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Refinements of some well-known numerical radius inequalities

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Abstract. In this article, we obtain some numerical radius inequalities that refine the well-known inequality $w(A) \leq \|A\|$ using the Cauchy-Schwarz inequality, a generalization of Buzano's inequality, and the mixed Schwarz inequality. Also, we prove some numerical radius inequalities for finite sums of operators that include functions of specific characters.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathbb{H}, \langle \cdot, \cdot \rangle)$. For every $A \in \mathbb{B}(\mathbb{H})$, we denote by $|A| = (A^*A)^{\frac{1}{2}}$ the positive square root of A^*A. The numerical range of A, denoted by $W(A)$, is defined by

$$W(A) = \{\langle Ax, x \rangle : x \in \mathbb{H}, \|x\| = 1\}.$$

The classical numerical radius $w(A)$, is defined by

$$w(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

Also, let the Crawford number $m(A)$ be defined as

$$m(A) = \inf\{|\langle Ax, x \rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

The usual operator norm of an operator A is defined to be

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{H}, \|x\| = 1\}.$$

It is known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for every $A \in \mathbb{B}(\mathbb{H})$, we have

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1}$$

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Many refinements of the inequalities (1) have been obtained. We refer the reader to [1], [8], [10], [11], [13], [17], [19], [21], [22], and references therein. For more generalizations and recent related results on the numerical radius $w(\cdot)$, the reader is referred to [2], [4], [5], [9], and [23].

Below, we list some results regarding the inequalities (1).

In [18], Kittaneh proved that for every $A \in \mathbb{B}(\mathbb{H})$,

$$w^2(A) \le \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\|. \tag{2}$$

The inequality (2) was improved in [14] as follows:

$$w(A) \le \frac{1}{2}\sqrt{\left\| |A|^2 + |A^*|^2 \right\| + 2w(|A|\,|A^*|)}. \tag{3}$$

In [20], the authors gave a refinement of the inequality (3):

$$w(A) \le \frac{1}{2}\sqrt{\left\| |A|^2 + |A^*|^2 \right\| + 2\left\|Re(|A|\,|A^*|)\right\|}. \tag{4}$$

The Cauchy-Schwarz inequality states that for $x, y \in \mathbb{H}$,

$$\left|\langle x, y\rangle\right| \le \|x\|\,\|y\|. \tag{5}$$

A generalization of the Cauchy-Schwarz inequality is Buzano's inequality [12], which states that for $x, y, e \in \mathbb{H}$ with $\|e\| = 1$,

$$\left|\langle x, e\rangle\langle e, y\rangle\right| \le \frac{\left|\langle x, y\rangle\right| + \|x\|\,\|y\|}{2}. \tag{6}$$

Another generalization of the Cauchy-Schwarz inequality is the mixed Schwarz inequality , which states that for every $x, y \in \mathbb{H}$ and $A \in \mathbb{B}(\mathbb{H})$,

$$\left|\langle Ax, y\rangle\right|^2 \le \langle |A|\,x, x\rangle\,\langle |A^*|\,y, y\rangle. \tag{7}$$

A generalization of the mixed Schwarz inequality was introduced in [15], which states that for every $x, y \in \mathbb{H}, 0 \le \alpha \le 1$, and $A \in \mathbb{B}(\mathbb{H})$,

$$\left|\langle Ax, y\rangle\right|^2 \le \left\langle |A|^{2\alpha}\,x, x\right\rangle\left\langle |A^*|^{2(1-\alpha)}\,y, y\right\rangle. \tag{8}$$

In Section 2, we use the Cauchy-Schwarz inequality, a generalization of Buzano's inequality, and the mixed Schwarz inequality to obtain several refinements of the second inequality in (1). In Section 3, we prove some numerical radius inequalities that include functions of specific characters that refine the second inequality in (1) and the inequality (2).

2. Numerical radius inequalities using a generalization of Buzano's inequality

We begin our study by the following two lemmas. The first one (see [8]) gives a generalization of Buzano's inequality (6). For the second lemma, we refer to [3].

Lemma 2.1. *Let $x_1, x_2, ..., x_n, e \in \mathbb{H}$, where $\|e\| = 1$. Then*

$$\left|\prod_{k=1}^{n}\langle x_k, e\rangle\right| \le \frac{\left|\langle x_1, x_2\rangle \prod\limits_{k=3}^{n}\langle x_k, e\rangle\right| + \prod\limits_{k=1}^{n}\|x_k\|}{2}.$$

Lemma 2.2. *Let $a, b \in [0, \infty)$. Then*

$$(a+b)^r \le a^r + b^r + (2^r - 2)\min\{a^r, b^r\} \quad \text{for } 0 < r \le 1$$

and

$$a^r + b^r + (2^r - 2)\min\{a^r, b^r\} \le (a+b)^r \quad \text{for } r \ge 1.$$

Our first result in this section can be stated as follows.

Theorem 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$w(A) \le \frac{1}{\sqrt[3]{4}}\left[\left(\sqrt[3]{2}-1\right)\left(w(A^3) + \left\|A^2\right\| \|A\|\right)^{\frac{1}{3}} + \|A\|\right]. \tag{9}$$

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. From Lemma 2.1 (for $n = 3$), we have

$$\begin{aligned}
|\langle Ax, x\rangle| &= |\langle Ax, x\rangle \langle A^*x, x\rangle \langle A^*x, x\rangle|^{\frac{1}{3}} \\
&\le \left(\frac{|\langle Ax, A^*x\rangle \langle A^*x, x\rangle| + \|Ax\| \|A^*x\|^2}{2}\right)^{\frac{1}{3}} \\
&= \left(\frac{\left|\left\langle A^2x, x\right\rangle \langle A^*x, x\rangle\right| + \|Ax\| \|A^*x\|^2}{2}\right)^{\frac{1}{3}} \\
&\le \left(1 - \frac{1}{\sqrt[3]{2}}\right)\left|\left\langle A^2x, x\right\rangle \langle A^*x, x\rangle\right|^{\frac{1}{3}} + \left(\frac{\|Ax\| \|A^*x\|^2}{2}\right)^{\frac{1}{3}} \text{ (by Lemma 2.2).}
\end{aligned}$$

Also, by Buzano's inequality (6), we have

$$\begin{aligned}
\left|\left\langle A^2x, x\right\rangle \langle A^*x, x\rangle\right| &\le \frac{\left|\left\langle A^2x, A^*x\right\rangle\right| + \left\|A^2x\right\| \|A^*x\|}{2} \\
&= \frac{\left|\left\langle A^3x, x\right\rangle\right| + \left\|A^2x\right\| \|A^*x\|}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\langle Ax, x\rangle| &\le \left(1 - \frac{1}{\sqrt[3]{2}}\right)\left|\frac{\left|\left\langle A^3x, x\right\rangle\right| + \left\|A^2x\right\| \|A^*x\|}{2}\right|^{\frac{1}{3}} + \left(\frac{\|Ax\| \|A^*x\|^2}{2}\right)^{\frac{1}{3}} \\
&\le \left(1 - \frac{1}{\sqrt[3]{2}}\right)\left(\frac{w(A^3) + \left\|A^2\right\| \|A\|}{2}\right)^{\frac{1}{3}} + \frac{\|A\|}{\sqrt[3]{2}} \\
&= \frac{1}{\sqrt[3]{4}}\left[\left(\sqrt[3]{2}-1\right)\left(w(A^3) + \left\|A^2\right\| \|A\|\right)^{\frac{1}{3}} + \sqrt[3]{2}\,\|A\|\right].
\end{aligned}$$

Therefore, taking the supremum over $\|x\| = 1$, we get the desired inequality. □

We can use the inequalities $w(A^3) \le \left\|A^3\right\| \le \|A\|^3$ and $\left\|A^2\right\| \le \|A\|^2$ to show that the inequality (9) is an improvement of the second inequality in (1).

To show a non-trivial improvement, we consider the matrix $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then $A^3 = 0$, and so

$$\frac{1}{\sqrt[3]{4}}\left[\left(\sqrt[3]{2}-1\right)\left(w(A^3)+\left\|A^2\right\|\|A\|\right)^{\frac{1}{3}}+\sqrt[3]{2}\,\|A\|\right]<\|A\|=2.$$

Now, we prove the following numerical radius inequality.

Theorem 2.4. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$w(A) \le \left(1-\frac{1}{\sqrt[3]{2}}\right)w^{\frac{1}{3}}\left(A^2A^*\right)+\frac{\|A\|}{\sqrt[3]{2}}. \tag{10}$$

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. By the Cauchy-Schwarz inequality (5), we have

$$|\langle Ax, x\rangle|^3 \le \|A^*x\|^2\,|\langle Ax, x\rangle| = \left|\left\langle |A^*|^2 x, x\right\rangle \langle x, A^*x\rangle\right|.$$

Now, by Buzano's inequality (6), we have

$$\begin{aligned}
|\langle Ax, x\rangle| &\le \left(\frac{\left|\left\langle |A^*|^2 x, A^*x\right\rangle\right|+\left\||A^*|^2 x\right\|\|A^*x\|}{2}\right)^{\frac{1}{3}} \\
&= \frac{1}{\sqrt[3]{2}}\left(\left|\left\langle A\,|A^*|^2 x, x\right\rangle\right|+\left\||A^*|^2 x\right\|\|A^*x\|\right)^{\frac{1}{3}} \\
&\le \left(1-\frac{1}{\sqrt[3]{2}}\right)\left|\left\langle A\,|A^*|^2 x, x\right\rangle\right|^{\frac{1}{3}}+\left(\frac{\left\||A^*|^2 x\right\|\|A^*x\|}{2}\right)^{\frac{1}{3}} \quad \text{(by Lemma 2.2)} \\
&\le \left(1-\frac{1}{\sqrt[3]{2}}\right)\left|\left\langle A\,|A^*|^2 x, x\right\rangle\right|^{\frac{1}{3}}+\left(\frac{\|A\|^3}{2}\right)^{\frac{1}{3}} \\
&\le \left(1-\frac{1}{\sqrt[3]{2}}\right)w^{\frac{1}{3}}\left(A\,|A^*|^2\right)+\frac{\|A\|}{\sqrt[3]{2}}.
\end{aligned}$$

Therefore, taking the supremum over $\|x\| = 1$, we get the desired inequality. □

Replacing A by A^* in the inequality (10), we get

$$w(A) \le \left(1-\frac{1}{\sqrt[3]{2}}\right)w^{\frac{1}{3}}\left(A^*A^2\right)+\frac{\|A\|}{\sqrt[3]{2}}. \tag{11}$$

Consider the matrix $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then, by direct computations, we see that

$$w(A^2A^*) = 1 < w(A^*A^2) = 2.$$

Now, by Theorem 2.4 and the inequality (11), we have the following corollary.

Corollary 2.5. *Let $A \in \mathbb{B}(\mathbb{H})$. If $w(A) = \|A\|$, then*

$$w(A) \le \left(1-\frac{1}{\sqrt[3]{2}}\right)\min\{w^{\frac{1}{3}}\left(A^2A^*\right), w^{\frac{1}{3}}\left(A^*A^2\right)\}+\frac{\|A\|}{\sqrt[3]{2}}.$$

The inequality in Corollary 2.5 gives an improvement of the second inequality in (1). This can be shown easily since $w\left(A^2A^*\right) \leq \left\|A^2\right\| \|A\| \leq \|A\|^3$ and $w\left(A^*A^2\right) \leq \left\|A^2\right\| \|A\| \leq \|A\|^3$.

Now, we give necessary conditions for the equality $w(A) = \|A\|$.

Corollary 2.6. *Let $A \in \mathbb{B}(\mathbb{H})$ be such that $w(A) = \|A\|$. Then*

$$w\left(A^2A^*\right) = w\left(A^*A^2\right) = \|A\|^3 .$$

Proof. Th result follows from Theorem 2.4 together with the inequality (11). □

In the following corollary, we show that the necessary conditions for the equality $w(A) = \|A\|$ are also sufficient.

Corollary 2.7. *Let $A \in \mathbb{B}(\mathbb{H})$. If any one of the following conditions holds*

(a) $w\left(A^2A^*\right) = \|A\|^3$,

(b) $w\left(A^*A^2\right) = \|A\|^3$,

then $w(A) = \|A\|$.

Proof. First observe that $\left|\langle Ay, y\rangle\right| \leq w(A) \left\|y\right\|^2$ for every $y \in \mathbb{H}$.

For part (a), we have

$$\left|\left\langle A^2A^*x, x\right\rangle\right| = |\langle AA^*x, A^*x\rangle| \leq w(A) \|A^*x\|^2 .$$

Taking the supremum over $\|x\| = 1$, we get

$$\|A\|^3 = w\left(A^2A^*\right) \leq w(A) \|A\|^2 .$$

So,

$$\|A\| \leq w(A),$$

which implies that $w(A) = \|A\|$ when $w\left(A^2A^*\right) = \|A\|^3$.

The sufficiency of part (b) can be proved similarly. □

Our next theorem reads as follows.

Theorem 2.8. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$w(A) \leq \left(1 - \frac{1}{\sqrt[3]{2}}\right)\left(\frac{w(|A|\, A\, |A^*|) + \left\|A^2\right\| \|A\|}{2}\right)^{\frac{1}{3}} + \frac{\|A\|}{\sqrt[3]{2}}.$$

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. By the mixed Schwarz inequality (7), we have

$$|\langle Ax, x\rangle|^3 \leq \langle |A^*|\, x, x\rangle \langle A^*x, x\rangle \langle |A|\, x, x\rangle .$$

From Lemma 2.1 (for $n = 3$), we have

$$\begin{aligned}
|\langle Ax, x\rangle| &\leq \left(\frac{|\langle |A^*|\,x, A^*x\rangle \langle |A|\,x, x\rangle| + \||A^*|\,x\|\,\|A^*x\|\,\||A|\,x\|}{2}\right)^{\frac{1}{3}} \\
&= \frac{1}{\sqrt[3]{2}} \left(|\langle |A^*|\,x, A^*x\rangle \langle |A|\,x, x\rangle| + \||A^*|\,x\|\,\|A^*x\|\,\||A|\,x\|\right)^{\frac{1}{3}} \\
&\leq \left(1 - \frac{1}{\sqrt[3]{2}}\right) |\langle A\,|A^*|\,x, x\rangle \langle |A|\,x, x\rangle|^{\frac{1}{3}} \\
&\quad + \left(\frac{\||A^*|\,x\|\,\|A^*x\|\,\||A|\,x\|}{2}\right)^{\frac{1}{3}} \quad \text{(by Lemma 2.2)} \\
&= \left(1 - \frac{1}{\sqrt[3]{2}}\right) \left(|\langle A\,|A^*|\,x, x\rangle| \langle |A|\,x, x\rangle\right)^{\frac{1}{3}} + \left(\frac{\||A^*|\,x\|\,\|A^*x\|\,\||A|\,x\|}{2}\right)^{\frac{1}{3}}.
\end{aligned}$$

By Buzano's inequality (6), we have

$$\begin{aligned}
|\langle A\,|A^*|\,x, x\rangle| \langle |A|\,x, x\rangle &\leq \frac{|\langle A\,|A^*|\,x, |A|\,x\rangle| + \|A\,|A^*|\,x\|\,\||A|\,x\|}{2} \\
&= \frac{|\langle |A|\,A\,|A^*|\,x, x\rangle| + \|A\,|A^*|\,x\|\,\||A|\,x\|}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\langle Ax, x\rangle| &\leq \left(1 - \frac{1}{\sqrt[3]{2}}\right) \left(\frac{|\langle |A|\,A\,|A^*|\,x, x\rangle| + \|A\,|A^*|\,x\|\,\||A|\,x\|}{2}\right)^{\frac{1}{3}} \\
&\quad + \left(\frac{\||A^*|\,x\|\,\|A^*x\|\,\||A|\,x\|}{2}\right)^{\frac{1}{3}} \\
&\leq \left(1 - \frac{1}{\sqrt[3]{2}}\right) \left(\frac{w(|A|\,A\,|A^*|) + \|A\,|A^*|\|\,\|A\|}{2}\right)^{\frac{1}{3}} + \frac{\|A\|}{\sqrt[3]{2}}.
\end{aligned}$$

From the fact that $\|X^*X\| = \|XX^*\| = \|X\|^2$ for every $X \in \mathbb{B}(\mathbb{H})$, it is easy to verify that $\|A\,|A^*|\| = \left\|A^2\right\|$. So,

$$|\langle Ax, x\rangle| \leq \left(1 - \frac{1}{\sqrt[3]{2}}\right) \left(\frac{w(|A|\,A\,|A^*|) + \left\|A^2\right\|\,\|A\|}{2}\right)^{\frac{1}{3}} + \frac{\|A\|}{\sqrt[3]{2}}.$$

Therefore, taking the supremum over $\|x\| = 1$, we get the desired inequality. □

Now, combining Theorem 2.8 and Corollary 2.5, we obtain the following corollary.

Corollary 2.9. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$w(A) \leq \left(1 - \frac{1}{\sqrt[3]{2}}\right) \min \left\{ \begin{array}{c} w^{\frac{1}{3}}\left(A^2A^*\right), w^{\frac{1}{3}}\left(A^*A^2\right) \\ , \left(\frac{w(|A|A|A^*|) + \left\|A^2\right\|\|A\|}{2}\right)^{\frac{1}{3}} \end{array} \right\} + \frac{\|A\|}{\sqrt[3]{2}}.$$

Clearly, the inequality in Corollary 2.9 is stronger than the second inequality in (1).
In the next two results, we give refinements of the second inequality in (1).

Theorem 2.10. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$w(A) \leq \frac{\left(\left(\sqrt[4]{2} - 1\right) w^{\frac{1}{4}}(A\,|A|) + \|A\|^{\frac{1}{2}}\right)\left(\left(\sqrt[4]{2} - 1\right) w^{\frac{1}{4}}(A^*\,|A^*|) + \|A\|^{\frac{1}{2}}\right)}{\sqrt{2}}.$$

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. By the mixed Schwarz inequality (7), we have

$$|\langle Ax, x\rangle|^2 \leq \langle |A|\, x, x\rangle \langle |A^*|\, x, x\rangle .$$

Hence,

$$\begin{aligned} |\langle Ax, x\rangle| &\leq \left(\langle |A|\, x, x\rangle \langle |A^*|\, x, x\rangle\, |\langle Ax, x\rangle \langle A^* x, x\rangle|\right)^{\frac{1}{4}} . \\ &= |\langle |A|\, x, x\rangle \langle A^* x, x\rangle|^{\frac{1}{4}}\, |\langle |A^*|\, x, x\rangle \langle Ax, x\rangle|^{\frac{1}{4}} \end{aligned}$$

By Buzano's inequality (6), we have

$$\begin{aligned} &|\langle |A|\, x, x\rangle \langle A^* x, x\rangle|^{\frac{1}{4}} \\ &\leq \left(\frac{|\langle |A|\, x, A^* x\rangle| + \||A|\, x\|\, \|A^* x\|}{2}\right)^{\frac{1}{4}} \\ &= \frac{1}{\sqrt[4]{2}} \left(|\langle |A|\, x, A^* x\rangle| + \||A|\, x\|\, \|A^* x\|\right)^{\frac{1}{4}} \\ &\leq \left(1 - \frac{1}{\sqrt[4]{2}}\right) |\langle |A|\, x, A^* x\rangle|^{\frac{1}{4}} + \left(\frac{\||A|\, x\|\, \|A^* x\|}{2}\right)^{\frac{1}{4}} \quad \text{(by Lemma 2.2)} \\ &= \left(1 - \frac{1}{\sqrt[4]{2}}\right) |\langle A\, |A|\, x, x\rangle|^{\frac{1}{4}} + \frac{\|A\|^{\frac{1}{2}}}{\sqrt[4]{2}}. \\ &\leq \left(1 - \frac{1}{\sqrt[4]{2}}\right) w^{\frac{1}{4}}(A\, |A|) + \frac{\|A\|^{\frac{1}{2}}}{\sqrt[4]{2}}. \end{aligned}$$

Similarly,

$$|\langle |A^*|\, x, x\rangle \langle Ax, x\rangle|^{\frac{1}{4}} \leq \left(1 - \frac{1}{\sqrt[4]{2}}\right) w^{\frac{1}{4}}(A^*\, |A^*|) + \frac{\|A\|^{\frac{1}{2}}}{\sqrt[4]{2}}.$$

Thus,

$$|\langle Ax, x\rangle| \leq \frac{\left(\left(\sqrt[4]{2} - 1\right) w^{\frac{1}{4}}(A\, |A|) + \|A\|^{\frac{1}{2}}\right)\left(\left(\sqrt[4]{2} - 1\right) w^{\frac{1}{4}}(A^*\, |A^*|) + \|A\|^{\frac{1}{2}}\right)}{\sqrt{2}}.$$

Therefore, taking the supremum over $\|x\| = 1$, we get the desired inequality. □

Using a similar technique as in Theorem 2.10, we can prove the following result.

Theorem 2.11. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$w(A) \leq \frac{\left(\left(\sqrt[4]{2} - 1\right) w^{\frac{1}{4}}(A^*\, |A|) + \|A\|^{\frac{1}{2}}\right)\left(\left(\sqrt[4]{2} - 1\right) w^{\frac{1}{4}}(A\, |A^*|) + \|A\|^{\frac{1}{2}}\right)}{\sqrt{2}}.$$

3. Upper bounds for numerical radius inequalities for finite sums of operators

We begin this section by the following lemmas. The first lemma can be found in [3]. The second lemma is known as Minkowski's inequality, and for the third lemma, we refer the reader to [7, p. 87–88]. The fourth lemma is a well-known result that can be proved by using the spectral theorem and Jensen's inequality. The inequalities in this lemma are of the Peierls-Bogoliubov type (see, e.g., [7, p. 281]). The last lemma (see [16]) presents a generalized formulation of the mixed Schwarz inequality.

Lemma 3.1. *Let $a, b \in \mathbb{R}$ and $r \geq 2$. Then*

$$|a+b|^r + |a-b|^r \geq 2\left(|a|^r + |b|^r + c_r(a,b)\right),$$

where $c_r(a,b) = (2^{\frac{r}{2}} - 2)\min\{|a|^r, |b|^r\}$.

Lemma 3.2. *Let $a_i, b_i > 0$ for $i = 1, 2, ..., n$. Then for $r \geq 1$,*

$$\left(\sum_{i=1}^{n}(a_i + b_i)^r\right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n} b_i^r\right)^{\frac{1}{r}}.$$

Lemma 3.3. *If f is a convex function on an interval J and if $a_i, i = 1, 2, ..., n$, are non–negative real numbers such that $\sum_{i=1}^{n} a_i = 1$, then*

$$f\left(\sum_{i=1}^{n} a_i t_i\right) \leq \sum_{i=1}^{n} a_i f(t_i) \text{ for all } t_i \in J.$$

In particular,

$$\left|\sum_{i=1}^{n} t_i\right|^r \leq n^{r-1}\sum_{i=1}^{n}|t_i|^r \text{ for all } t_i \in \mathbb{R} \text{ and } r \geq 1.$$

Lemma 3.4. *Let $A \in \mathbb{B}(\mathbb{H})$ and $x \in \mathbb{H}$ be such that $\|x\| = 1$. Then*

(a) $\langle |A|\, x, x\rangle^r \leq \langle |A|^r x, x\rangle$ *for* $r \geq 1$.
(b) $\langle |A|^r x, x\rangle \leq \langle |A|\, x, x\rangle^r$ *for* $0 < r \leq 1$.
(c) $|\langle Ax, x\rangle| \leq \langle |A|\, x, x\rangle$ *where A is self-adjoint.*

Lemma 3.5. *Let $A \in \mathbb{B}(\mathbb{H})$ and $x, y \in \mathbb{H}$ be any vectors. If f, g are non–negative continuous functions on $[0, \infty)$ with $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\left|\langle Ax, y\rangle\right|^2 \leq \left\langle f^2\left(|A|\right)x, x\right\rangle \left\langle g^2\left(|A^*|\right)y, y\right\rangle.$$

Our first result in this section can be stated as follows.

Theorem 3.6. *Let $R_i, S_i \in \mathbb{B}(\mathbb{H}), i = 1, ..., n$, and let f, g be non–negative continuous functions on $[0, \infty)$ with $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for $r \geq 1$, we have*

$$w^r\left(\sum_{i=1}^{n}(R_i + S_i)\right) \leq \left(n^{r-1}\right)\left(2^{r-2}\right)\left\|\sum_{i=1}^{n}(f^{2r}\left(|R_i|\right) + f^{2r}\left(|S_i|\right) + g^{2r}\left(\left|R_i^*\right|\right) + g^{2r}\left(\left|S_i^*\right|\right))\right\|.$$

Proof. For a unit vector $x \in \mathbb{H}$, we have

$$\left|\left\langle\sum_{i=1}^{n}(R_i + S_i)x, x\right\rangle\right|^r$$

$$
\begin{aligned}
&\le n^{r-1}\sum_{i=1}^{n}\left|\left\langle (R_i+S_i)x,x\right\rangle\right|^r \qquad \text{(by Lemma 3.3)}\\
&= n^{r-1}\sum_{i=1}^{n}\left|\left\langle R_i x,x\right\rangle+\left\langle S_i x,x\right\rangle\right|^r\\
&\le n^{r-1}\sum_{i=1}^{n}\left(\left|\left\langle R_i x,x\right\rangle\right|+\left|\left\langle S_i x,x\right\rangle\right|\right)^r \quad \text{(by the triangle inequality)}\\
&\le n^{r-1}\left[\left(\sum_{i=1}^{n}\left|\left\langle R_i x,x\right\rangle\right|^r\right)^{\frac{1}{r}}+\left(\sum_{i=1}^{n}\left|\left\langle S_i x,x\right\rangle\right|^r\right)^{\frac{1}{r}}\right]^r \text{(by Minkowski's inequality)}\\
&\le (2n)^{r-1}\left[\sum_{i=1}^{n}\left|\left\langle R_i x,x\right\rangle\right|^r+\sum_{i=1}^{n}\left|\left\langle S_i x,x\right\rangle\right|^r\right] \text{(by the convexity of } f(t)=t^r, r\ge 1)\\
&\le (2n)^{r-1}\left[\begin{array}{c}\sum_{i=1}^{n}\left\langle f^2\left(|R_i|\right)x,x\right\rangle^{\frac{r}{2}}\left\langle g^2\left(\left|R_i^*\right|\right)x,x\right\rangle^{\frac{r}{2}}\\ +\sum_{i=1}^{n}\left\langle f^2\left(|S_i|\right)x,x\right\rangle^{\frac{r}{2}}\left\langle g^2\left(\left|S_i^*\right|\right)x,x\right\rangle^{\frac{r}{2}}\end{array}\right] \text{(by Lemma 3.5)}\\
&\le \frac{(2n)^{r-1}}{2}\left[\begin{array}{c}\sum_{i=1}^{n}\left(\left\langle f^2\left(|R_i|\right)x,x\right\rangle^{r}+\left\langle g^2\left(\left|R_i^*\right|\right)x,x\right\rangle^{r}\right)\\ +\sum_{i=1}^{n}\left(\left\langle f^2\left(|S_i|\right)x,x\right\rangle^{r}+\left\langle g^2\left(\left|S_i^*\right|\right)x,x\right\rangle^{r}\right)\end{array}\right] \text{(by the AM-GM inequality)}\\
&\le \frac{(2n)^{r-1}}{2}\left[\begin{array}{c}\sum_{i=1}^{n}\left(\left\langle f^{2r}\left(|R_i|\right)x,x\right\rangle+\left\langle g^{2r}\left(\left|R_i^*\right|\right)x,x\right\rangle\right)\\ +\sum_{i=1}^{n}\left(\left\langle f^{2r}\left(|S_i|\right)x,x\right\rangle+\left\langle g^{2r}\left(\left|S_i^*\right|\right)x,x\right\rangle\right)\end{array}\right] \text{(by Lemma 3.4)}\\
&= \frac{(2n)^{r-1}}{2}\left[\sum_{i=1}^{n}\left\langle\left(\begin{array}{c}f^{2r}\left(|R_i|\right)+f^{2r}\left(|S_i|\right)\\ +g^{2r}\left(\left|R_i^*\right|\right)+g^{2r}\left(\left|S_i^*\right|\right)\end{array}\right)x,x\right\rangle\right].
\end{aligned}
$$

Taking the supremum over all unit vectors $x\in\mathbb{H}$, we get

$$
w^r\left(\sum_{i=1}^{n}(R_i+S_i)\right)\le\frac{(2n)^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(f^{2r}\left(|R_i|\right)+f^{2r}\left(|S_i|\right)+g^{2r}\left(\left|R_i^*\right|\right)+g^{2r}\left(\left|S_i^*\right|\right)\right)\right\|.
$$

□

We can obtain the following corollary from Theorem 3.6 by taking $n=1$.

Corollary 3.7. *Let $R,S\in\mathbb{B}(\mathbb{H})$, and let f,g be non–negative continuous functions on $[0,\infty)$ with $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then for $r\ge 1$, we have*

$$
w^r(R+S)\le 2^{r-2}\left\|f^{2r}\left(|R|\right)+f^{2r}\left(|S|\right)+g^{2r}\left(|R^*|\right)+g^{2r}\left(|S^*|\right)\right\|.
$$

Note that by taking $R=S$ and $f(t)=g(t)=\sqrt{t}$ in Corollary 3.7, we get

$$
w^r(S)\le\frac{1}{2}\left\||S|^r+|S^*|^r\right\|. \tag{12}
$$

Taking $r=2$, we get the inequality (2). By taking $r=1$ in the inequality (12), we get

$$
w(S)\le\frac{1}{2}\left\||S|+|S^*|\right\|, \tag{13}
$$

which refines the second inequality in (1). It should be mentioned that the inequality (13) was proved by Kittaneh in [17].

The following result gives a refinement for Theorem 2.2 given in [6].

Theorem 3.8. *Let $R, S \in \mathbb{B}(\mathbb{H})$, and let f, g be non–negative continuous functions on $[0,\infty)$ with $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then for $r \geq 2$, we have*

$$w^r(R+S) + 2^{r-1}c_r(m(R), m(S))$$
$$\leq 2^{r-3}\left\|f^{2r}(|R+S|) + g^{2r}\left(\left|(R+S)^*\right|\right) + f^{2r}(|R-S|) + g^{2r}\left(\left|(R-S)^*\right|\right)\right\|.$$

Proof. We have

$$|\langle (R+S)x,x\rangle|^r + 2^{r-1}c_r(|\langle Rx,x\rangle|, |\langle (Sx,x\rangle|)$$

$$\begin{aligned}
&= (|\langle Rx,x\rangle| + |\langle Sx,x\rangle|)^r + 2^{r-1}c_r(|\langle Rx,x\rangle|, |\langle (Sx,x\rangle|) \\
&\leq 2^{r-1}\left(|\langle Rx,x\rangle|^r + |\langle Sx,x\rangle|^r + c_r(|\langle Rx,x\rangle|, |\langle (Sx,x\rangle|)\right) \\
&\leq 2^{r-2}\left(|\langle (R+S)x,x\rangle|^r + |\langle (R-S)x,x\rangle|^r\right) \ \text{(by Lemma 3.1)} \\
&\leq 2^{r-2}\begin{pmatrix} \left\langle f^2(|R+S|)x,x\right\rangle^{r/2}\left\langle g^2\left(\left|(R+S)^*\right|\right)x,x\right\rangle^{r/2} + \\ \left\langle f^2(|R-S|)x,x\right\rangle^{r/2}\left\langle g^2\left(\left|(R-S)^*\right|\right)x,x\right\rangle^{r/2} \end{pmatrix} \ \text{(by Lemma 3.5)} \\
&\leq 2^{r-2}\begin{pmatrix} \left\langle f^r(|R+S|)x,x\right\rangle\left\langle g^r\left(\left|(R+S)^*\right|\right)x,x\right\rangle + \\ \left\langle f^r(|R-S|)x,x\right\rangle\left\langle g^r\left(\left|(R-S)^*\right|\right)x,x\right\rangle \end{pmatrix} \ \text{(by Lemma 3.4)} \\
&\leq 2^{r-3}\begin{pmatrix} \left\langle f^{2r}(|R+S|)x,x\right\rangle + \left\langle g^{2r}\left(\left|(R+S)^*\right|\right)x,x\right\rangle + \\ \left\langle f^{2r}(|R-S|)x,x\right\rangle + \left\langle g^{2r}\left(\left|(R-S)^*\right|\right)x,x\right\rangle \end{pmatrix} \ \text{(by the AM-GM inequality)} \\
&= 2^{r-3}\left\langle\begin{pmatrix} f^{2r}(|R+S|) + g^{2r}\left(\left|(R+S)^*\right|\right) + \\ f^{2r}(|R-S|) + g^{2r}\left(\left|(R-S)^*\right|\right) \end{pmatrix}x,x\right\rangle.
\end{aligned}$$

So,

$$|\langle (R+S)x,x\rangle|^r + 2^{r-1}c_r(m(R), m(S))$$
$$\leq 2^{r-3}\left\langle\begin{pmatrix} f^{2r}(|R+S|) + g^{2r}\left(\left|(R+S)^*\right|\right) + \\ f^{2r}(|R-S|) + g^{2r}\left(\left|(R-S)^*\right|\right) \end{pmatrix}x,x\right\rangle.$$

Taking the supremum over all unit vectors $x \in \mathbb{H}$, we get

$$w^r(R+S) + 2^{r-1}c_r(m(R), m(S))$$
$$\leq 2^{r-3}\left\|f^{2r}(|R+S|) + g^{2r}\left(\left|(R+S)^*\right|\right) + f^{2r}(|R-S|) + g^{2r}\left(\left|(R-S)^*\right|\right)\right\|.$$

□

When $R = S$ and $f(t) = g(t) = \sqrt{t}$ for all $t \in [0,\infty)$, we obtain the following corollary from Theorem 3.8.

Corollary 3.9. *Let $S \in \mathbb{B}(\mathbb{H})$. Then for $r \geq 2$, we have*

$$w^r(S) + (2^{\frac{r}{2}-1} - 1)m^r(S) \leq 2^{r-3}\left\||S|^r + |S^*|^r\right\|.$$

We conclude this section with the following related norm inequality.

Corollary 3.10. *Let $R, S \in \mathbb{B}(\mathbb{H})$ be self adjoint operators and $r \geq 2$. Then*

$$\|R+S\|^r + 2^{r-1}c_r(m(R), m(S)) \leq 2^{r-2}\left\||R+S|^r + |R-S|^r\right\|. \tag{14}$$

Proof. Let $f(t) = g(t) = \sqrt{t}$ in Theorem 3.8. Then, we get

$$\begin{aligned}
&\|R+S\|^r + 2^{r-1} c_r(m(R), m(S)) \\
&\leq 2^{r-3} \left\| |R+S|^r + \left|(R+S)^*\right|^r + |R-S|^r + \left|(R-S)^*\right|^r \right\| \\
&= 2^{r-2} \left\| |R+S|^r + |R-S|^r \right\|.
\end{aligned}$$

□

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