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T-\lambda)x\| \leq \varepsilon \right\}$$

In [14], M. P. H. Wolff utilized ε-approximate pseudospectrum sets for the discrete approximation of the spectrum of an operator. In the same paper, it is demonstrated that a sequence of ε-approximate pseudospectrum sets provides a discrete approximation to the spectrum from below. In [13], the authors discuss various necessary and sufficient conditions for the continuity nature of the ε-approximate pseudospectrum set.

The organization of this article is as follows: Basic concepts, ideas, and known results are provided in section 2. These are crucial to achieve our main objective. Section 3 deals with the main results of this manuscript. Here we mainly focus on the mapping theorem of the approximate pseudospectrum. To provide more insights into the established results, we have included illustrations.

2. Basic definitions and notions

This section deals with some basic definitions, notions, and existing results that are required to prove our main results. We start with a theorem, which is the equivalent characterization of the set defined in (1).

Theorem 2.1 (Theorem 3.3 in [1]). *Let $T \in B(X)$ and $\varepsilon > 0$. Then, $\lambda \in \sigma_{ap,\varepsilon}(T)$ if and only if there exists $D \in B(X)$ such that $\|D\| < \varepsilon$ such that $\lambda \in \sigma_{ap}(T+D)$.*

Next, we list out a few fundamental properties of the approximate pseudospectrum.

Proposition 2.2 (Proposition 2.6 in [1]). *If $T \in B(X)$ and $\varepsilon > 0$, then the following holds*

1. *$\Sigma_{ap,\varepsilon}(T)$ is a non empty compact susbet of $\mathbb{C}$.*
2. *$\sigma_{ap}(T) = \bigcap_{\varepsilon>0} \Sigma_{ap,\varepsilon}(T)$.*
3. *If $\varepsilon_1 < \varepsilon_2$ then $\Sigma_{ap,\varepsilon_1}(T) \subseteq \Sigma_{ap,\varepsilon_2}(T)$.*
4. *If $\lambda \in \Sigma_{ap,\varepsilon}(T)$ then $|\lambda| \leq \varepsilon + \|T\|$.*

In order to get some interesting properties of approximate pseudospectrum one needs to know about the limit inferior and superior, limits of sequence of sets.

Definition 2.3 (see page 2 in [8]). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a metric space X. We say that the subset*

$$\limsup_{n\to\infty} K_n := \{\lambda \in \mathbb{C} : \forall\varepsilon > 0, \text{ there exists an infinite subset } J \subseteq \mathbb{N} \text{ such that } B(\lambda,\varepsilon) \cap K_n \neq \emptyset, \ \forall n \in J\}$$

is the upper limit of the sequence K_n and that the subset

$$\liminf_{n\to\infty} K_n := \{\lambda \in \mathbb{C} : \text{ for every } \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } B(\lambda,\varepsilon) \cap K_n \neq \emptyset \ , \forall n \geq N\}$$

is its lower limit. A subset K is said to be the limits or the set limit of the sequence K_n if

$$K = \limsup_{n\to\infty} K_n = \liminf_{n\to\infty} K_n =: \lim_{n\to\infty} K_n.$$

The following theorem is obtained in [1].

Theorem 2.4. *Let $T \in B(X)$ and $\varepsilon_0 > 0$. If $\lim\limits_{n\to\infty} \varepsilon_n = \varepsilon_0$, then $\liminf\limits_{n\to\infty} \Sigma_{ap,\varepsilon_n}(T) = \Sigma_{ap,\varepsilon_0}(T)$ and if $\lim\limits_{n\to\infty} \varepsilon_n = 0$, then $\lim\limits_{n\to\infty} \Sigma_{ap,\varepsilon_n}(T) = \sigma_{ap}(T)$*

The primary goal of this note is to investigate the mapping theorem of approximate pseudospectrum. We define $f(T)$ here for any holomorphic function f on an open subset of the complex plane because the spectral mapping theorem uses the concept of functional calculus of the underlying operator T.

Definition 2.5 (Definition 10.26 in [7]). *Let $T \in B(X)$, Ω be an open subset of $\mathbb{C}$ and $H(\Omega)$ is the algebra of all complex holomorphic functions in Ω. Then*

$$f(T) = \frac{1}{2\pi i}\int_C f(\lambda)(T-\lambda)^{-1}\mathrm{d}\lambda.$$

where C is any contour that surrounds $\sigma(T)$ in Ω.

We apply the inverse mapping theorem in one of our spectral mapping theorems. The definition of *Fréchet* derivative, which is used in the inverse mapping theorem, is as follows:

Definition 2.6 (Definition 10.34 in [7]). *Let X and Y be two Banach space and Ω be an open subset of X, F maps Ω into Y, and $a \in \Omega$. If there exists $G \in B(X, Y)$ such that*

$$\lim_{x\to 0} \frac{\|F(a+x) - F(a) - Gx\|}{\|x\|} = 0,$$

then G is called the Fréchet derivative of F at a. We use the notation $(DF)_a$ for Fréchet derivative of F at a.

Note 2.7. *If $(DF)_a$ exists for every $a \in \Omega$, and if $a \mapsto (DF)_a$ is a continuous mapping of Ω into $B(X, Y)$, then F is said to be continuously differentiable in Ω.*

Theorem 2.8 (Theorem 10.36 in [7]). *Suppose*

1. *W is an open subset of a Banach space X.*
2. *$F : W \to X$ is continuously differentiable.*
3. *For every $a \in W$, $(DF)_a$ is an invertible member of $B(X)$*

then there exists a neighbourhood U of a such that

1. *F is one-to-one on U.*
2. *$F(U) = V$ is an open subset of X.*
3. *$F^{-1} : V \to U$ is continuously differentiable .*

Next, we state the spectral mapping theorem of approximate spectrum.

Theorem 2.9 (Theorem 1.2 in [3]). *If f is a complex valued function which is holomorphic on an open set containing the spectrum of the linear operator $T \in B(X)$ then, $\sigma_{ap}(f(T)) = f(\sigma_{ap}(T))$*

Main results

The mapping theorem of approximate pseudospectrum is the main topic of this section. We start with the following Theorem, in which we show that, every point in the approximate spectrum is an interior point of the approximate pseudospectrum.

Proposition 2.10. *Let $T \in B(X)$ and $\varepsilon > 0$. Then for every $\lambda \in \sigma_{ap}(T)$, $B(\lambda, \varepsilon) \subseteq \sigma_{ap,\varepsilon}(T)$.*

Proof. *Let $\mu \in B(\lambda, \varepsilon)$ where $\lambda \in \sigma_{ap}(T)$. Consider the operator $S = (\mu - \lambda)I$. Clearly $\|S\| < \varepsilon$ and $T + S - \mu = T - \lambda$. By using Theorem 2.1, we have $\mu \in \sigma_{ap,\varepsilon}(T)$.*

The following example illustrates that the spectral mapping theorem does not hold in the form stated in Theorem 2.9.

Example 2.11. *Consider the Hilbert space $\ell^2(\mathbb{N})$. Define,*

$$T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \text{ by } T(e_{2i+1}) = e_{2i+1} \text{ and } T(e_{2i}) = -e_{2i}.$$

We note that, $\{1, -1\} \in \sigma_{ap}(T) \subseteq \Sigma_{ap,\varepsilon}(T)$. By Proposition 2.10, $B(1, \varepsilon) \subseteq \Sigma_{ap,\varepsilon}(T)$ and $B(-1, \varepsilon) \subseteq \Sigma_{ap,\varepsilon}(T)$. Let $\mathcal{D}$ be an open subset of $\mathbb{C}$ such that $\mathcal{D}$ contains $\sigma(T)$. Now, consider the function

$$f : \mathcal{D} \to \mathbb{C} \text{ such that } f(z) = z^2.$$

Clearly f is analytic, $f(T) = T^2$ and $T^2 = I$. For any $\varepsilon > 0$, we have

$$\Sigma_{ap,\varepsilon}(f(T)) = \Sigma_{ap,\varepsilon}(T^2) = \Sigma_{ap,\varepsilon}(I) = B(1, \varepsilon)$$

and

$$f\left(\Sigma_{ap,\varepsilon}\right)(T) \supseteq f\left(B(1, \varepsilon) \cup B(-1, \varepsilon)\right) = \left\{z^2 \in \mathbb{C} : z \in B(1, \varepsilon) \cup B(-1, \varepsilon)\right\} \supset \Sigma_{ap,\varepsilon}(f(T)).$$

Hence $f\left(\Sigma_{ap,\varepsilon}(T)\right) \neq \Sigma_{ap,\varepsilon} f(T)$.

Theorem 2.12. *Let $T \in B(X)$. For $\varepsilon > 0$, let Ω be a bounded open subset of $\mathbb{C}$ containing $\Lambda_\varepsilon(T)$ and f be an analytic function on Ω. Denote by*
$\phi(\varepsilon) := \sup\limits_{\lambda \in \Sigma_{ap,\varepsilon}(T)} \left\{ \inf\limits_{\|x\|=1} \|(f(T) - f(\lambda))x\| \right\}$. *If the map $E_1 : \sigma(T) \setminus \sigma_{ap}(T) \to \mathbb{R}$ defined by $E_1(\lambda) = \inf\limits_{\|x\|=1} \|(f(T) - f(\lambda))x\|$ is continuous, then $\phi(\varepsilon)$ is well defined, $\lim\limits_{\varepsilon \to 0} \phi(\varepsilon) = 0$ and $f\left(\Sigma_{ap,\varepsilon}(T)\right) \subseteq \Sigma_{ap,\phi(\varepsilon)} f(T)$.*

Proof. *Define $g : \mathbb{C} \to \mathbb{R}$ by*

$$g(\lambda) = \begin{cases} \inf_{\|x\|=1} \|(f(T) - f(\lambda))x\| \text{ for } \lambda \notin \sigma_{ap}(T). \\ 0 \text{ for } \lambda \in \sigma_{ap}(T). \end{cases}$$

We first show that g is continuous. Let $\lambda_0 \in \mathbb{C}$ and there is a sequence $\{\lambda_n\}$ in $\mathbb{C}$ such that $\lambda_n \to \lambda_0$. Assume that $\lambda_0 \notin \sigma(T)$. Then, by spectral mapping theorem, $f(\lambda_0) \notin \sigma(f(T))$, and hence $[f(\lambda_0) - f(T)]^{-1}$ exists. Further,

$$\frac{1}{\left\|[f(\lambda_0) - f(T)]^{-1}\right\|} = \inf_{\|x\|=1} \|(f(T) - f(\lambda_0))x\|.$$

Since invertible operators forms an open set $[f(\lambda_n) - f(T)]^{-1}$ exists for infinitely many n and

$$\frac{1}{\left\|[f(\lambda_n) - f(T)]^{-1}\right\|} = \inf_{\|x\|=1} \|(f(T) - f(\lambda_n))x\|.$$

The map $T \mapsto T^{-1}$ is continuous, so $g(\lambda_n) \to g(\lambda_0)$. Next, assume that $\lambda_0 \in \sigma_{ap}(T)$ then $g(\lambda_0) = 0$. By Theorem 2.9, $f(\lambda_0) \in \sigma_{ap}(f(T))$. Consider $\delta > 0$. For every $x \in X$ with $\|x\| = 1$,

$$(f(T) - f(\lambda_n))x \to (f(T) - f(\lambda_0))x.$$

For $\frac{\delta}{2}$, there exists $y \in H$ with $\|y\| = 1$ such that

$$\left\|(f(T) - f(\lambda_0)y\right\| < \frac{\delta}{2}.$$

For $\frac{\delta}{2}$, there exists $N > 0$ such that

$$\left|\left\|(f(T) - f(\lambda_n))y\right\| - \left\|(f(T) - f(\lambda_0)y\right\|\right| < \frac{\delta}{2}$$

for all $n \geq N$. For any $n > N$,

$$\begin{array}{rcl} \left\|(f(T) - f(\lambda_n))y\right\| & = & \left|\left\|(f(T) - f(\lambda_n))y\right\| - \left\|(f(T) - f(\lambda_0)y\right\| + \left\|(f(T) - f(\lambda_0)y\right\|\right| \\ & \leq & \left|\left\|(f(T) - f(\lambda_n))y\right\| - \left\|(f(T) - f(\lambda_0)y\right\|\right| + \left\|(f(T) - f(\lambda_0)y\right\| \\ & < & \dfrac{\delta}{2} + \dfrac{\delta}{2} \\ & < & \delta. \end{array}$$

Hence $\inf\limits_{\|x\|=1} \left\|(f(T) - f(\lambda_n))x\right\| \to 0$. Consequently, $g(\lambda_n) \to 0$. Suppose, $\lambda_0 \in \sigma(T) \setminus \sigma_{ap}(T)$ then by our assumption, the map E_1 is continuous we have $g(\lambda_n) \to g(\lambda_0)$. Thus g is continuous.

For $\varepsilon > 0, \Sigma_{ap,\varepsilon}(T)$ is compact, and so $\phi(\varepsilon) = \sup\{g(\lambda) : \lambda \in \Sigma_{ap,\varepsilon}(T)\}$ exists. Thus $\phi(\varepsilon)$ is well defined.

We show that $\lim\limits_{\varepsilon\to 0} \phi(\varepsilon) = 0$. Let $\varepsilon_n > 0$ be a sequence converging to 0. By compactness of $\Sigma_{ap,\varepsilon_n}(T)$ there exists $\lambda_n \in \Sigma_{ap,\varepsilon_n}(T)$ such that $g(\lambda_n) = \phi(\varepsilon_n)$. Since $\Sigma_{ap,\varepsilon_1}(T)$ is compact, there exists a subsequence λ_{n_k} of λ_n such that $\lambda_{n_k} \to \lambda$. By Definition 2.3 and Theorem 2.4, one can see that $\lambda \in \sigma_{ap}(T)$

Because, g is continuous, $g(\lambda_{n_k}) \to 0$. Since $\phi(\varepsilon_n)$ is monotonically increasing, we have $\phi(\varepsilon_n) \to 0$.

Let $\varepsilon > 0$ and $\lambda \in \Sigma_{ap,\varepsilon}(T)$. Then $g(\lambda) \leq \phi(\varepsilon)$. Hence

$$\inf_{\|x\|=1} \|(f(T) - f(\lambda))x\| \leq \phi(\varepsilon).$$

This means $f(\lambda) \in \Sigma_{ap,\phi(\varepsilon)}(f(T))$. Thus

$$f\left(\Sigma_{ap,\varepsilon}(T)\right) \subseteq \Sigma_{ap,\phi(\varepsilon)} f(T).$$

Theorem 2.13. *Let $T \in B(X)$. For $\varepsilon > 0$, let Ω be a bounded open subset of $\mathbb{C}$ containing $\Lambda_\varepsilon(T)$ and f be an analytic function on Ω. Suppose there exists $\varepsilon_0 > 0$ such that $\Sigma_{ap,\varepsilon_0}(f(T)) \subseteq f(\Omega)$. For $0 < \varepsilon \leq \varepsilon_0$, define*

$$\psi(\varepsilon) = \sup_{\mu \in f^{-1}(\Sigma_{ap,\varepsilon}(f(T)))} \left\{ \inf_{\|x\|=1} \|(T - \mu)x\| \right\}.$$

If the map $E_2 : \sigma(T) \setminus \sigma_{ap}(T) \to \mathbb{R}$ defined by $E_2(\lambda) = \inf\limits_{\|x\|=1} \|(T - \lambda)x\|$ is continuous, then $\psi(\varepsilon)$ is well defined, $\lim\limits_{\varepsilon\to 0} \psi(\varepsilon) = 0$ and $\Sigma_{ap,\varepsilon}(f(T)) \subseteq f\left(\Sigma_{ap,\psi(\varepsilon)}(T)\right)$.

Proof. *Define $h : \mathbb{C} \to \mathbb{R}$ by*

$$h(\lambda) = \begin{cases} \inf_{\|x\|=1} \|(T - \lambda)x\| \text{ for } \lambda \notin \sigma_{ap}(T) \\ 0 \text{ for } \lambda \in \sigma_{ap}(T). \end{cases}$$

We prove that h is continuous. Let $\lambda_0 \in \mathbb{C}$ and there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_0$. If $\lambda_0 \notin \sigma(T)$ then $(\lambda_0 - T)^{-1}$ exists. and

$$\frac{1}{\left\|(\lambda_0 - T)^{-1}\right\|} = \inf_{\|x\|=1} \|(T-\lambda_0)x\|.$$

Since invertible operators form an open set, $(\lambda_n - T))^{-1}$ exists for infinitely many n and

$$\frac{1}{\left\|[\lambda_n - T]^{-1}\right\|} = \inf_{\|x\|=1} \|(T-\lambda_n)x\|.$$

The map $T \mapsto T^{-1}$ is continuous, so $h(\lambda_n) \to h(\lambda_0)$. Next, assume that $\lambda_0 \in \sigma(T)$ then $h(\lambda_0) = 0$. Consider $\delta > 0$. For every $x \in H$ with $\|x\| = 1$,

$$(T-\lambda_n)x \to (T-\lambda_0)x.$$

For $\frac{\delta}{2}$, there exists $y \in H$ with $\|y\| = 1$ such that

$$\left\|(T-\lambda_0)y\right\| < \frac{\delta}{2}.$$

For $\frac{\delta}{2}$, there exists $N > 0$ such that

$$\left|\left\|(T-\lambda_n)y\right\| - \left\|(T-\lambda_0)y\right\|\right| < \frac{\delta}{2}$$

for all $n \geq N$. For any $n > N$,

$$\begin{aligned}
\left\|(T-\lambda_n)y\right\| &= \left|\left\|(T-\lambda_n)y\right\| - \left\|(T-\lambda_0)y\right\| + \left\|(T-\lambda_0)y\right\|\right| \\
&\leq \left|\left\|(T-\lambda_n)y\right\| - \left\|(T-\lambda_0)y\right\|\right| + \left\|(T-\lambda_0)y\right\| \\
&< \frac{\delta}{2} + \frac{\delta}{2} \\
&< \delta.
\end{aligned}$$

Hence $\inf\limits_{\|x\|=1} \|(T-\lambda_n)x\| \to 0$. Thus $h(\lambda_n) \to 0$. Suppose, $\lambda_0 \in \sigma(T) \setminus \sigma_{ap}(T)$. Since, the map E_2 is continuous and so for any $\lambda_0 \in \sigma(T) \setminus \sigma_{ap}(T)$, we have $h(\lambda_n) \to h(\lambda_0)$. Thus h is continuous. Since f is injective and Ω is bounded, $f^{-1}(\Sigma_{ap,\varepsilon}(f(T)))$ is closed and bounded for all $0 \leq \varepsilon \leq \varepsilon_0$. Hence $\psi(\varepsilon)$ is well defined.

We show that $\lim\limits_{\varepsilon \to 0} \psi(\varepsilon) = 0$. Let $\varepsilon_n > 0$ be a sequence converging to 0. By compactness of $f^{-1}(\Sigma_{ap,\varepsilon_n}(f(T)))$ there exists $\lambda_n \in f^{-1}(\Sigma_{ap,\varepsilon_n}(f(T)))$ such that $h(\lambda_n) = \psi(\varepsilon_n)$. Since $f^{-1}(\Sigma_{ap,\varepsilon_0}(f(T)))$ is compact, there exists a subsequence λ_{n_k} of λ_n such that $\lambda_{n_k} \to \lambda$. Since $\lambda_{n_k} \in f^{-1}(\Sigma_{ap,\varepsilon_{n_k}}(f(T)))$ gives us $f(\lambda_{n_k}) \in \Sigma_{ap,\varepsilon_{n_k}}(f(T))$. By the Definition 2.3 and by Theorem 2.4, one can see that $f(\lambda) \in \sigma_{ap}(f(T))$. Since f is injective, $\lambda \in \sigma_{ap}(T)$. Since h is continuous $h(\lambda_{n_k}) \to 0$. This gives $\psi(\varepsilon_{n_k}) \to 0$. Since $\psi(\varepsilon_n)$ is monotonically increasing, we have $\psi(\varepsilon_n) \to 0$.

Let $\varepsilon > 0$ and $\mu \in \Sigma_{ap,\varepsilon}(f(T)) \subseteq \Sigma_{ap,\varepsilon_0}(f(T)) \subseteq f(\Omega)$. Consider $\lambda \in \Omega$ such that $\mu = f(\lambda)$. Then $\lambda \in f^{-1}(\Sigma_{ap,\varepsilon}(f(T)))$ and $h(\lambda) \leq \psi(\varepsilon)$. Hence

$$\inf_{\|x\|=1} \|(T-\lambda)x\| \leq \phi(\varepsilon).$$

This means $\lambda \in \Sigma_{ap,\psi(\varepsilon)}(T)$. It follows that, $\mu = f(\lambda) \in f(\Sigma_{ap,\psi(\varepsilon)}(T))$. Thus

$$\Sigma_{ap,\varepsilon}(f(T)) \subseteq f(\Sigma_{ap,\psi(\varepsilon)}(T)).$$

Remark 2.14. *Under the assumption in Theorem 2.12 and Theorem 2.13, we have the following conclusions,*

$$f\left(\Sigma_{ap,\varepsilon}(T)\right) \subseteq \Sigma_{ap,\phi(\varepsilon)} f(T) \subseteq f(\Sigma_{ap,\psi(\phi(\varepsilon))}(T)).$$

and

$$\Sigma_{ap,\varepsilon}(f(T)) \subseteq f(\Sigma_{ap,\psi(\varepsilon)}(T)) \subseteq \Sigma_{ap,\phi(\psi(\varepsilon))}(f(T)).$$

Remark 2.15. *For any $T \in B(X)$ and an analytic function f which satisfies the hypothesis of Theorem2.12 and Theorem 2.13, we have $\lim_{f^{-1}\to 0} \phi(\varepsilon) = 0$, $\lim_{f^{-1}\to 0} \psi(\varepsilon) = 0$, $\sigma_{ap}(T) = \bigcap_{\varepsilon>0} \Sigma_{ap,\varepsilon}(T)$ and ϕ, ψ are monotonically increasing functions. Moreover, Theorem 2.9 can be deduced from Theorem 2.13.*

The example that follows is constructed to demonstrate that there are operators and analytical functions that adhere to the presumptions made by the theorem that was proven above. In addition, we determine $\phi(\varepsilon)$ and $\psi(\varepsilon)$ in this situation.

Example 2.16. *Consider the Banach space $\ell^2(\mathbb{N})$. Let*

$$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \text{ by } S(e_i) = e_{i+1} \ \ \forall i \in \mathbb{N}.$$

The following holds for S:

$$\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}, \|S\| = 1, \|Sx\| = \|x\|$$

and

$$\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Moreover, for any $\lambda \in \sigma(S)$ and for any $x \in \ell^2(\mathbb{N})$ with $\|x\| = 1$, it is easy to observe that,

$$\inf_{\|x\|=1} \|(S - \lambda)x\| \geq 1 - |\lambda|. \tag{2}$$

For $0 < \varepsilon \leq 1$, in [14], it is shown that

$$\{\lambda \in \mathbb{C} : 1 - \varepsilon \leq |\lambda| \leq 1\} \subseteq \Sigma_{ap,\varepsilon}(S). \tag{3}$$

Consider $r \in (0, 1]$. For $\varepsilon_0 = 1 - r$, from equation (3),

$$\{\lambda \in \mathbb{C} : r \leq |\lambda| \leq 1\} \subseteq \Sigma_{ap,\varepsilon_0}(S). \tag{4}$$

Thus for any $\lambda_0 \in \sigma(S)$ with $|\lambda_0| = r$,

$$\inf_{\|x\|=1} \|(S - \lambda_0)x\| \leq \varepsilon_0 = 1 - r = 1 - |\lambda_0|. \tag{5}$$

From, equation (2) and (5),

$$\inf_{\|x\|=1} \|(S - \lambda_0)x\| = 1 - |\lambda_0|. \tag{6}$$

Since

$$\inf_{\|x\|=1} \|(S - \lambda)x\| = 1 - |\lambda|$$

for all $\lambda \in \sigma(S)$, the map

$$E_2 : \sigma(S) \setminus \sigma_{ap}(S) \to \mathbb{R} \text{ defined by } E_2(\lambda) = \inf_{\|x\|=1} \|(S - \lambda)x\|$$

is continuous. Let Ω be a bounded open subset of $\mathbb{C}$ containing $\Lambda_\varepsilon(S)$. For $\alpha \neq 0$, consider the affine function,

$$f : \Omega \to \mathbb{C} \text{ defined by } f(z) = \alpha z + \beta$$

for some $\alpha, \beta \in \mathbb{C}$. By spectral mapping theorem,

$$\sigma(f(S)) = \{\alpha\lambda + \beta \in \mathbb{C} : |\lambda| \leq 1\},$$

and by Theorem 2.9,

$$\sigma_{ap}(f(S)) = \{\alpha\lambda + \beta \in \mathbb{C} : |\lambda| = 1\}.$$

The map

$$E_1 : \sigma(S) \setminus \sigma_{ap}(S) \to \mathbb{R} \text{ defined by } E_1(\lambda) = \inf_{\|x\|=1} \|(f(S) - f(\lambda))x\|$$

is continuous because $\inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| = |\alpha| E_2(\lambda)$*. Consider* $\varepsilon \in (0, 1]$*. Now,*

$$\begin{aligned}\phi(\varepsilon) &= \sup_{\lambda \in \Sigma_{ap,\varepsilon}(S)} \left\{ \inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| \right\} \\ &= |\alpha| \sup_{\lambda \in \Sigma_{ap,\varepsilon}(S)} \left\{ \inf_{\|x\|=1} \|(S - \lambda)\, x\| \right\}.\end{aligned}$$

We note that, if $\lambda \in \Sigma_{ap,\varepsilon}(S) \setminus \sigma(S)$*, then* $\lambda \in \Lambda_\varepsilon(S) \setminus \sigma(S)$*. In this case*

$$\inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| = |\alpha| \inf_{\|x\|=1} \|(S - \lambda)\, x\| = |\alpha| \frac{1}{\|(T - \lambda)^{-1}\|} \leq |\alpha|\varepsilon.$$

If $\lambda \in \sigma(S) \cap \Sigma_{ap,\varepsilon}(S)$*, then from equation (3),*

$$\inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| = |\alpha| \inf_{\|x\|=1} \|(S - \lambda)\, x\| \leq |\alpha|\varepsilon.$$

Moreover, for $|\lambda| = 1 - \varepsilon$*, from equation 6, we have*

$$\inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| = |\alpha| \inf_{\|x\|=1} \|(S - \lambda)\, x\| = |\alpha|\varepsilon.$$

Thus $\phi(\varepsilon) = |\alpha|\varepsilon$*. Next,*

$$\begin{aligned}\psi(\varepsilon) &= \sup_{\lambda \in f^{-1}(\Sigma_{ap,\varepsilon}(f(S)))} \left\{ \inf_{\|x\|=1} \|(S - \lambda)x\| \right\} \\ &= \frac{1}{|\alpha|} \sup_{\lambda \in f^{-1}(\Sigma_{ap,\varepsilon}(f(S)))} \left\{ \inf_{\|x\|=1} \left\|(f(S) - f(\lambda))\, x\right\| \right\}.\end{aligned}$$

Let $\lambda \in f^{-1}(\Sigma_{ap,\varepsilon}(f(S))$*. If* $f(\lambda) \in \Sigma_{ap,\varepsilon}(f(S)) \setminus \sigma(f(S))$*, then* $f(\lambda) \in \Lambda_\varepsilon(f(S)) \setminus \sigma(f(S))$*. In this case*

$$\inf_{\|x\|=1} \|(S - \lambda)\, x\| = \frac{1}{|\alpha|} \inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| = \frac{1}{|\alpha|} \frac{1}{\|(f(T) - f(\lambda))^{-1}\|} \leq \frac{\varepsilon}{|\alpha|}.$$

If $f(\lambda) \in \sigma(f(S)) \cap \Sigma_{ap,\varepsilon}(f(S))$*, then*

$$\inf_{\|x\|=1} \|(S - \lambda)\, x\| = \frac{1}{|\alpha|} \inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| \leq \frac{\varepsilon}{|\alpha|}.$$

Moreover, for $|\lambda| = 1 - \varepsilon$*, from equation 6*

$$\inf_{\|x\|=1} \|(S - \lambda)\, x\| = \frac{1}{|\alpha|} \inf_{\|x\|=1} \|(f(S) - f(\lambda))x\| = \frac{\varepsilon}{|\alpha|}.$$

Thus $\psi(\varepsilon) = \frac{\varepsilon}{|\alpha|}$ *and*

$$f(\Sigma_{ap,\varepsilon}(S)) = \Sigma_{ap,\varepsilon}(f(S)).$$

The following is another version of the spectral mapping theorem. It is proved for the set $\sigma_{ap,\varepsilon}(T)$

Theorem 2.17. *Let $T \in B(X)$. Given an arbitrarily small $\varepsilon' > 0$, let Ω be an open subset of $\mathbb{C}$ containing $\Lambda_{\varepsilon'}(T)$ and $A_\Omega = \{D \in B(X) : \sigma(D) \subseteq \Omega\}$. Suppose*

1. *f be an injective analytic function defined on Ω, the map*

   $$\Gamma : A_\Omega \to B(X) \text{ defined by } \Gamma(T) = f(T)$$

   has the invertible Fréchet derivative $D\Gamma$ at T,
2. $\displaystyle\bigcup_{\{D\in B(X):\|D\|\le\varepsilon\}} \sigma_{ap}(T+D) = \Sigma_{ap,\varepsilon}(T)$ *for all $\varepsilon > 0$,*
3. $\displaystyle\bigcup_{\{D\in B(X):\|D\|\le\varepsilon\}} \sigma_{ap}(f(T)+D) = \Sigma_{ap,\varepsilon}(f(T))$ *for all $\varepsilon > 0$,*

then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$, we define the following sets

$$\begin{aligned} \gamma_\varepsilon &:= \sup\{\|f(T+S) - f(T)\| : \|S\| \le \varepsilon\} \\ \delta_\varepsilon &:= \sup\{\|S\| : \|f(T+S) - f(T)\| \le \varepsilon\}. \end{aligned}$$

The following holds,

1. $\lim\limits_{\varepsilon\to 0} \gamma_\varepsilon = 0.$
2. $f(\Sigma_{ap,\varepsilon}(T)) \subseteq \Sigma_{ap,\gamma_\varepsilon}(f(T)).$
3. $\lim\limits_{\varepsilon\to 0} \delta_\varepsilon = 0.$
4. $\Sigma_{ap,\varepsilon}(f(T)) \subseteq f\left(\Sigma_{ap,\delta_\varepsilon}(T)\right).$

Proof. *We first prove that $\lim\limits_{\varepsilon\to 0} \gamma_\varepsilon = 0$. The map*

$$\Gamma : A_\Omega \to B(X) \text{ defined by } \Gamma(T) = f(T)$$

is continuous at T by the definition of $f(T)$. Consider $\delta > 0$. There exists $\delta' > 0$ such that

$$\|f(T+S) - f(T)\| < \delta \text{ for all } \|S\| < \delta'.$$

From the above equation it is clear that $\gamma_\varepsilon < \delta$ whenever $\varepsilon < \delta'$.

Take $\varepsilon_0 = \varepsilon'$. If $\mu \in \Sigma_{ap,\varepsilon}(T)$, then by our assumption there exists $D \in B(X)$ with $\|D\| \le \varepsilon$ such that $\mu \in \sigma_{ap}(T+D)$. From Theorem 2.9, $f(\mu) \in \sigma_{ap}(f(T+D))$. By the definition of γ_ε,

$$\|f(T+D) - f(T)\| \le \gamma_\varepsilon.$$

If $E = f(T+D) - f(T)$, then $f(\mu) \in \sigma_{ap}(f(T)+E)$ and $\|E\| \le \gamma_\varepsilon$. By our assumption, $f(\mu) \in \sigma_{ap,\gamma_\varepsilon}(f(T))$ and hence

$$f(\Sigma_{ap,\varepsilon}(T)) \subseteq \Sigma_{ap,\gamma_\varepsilon}(f(T)).$$

Next, we show that, $\lim\limits_{\varepsilon\to 0} \delta_\varepsilon = 0$. Consider the inverse map

$$g : f(\Omega) \to \Omega \text{ defined by } g(\lambda) = f^{-1}(\lambda).$$

The map g is well defined. Because f is injective, we have g is analytic. Consider $B_{f(\Omega)} = \{S \in B(X) : \sigma(S) \subseteq f(\Omega)\}$. Since $\sigma(T) \subseteq \Omega$, by spectral mapping theorem $f(\sigma(T)) \subseteq f(\Omega)$ and hence, $f(T) \in B_{f(\Omega)}$. The map

$$\Gamma' : B_{f(\Omega)} \to B(X) \text{ defined by } \Gamma'(S) = g(S)$$

is continuous by the definition of $g(S)$. In particular, Γ' is continuous at $f(T)$. Consider $\delta > 0$. There exists $\delta' > 0$ such that

$$\|\Gamma'(f(T)) - \Gamma'(E)\| < \delta \text{ for all } \|E - f(T)\| < \delta'.$$

If $\varepsilon < \delta'$, then by the last equation, we have $\delta_\varepsilon < \delta$. Thus $\lim_{\varepsilon\to 0} \delta_\varepsilon = 0$.

By our assumption, the Fréchet derivate $D\Gamma$ at T is invertible, so by inverse mapping theorem, there exists an open set U which contains T such that the map $U \mapsto \Gamma(U)$ is invertible and $\Gamma(U)$ is open. Since $f(T) \in \Gamma(U)$, there exists δ_1 such that $U' = \{N \in B(X) : \|f(T) - N\| < \delta_1\} \subseteq \Gamma(U)$. Choose $\varepsilon_0 < \min\{\varepsilon', \delta_1\}$.

For any $\varepsilon \le \varepsilon_0$, if $\mu \in \Sigma_{ap,\varepsilon}(f(T))$, then by our assumption there exists $D \in B(X)$ with $\|D\| \le \varepsilon$ such that $\mu \in \Sigma_{ap}(f(T) + D)$. By inverse mapping theorem, there exists $E \in U$ such that

$$f(T + E) = f(T) + D.$$

We have $\mu \in \Sigma_{ap}(f(T + E))$. By Theorem 2.9, and f is injective, there exists $\lambda \in \Sigma_{ap}(T + E)$ such that $f(\lambda) = \mu$. Next,

$$\|f(T + E) - f(T)\| \le \varepsilon \text{ and } \|E\| < \delta_\varepsilon.$$

implies,

$$\lambda \in \Sigma_{ap,\delta_\varepsilon}(T).$$

Hence, $f(\lambda) \in f(\Sigma_{ap,\delta_\varepsilon}(T))$.

Remark 2.18. *Note that in Theorem 2.17,the proof of the existence of γ_ε, the fact $\lim_{\varepsilon\to 0} \gamma_\varepsilon = 0$ and the fact $f(\Sigma_{ap,\varepsilon}(T)) \subseteq \Sigma_{ap,\gamma_\varepsilon}(f(T))$ does not require the injectivity of f, $\bigcup_{\{D\in B(X):\|D\|\le\varepsilon\}} \sigma_{ap}(f(T) + D) = \Sigma_{ap,\varepsilon}(f(T))$ and the Fréchet derivative of Γ at T is invertible.*

In the following example, we find γ_ε based on the conclusions given in the above remark.

Example 2.19. *Consider the operator S given in example 2.16 and Ω be an open subset of $\mathbb{C}$ containing $\Lambda_\varepsilon(S)$ for some $\varepsilon > 0$. We define the map*

$$f : \Omega \to \mathbb{C} \text{ by } f(z) = z^2.$$

For any $E \in B(\ell^2(\mathbb{N}))$ with $\|E\| \le \varepsilon$, the following observation has been made

$$f(S + E) - f(S) = (S + E)^2 - S^2 = SE + ES + E^2.$$

Since, $\|S\| = 1$, we have

$$\|f(S + E) - f(S)\| = \|SE + ES + E^2\| \le 2\varepsilon + \varepsilon^2.$$

Hence $\gamma_\varepsilon = \sup\{\|f(T + E) - f(T)\| : \|E\| \le \varepsilon\} \le 2\varepsilon + \varepsilon^2$. For the operator $E_0 = \varepsilon S$, we have $\|E_0\| = \varepsilon$ and

$$\|f(S + E_0) - f(S)\| = \|SE_0 + E_0S + E_0^2\| = (2\varepsilon + \varepsilon^2)\|S^2\|$$

It is easy to see that $\|S^2\| = 1$. Thus $\gamma_\varepsilon = (2\varepsilon + \varepsilon^2)$.

In the next example we calculate δ_ε.

Example 2.20. *Consider the operator S given in example 2.16, Ω be an open subset of $\mathbb{C}$ containing $\Lambda_\varepsilon(S)$ for some $\varepsilon > 0$. For $\alpha, \beta \in \mathbb{C}$, take the map*

$$f : \Omega \to \mathbb{C} \text{ by } f(z) = \alpha z + \beta.$$

Then f is injective, the Fréchet derivative of Γ at S is invertible. For any $E \in B(\ell^2(\mathbb{N}))$, if $\|f(S+E) - f(S)\| \leq \varepsilon$, then

$$\|\alpha S + \alpha E + \beta - (\alpha S + \beta)\| \leq \varepsilon.$$

Hence, $\|E\| \leq \frac{\varepsilon}{|\alpha|}$. It follows that

$$\delta_\varepsilon = \sup\{\|E\| : \|f(T+E) - f(T)\| \leq \varepsilon\} \leq \frac{\varepsilon}{|\alpha|}.$$

Since $\|S\| = 1$, for the operator $E_0 = \frac{\varepsilon}{|\alpha|}S$, we have,

$$\|f(S+E_0) - f(S)\| = \varepsilon.$$

Thus $\delta_\varepsilon = \frac{\varepsilon}{|\alpha|}$.

Acknowledgments

The authors express their gratitude to the anonymous referees and editors for their valuable suggestions and comments, which have significantly enhanced the quality and presentation of the manuscript.

Compliance with ethical standards

Conflict of interest:

The authors have equally contributed and give their consent for publication. The authors declare that they have no conflict of interest.

Research involving human participants and/or animals:

This paper does not contain any studies involving with human participants/ animals.

References

[1] Ammar, Aymen; Jeribi, Aref; Mahfoudhi, Kamel. *The essential approximate pseudospectrum and related results.* Filomat 32 (2018), no. 6, 2139–2151.

[2] Davies, E. Brian. *Linear operators and their spectra.* Cambridge Studies in Advanced Mathematics, 106. Cambridge University Press, Cambridge, 2007. xii+451 pp. ISBN: 978-0-521-86629-3; 0-521-86629-4.

[3] Kimura, F. *Spectral mapping theorem for approximate spectra and its applications.* Nihonkai Math. J. **13** (2002), no. 2, 183–189.

[4] Krishna Kumar, G.; Kulkarni, S. H. *An analogue of the spectral mapping theorem for condition spectrum.* Concrete operators, spectral theory, operators in harmonic analysis and approximation, 299–316, Oper. Theory Adv. Appl., 236, Birkhäuser/Springer, Basel, 2014.

[5] Nevanlinna, Olavi. *Convergence of iterations for linear equations.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993. viii+177 pp. ISBN: 3-7643-2865-7.

[6] Reddy, Satish C.; Trefethen, Lloyd N. *Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues. Spectral and high order methods for partial differential equations (Como, 1989).* Comput. Methods Appl. Mech. Engrg. 80 (1990), no. 1-3, 147–164.

[7] Rudin, Walter. *Functional analysis.* Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991. xviii+424 pp. ISBN: 0-07-054236-8.

[8] Sánchez-Perales, S.; Djordjević, S. V. *Continuity of spectrum and approximate point spectrum on operator matrices.* J. Math. Anal. Appl. 378 (2011), no. 1, 289–294.

[9] Trefethen, L. N. *Pseudospectra of linear operators.* SIAM Rev. 39 (1997), no. 3, 383–406.

[10] Trefethen, L. N. *Approximation theory and numerical linear algebra.* Algorithms for approximation, II (Shrivenham, 1988), 336–360, Chapman and Hall, London, 1990.

[11] Trefethen, N.; Embree, M. *Spectra and pseudospectra. The behavior of nonnormal matrices and operators.* Princeton University Press, Princeton, NJ, 2005. xviii+606 pp. ISBN: 978-0-691-11946-5; 0-691-11946-5.

[12] Varah, J. M. *The computation of bounds for the invariant subspaces of a general matrix operator, Ph.D. thesis, Stanford University.* ProQuest LLC, Ann Arbor, (1967).

[13] Veeramani, S.; Ganesh, J. *Some remarks on the stability of approximate pseudospectrum.* Indian J Pure Appl Math (2024).

[14] Wolff, M. P. H. *Discrete approximation of unbounded operators and approximation of their spectra.* J. Approx. Theory 113 (2001), no. 2, 229–244.